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Affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups

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Abstract: In this paper, we compute the Bott connection and their curvature on three-dimensional Lorentzian Lie groups with three different distributions, then we classify affine Ricci solitons associated to the Bott connection on the spaces under study.

Key words: Affine Ricci solitons, the Bott connection; three-dimensional Lorentzian Lie groups

1. Introduction

In geometry, Einstein metrics have been studied in three-dimensional Lorentzian manifolds. As a natural generalization of Einstein metrics, the definition of the Ricci soliton was introduced by Hamilton in [11]. Therefore, study of Ricci solitons over different geometric spaces has become more significant. Einstein manifolds associated to affine connections have been studied by many geometers. In [12] and [17], Wang studied Einstein manifolds associated to semisymmetric nonmetric connections and semisymmetric metric connections respectively.

Naturally, mathematicians start to study Ricci solitons associated to different affine connections. In [9], Crasmareanu gave several generalizations of Ricci solitons with linear connections. In [10], it is proved the equivalent conditions of the Ricci soliton in Kenmotsu manifold associated to the Schouten–Van Kampen connection to be steady. Hui-Prasad-Chakraborty studied Ricci solitons on Kenmotsu manifolds with respect to quarter symmetric nonmetric φ-connection in [14]. Perktas-Yildiz studied several soliton types on a quasi-Sasakian 3-manifold with respect to the Schouten–Van Kampen connection in [15]. In [18], Wang classify affine Ricci solitons associated to canonical connections and Kobayashi–Nomizu connections and perturbed canonical connections and perturbed Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups with some product structures. In [1, 4, 13], the definition of the Bott connection was introduced. In [5], F. Baudoin and E. Grong proved transverse Weitzenb$\ddot{o}$ck identities for the horizontal Laplacians of a totally geodesic foliation. In [6], the authors developed a variational theory of geodesics for the canonical variation of the metric of a totally geodesic foliation, where the Bott connection is a first natural connection on manifold M that respects the foliation structure. And because the Levi–Civita connection is not adapted to the study of follations because the horizontal and the vertical bundle may not be parallel. More adapted to the geometry of the foliation is the Bott connection. Therefore, we study the Bott connection on three-dimensional Lorentzian Lie groups. In this paper, we compute the Bott connection and their curvature on three-dimensional Lorentzian Lie groups

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with three different distributions, and we classify affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups with three different distributions.

In Sections 2 and 3, we recall the definition of the Bott connection, and give affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian unimodular and nonunimodular Lie groups with the first distribution. In Section 4, we give affine Ricci solitons associated to the perturbed Bott connection on three-dimensional Lorentzian Lie groups with the first distribution. In Section 5, we give affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups with the second distribution. In Section 6, we give affine Ricci solitons associated to the perturbed Bott connection on three-dimensional Lorentzian Lie groups with the second distribution. In Section 7, we give affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups with the third distribution. In Section 8, we give affine Ricci solitons associated to the perturbed Bott connection on three-dimensional Lorentzian Lie groups with the third distribution.

2. Affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian unimodular Lie groups with the first distribution

Firstly, throughout this paper, we shall by $\{G_i\}_{i=1,\cdots,7}$, denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g and having Lie algebra $\{\mathfrak{g}_i\}_{i=1,\cdots,7}$. Let ∇^L be the Levi–Civita connection of G_i and R^L its curvature tensor, then

$$R^L(X,Y)Z = \nabla^L_X\nabla^L_Y Z - \nabla^L_Y\nabla^L_X Z - \nabla^L_{[X,Y]}Z. \tag{2.1}$$

The Ricci tensor of (G_i, g) is defined by

$$\rho^L(X,Y) = -g(R^L(X,\widehat{e}_1)Y,\widehat{e}_1) - g(R^L(X,\widehat{e}_2)Y,\widehat{e}_2) + g(R^L(X,\widehat{e}_3)Y,\widehat{e}_3), \tag{2.2}$$

where $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Nextly, we recall the definition of the Bott connection ∇^B. Let M be a smooth manifold, and let $TM = span\{\widehat{e}_1,\widehat{e}_2,\widehat{e}_3\}$, then took the distribution $D = span\{\widehat{e}_1,\widehat{e}_2\}$ and $D^\perp = span\{\widehat{e}_3\}$.
The definition of the Bott connection ∇^B is given as follows: (see [4], [1], [13])

$$\nabla^B_X Y = \begin{cases} \pi_D(\nabla^L_X Y), & X,Y \in \Gamma^\infty(D) \\ \pi_D([X,Y]), & X \in \Gamma^\infty(D^\perp), Y \in \Gamma^\infty(D) \\ \pi_{D^\perp}([X,Y]), & X \in \Gamma^\infty(D), Y \in \Gamma^\infty(D^\perp) \\ \pi_{D^\perp}(\nabla^L_X Y), & X,Y \in \Gamma^\infty(D^\perp) \end{cases} \tag{2.3}$$

where π_D (resp. $\pi_D^\perp$) the projection on D (resp. $D^\perp$).
We define

$$R^B(X,Y)Z = \nabla^B_X\nabla^B_Y Z - \nabla^B_Y\nabla^B_X Z - \nabla^B_{[X,Y]}Z. \tag{2.4}$$

The Ricci tensor of (G_i, g) associated to the Bott connection ∇^B is defined by

$$\rho^B(X,Y) = -g(R^B(X,\widehat{e}_1)Y,\widehat{e}_1) - g(R^B(X,\widehat{e}_2)Y,\widehat{e}_2) + g(R^B(X,\widehat{e}_3)Y,\widehat{e}_3). \tag{2.5}$$

Let

$$\widetilde{\rho}^B(X,Y) = \frac{\rho^B(X,Y) + \rho^B(Y,X)}{2}. \tag{2.6}$$

We define:

$$(L_V^B g)(X,Y) := g(\nabla_X^B V, Y) + g(X, \nabla_Y^B V), \tag{2.7}$$

for vector X, Y, V.

Definition 2.1 *(G_i, g) is called the affine Ricci soliton associated to the Bott connection ∇^B if it satisfies*

$$(L_V^B g)(X,Y) + 2\widetilde{\rho}^B(X,Y) + 2\mu g(X,Y) = 0, \tag{2.8}$$

where μ is a real number and $V = \mu_1\widehat{e}_1 + \mu_2\widehat{e}_2 + \mu_3\widehat{e}_3$ and μ_1, μ_2, μ_3 are real numbers.

2.1 Affine Ricci solitons of G_1

By [3], we have the following Lie algebra of G_1 satisfies

$$[\widehat{e}_1, \widehat{e}_2] = \alpha\widehat{e}_1 - \beta\widehat{e}_3, \quad [\widehat{e}_1, \widehat{e}_3] = -\alpha\widehat{e}_1 - \beta\widehat{e}_2, \quad [\widehat{e}_2, \widehat{e}_3] = \beta\widehat{e}_1 + \alpha\widehat{e}_2 + \alpha\widehat{e}_3,\ \alpha \neq 0, \tag{2.9}$$

where $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 2.2 *([7],[3]) The Levi–Civita connection ∇^L of G_1 is given by*

$$\begin{aligned}
&\nabla^L_{\widehat{e}_1}\widehat{e}_1 = -\alpha\widehat{e}_2 - \alpha\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_2 = \alpha\widehat{e}_1 - \frac{\beta}{2}\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_3 = -\alpha\widehat{e}_1 - \frac{\beta}{2}\widehat{e}_2,\\
&\nabla^L_{\widehat{e}_2}\widehat{e}_1 = \frac{\beta}{2}\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_2 = \alpha\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_3 = \frac{\beta}{2}\widehat{e}_1 + \alpha\widehat{e}_2,\\
&\nabla^L_{\widehat{e}_3}\widehat{e}_1 = \frac{\beta}{2}\widehat{e}_2, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_2 = -\frac{\beta}{2}\widehat{e}_1 - \alpha\widehat{e}_3, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_3 = -\alpha\widehat{e}_2.
\end{aligned} \tag{2.10}$$

Lemma 2.3 *The Bott connection ∇^B of G_1 is given by*

$$\begin{aligned}
&\nabla^B_{\widehat{e}_1}\widehat{e}_1 = -\alpha\widehat{e}_2, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_2 = \alpha\widehat{e}_1, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_3 = 0,\\
&\nabla^B_{\widehat{e}_2}\widehat{e}_1 = 0, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_3 = \alpha\widehat{e}_3,\\
&\nabla^B_{\widehat{e}_3}\widehat{e}_1 = \alpha\widehat{e}_1 + \beta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_2 = -\beta\widehat{e}_1 - \alpha\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_3 = 0.
\end{aligned} \tag{2.11}$$

Lemma 2.4 *The curvature R^B of the Bott connection ∇^B of (G_1, g) is given by*

$$\begin{aligned}
&R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = \alpha\beta\widehat{e}_1 + (\alpha^2+\beta^2)\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = -(\alpha^2+\beta^2)\widehat{e}_1 - \alpha\beta\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = 0,\\
&R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = -3\alpha^2\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = -\alpha^2\widehat{e}_1, \quad R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = \alpha\beta\widehat{e}_3,\\
&R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = -\alpha^2\widehat{e}_1, \quad R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = \alpha^2\widehat{e}_2, \quad R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = -\alpha^2\widehat{e}_3.
\end{aligned} \tag{2.12}$$

By (2.5), we have

$$\begin{aligned}
&\rho^B(\widehat{e}_1, \widehat{e}_1) = -(\alpha^2+\beta^2), \quad \rho^B(\widehat{e}_1, \widehat{e}_2) = \alpha\beta, \quad \rho^B(\widehat{e}_1, \widehat{e}_3) = -\alpha\beta,\\
&\rho^B(\widehat{e}_2, \widehat{e}_1) = \alpha\beta, \quad \rho^B(\widehat{e}_2, \widehat{e}_2) = -(\alpha^2+\beta^2), \quad \rho^B(\widehat{e}_2, \widehat{e}_3) = \alpha^2,\\
&\rho^B(\widehat{e}_3, \widehat{e}_1) = \rho^B(\widehat{e}_3, \widehat{e}_2) = \rho^B(\widehat{e}_3, \widehat{e}_3) = 0.
\end{aligned} \tag{2.13}$$

Then,

$$\widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_1) = -(\alpha^2+\beta^2), \quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_2) = \alpha\beta, \quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_3) = -\frac{\alpha\beta}{2},$$
$$\widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_2) = -(\alpha^2+\beta^2), \quad \widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_3) = \frac{\alpha^2}{2}, \quad \widetilde{\rho}^B(\widehat{e}_3,\widehat{e}_3) = 0. \tag{2.14}$$

By (2.7), we have

$$(L_V^B g)(\widehat{e}_1,\widehat{e}_1) = 2\mu_2\alpha, \quad (L_V^B g)(\widehat{e}_1,\widehat{e}_2) = -\mu_1\alpha, \quad (L_V^B g)(\widehat{e}_1,\widehat{e}_3) = \mu_1\alpha - \mu_2\beta,$$
$$(L_V^B g)(\widehat{e}_2,\widehat{e}_2) = 0, \quad (L_V^B g)(\widehat{e}_2,\widehat{e}_3) = \mu_1\beta - (\mu_2+\mu_3)\alpha, \quad (L_V^B g)(\widehat{e}_3,\widehat{e}_3) = 0. \tag{2.15}$$

Then, if (G_1, g, V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following six equations:

$$\begin{cases} \mu_2\alpha - \alpha^2 - \beta^2 + \mu = 0 \\ 2\alpha\beta - \mu_1\alpha = 0 \\ \mu_1\alpha - \mu_2\beta - \alpha\beta = 0 \\ \alpha^2 + \beta^2 - \mu = 0 \\ (\mu_2+\mu_3)\alpha - \mu_1\beta - \alpha^2 = 0 \\ \mu = 0 \end{cases} \tag{2.16}$$

By solving (2.16) , we get $\alpha = 0$, there is a contradiction. So

Theorem 2.5 *(G_1, g, V) is not an affine Ricci soliton associated to the Bott connection ∇^B.*

2.2 Affine Ricci solitons of G_2

By [3], we have the following Lie algebra of G_2 satisfies

$$[\widehat{e}_1,\widehat{e}_2] = \gamma\widehat{e}_2 - \beta\widehat{e}_3, \quad [\widehat{e}_1,\widehat{e}_3] = -\beta\widehat{e}_2 - \gamma\widehat{e}_3, \quad [\widehat{e}_2,\widehat{e}_3] = \alpha\widehat{e}_1, \ \gamma \neq 0, \tag{2.17}$$

where $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 2.6 *([7],[3]) The Levi–Civita connection ∇^L of G_2 is given by*

$$\nabla^L_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_2 = (\frac{\alpha}{2} - \beta)\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_3 = (\frac{\alpha}{2} - \beta)\widehat{e}_2,$$
$$\nabla^L_{\widehat{e}_2}\widehat{e}_1 = -\gamma\widehat{e}_2 + \frac{\alpha}{2}\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_2 = \gamma\widehat{e}_1, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_3 = \frac{\alpha}{2}\widehat{e}_1,$$
$$\nabla^L_{\widehat{e}_3}\widehat{e}_1 = \frac{\alpha}{2}\widehat{e}_2 + \gamma\widehat{e}_3, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_2 = -\frac{\alpha}{2}\widehat{e}_1, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_3 = \gamma\widehat{e}_1. \tag{2.18}$$

Lemma 2.7 *The Bott connection ∇^B of G_2 is given by*

$$\nabla^B_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_3 = -\gamma\widehat{e}_3,$$
$$\nabla^B_{\widehat{e}_2}\widehat{e}_1 = -\gamma\widehat{e}_2, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_2 = \gamma\widehat{e}_1, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_3 = 0,$$
$$\nabla^B_{\widehat{e}_3}\widehat{e}_1 = \beta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_2 = -\alpha\widehat{e}_1, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_3 = 0. \tag{2.19}$$

Lemma 2.8 *The curvature R^B of the Bott connection ∇^B of (G_2, g) is given by*

$$R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=(\beta^2+\gamma^2)\widehat{e}_2,\quad R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=-(\gamma^2+\alpha\beta)\widehat{e}_1,\quad R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=0,$$
$$R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=0,\quad R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=\gamma(\beta-\alpha)\widehat{e}_1,\quad R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=0,$$
$$R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=\gamma(\beta-\alpha)\widehat{e}_1,\quad R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=\gamma(\alpha-\beta)\widehat{e}_2,\quad R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=\alpha\gamma\widehat{e}_3. \tag{2.20}$$

By (2.5), we have

$$\rho^B(\widehat{e}_1,\widehat{e}_1)=-(\beta^2+\gamma^2),\quad \rho^B(\widehat{e}_1,\widehat{e}_2)=0,\quad \rho^B(\widehat{e}_1,\widehat{e}_3)=0,$$
$$\rho^B(\widehat{e}_2,\widehat{e}_1)=0,\quad \rho^B(\widehat{e}_2,\widehat{e}_2)=-(\gamma^2+\alpha\beta),\quad \rho^B(\widehat{e}_2,\widehat{e}_3)=-\alpha\gamma,$$
$$\rho^B(\widehat{e}_3,\widehat{e}_1)=\rho^B(\widehat{e}_3,\widehat{e}_2)=\rho^B(\widehat{e}_3,\widehat{e}_3)=0. \tag{2.21}$$

Then,

$$\widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_1)=-(\beta^2+\gamma^2),\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_2)=0,\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_3)=0,$$
$$\widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_2)=-(\gamma^2+\alpha\beta),\quad \widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_3)=-\frac{\alpha\gamma}{2},\quad \widetilde{\rho}^B(\widehat{e}_3,\widehat{e}_3)=0. \tag{2.22}$$

By (2.7), we have

$$(L_V^Bg)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L_V^Bg)(\widehat{e}_1,\widehat{e}_2)=\mu_2\gamma,\quad (L_V^Bg)(\widehat{e}_1,\widehat{e}_3)=\mu_3\gamma-\mu_2\alpha,$$
$$(L_V^Bg)(\widehat{e}_2,\widehat{e}_2)=-2\mu_1\gamma,\quad (L_V^Bg)(\widehat{e}_2,\widehat{e}_3)=\mu_1\beta,\quad (L_V^Bg)(\widehat{e}_3,\widehat{e}_3)=0. \tag{2.23}$$

Then, if (G_2, g, V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following six equations:

$$\begin{cases}\beta^2+\gamma^2-\mu=0\\ \mu_2\gamma=0\\ \mu_3\gamma-\mu_2\alpha=0\\ \mu_1\gamma+\gamma^2+\alpha\beta-\mu=0\\ \mu_1\beta-\alpha\gamma=0\\ \mu=0\end{cases} \tag{2.24}$$

By solving (2.24), we get $\beta=\gamma=0$, there is a contradiction. So

Theorem 2.9 *(G_2, g, V) is not an affine Ricci soliton associated to the Bott connection ∇^B.*

2.3 Affine Ricci solitons of G_3

By [3], we have the following Lie algebra of G_3 satisfies

$$[\widehat{e}_1,\widehat{e}_2]=-\gamma\widehat{e}_3,\quad [\widehat{e}_1,\widehat{e}_3]=-\beta\widehat{e}_2,\quad [\widehat{e}_2,\widehat{e}_3]=\alpha\widehat{e}_1, \tag{2.25}$$

where $\widehat{e}_1,\widehat{e}_2,\widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 2.10 *([7],[3]) The Levi–Civita connection* ∇^L *of* G_3 *is given by*

$$\begin{aligned}&\nabla^L_{\widehat{e}_1}\widehat{e}_1=0,\quad \nabla^L_{\widehat{e}_1}\widehat{e}_2=m_1\widehat{e}_3,\quad \nabla^L_{\widehat{e}_1}\widehat{e}_3=m_1\widehat{e}_2,\\&\nabla^L_{\widehat{e}_2}\widehat{e}_1=m_2\widehat{e}_3,\quad \nabla^L_{\widehat{e}_2}\widehat{e}_2=0,\quad \nabla^L_{\widehat{e}_2}\widehat{e}_3=m_2\widehat{e}_1,\\&\nabla^L_{\widehat{e}_3}\widehat{e}_1=m_3\widehat{e}_2,\quad \nabla^L_{\widehat{e}_3}\widehat{e}_2=-m_3\widehat{e}_1,\quad \nabla^L_{\widehat{e}_3}\widehat{e}_3=0,\end{aligned}\tag{2.26}$$

where

$$m_1=\frac{1}{2}(\alpha-\beta-\gamma),\quad m_2=\frac{1}{2}(\alpha-\beta+\gamma),\quad m_3=\frac{1}{2}(\alpha+\beta-\gamma).\tag{2.27}$$

Lemma 2.11 *The Bott connection* ∇^B *of* G_3 *is given by*

$$\begin{aligned}&\nabla^B_{\widehat{e}_1}\widehat{e}_1=0,\quad \nabla^B_{\widehat{e}_1}\widehat{e}_2=0,\quad \nabla^B_{\widehat{e}_1}\widehat{e}_3=0,\\&\nabla^B_{\widehat{e}_2}\widehat{e}_1=0,\quad \nabla^B_{\widehat{e}_2}\widehat{e}_2=0,\quad \nabla^B_{\widehat{e}_2}\widehat{e}_3=0,\\&\nabla^B_{\widehat{e}_3}\widehat{e}_1=\beta\widehat{e}_2,\quad \nabla^B_{\widehat{e}_3}\widehat{e}_2=-\alpha\widehat{e}_1,\quad \nabla^B_{\widehat{e}_3}\widehat{e}_3=0.\end{aligned}\tag{2.28}$$

Lemma 2.12 *The curvature* R^B *of the Bott connection* ∇^B *of* (G_3,g) *is given by*

$$\begin{aligned}&R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=\beta\gamma\widehat{e}_2,\quad R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=-\alpha\gamma\widehat{e}_1,\quad R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=0,\\&R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=0,\quad R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=0,\quad R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=0,\\&R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=0,\quad R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=0,\quad R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=0.\end{aligned}\tag{2.29}$$

By (2.5), we have

$$\begin{aligned}&\rho^B(\widehat{e}_1,\widehat{e}_1)=-\beta\gamma,\quad \rho^B(\widehat{e}_1,\widehat{e}_2)=0,\quad \rho^B(\widehat{e}_1,\widehat{e}_3)=0,\\&\rho^B(\widehat{e}_2,\widehat{e}_1)=0,\quad \rho^B(\widehat{e}_2,\widehat{e}_2)=-\alpha\gamma,\quad \rho^B(\widehat{e}_2,\widehat{e}_3)=0,\\&\rho^B(\widehat{e}_3,\widehat{e}_1)=\rho^B(\widehat{e}_3,\widehat{e}_2)=\rho^B(\widehat{e}_3,\widehat{e}_3)=0.\end{aligned}\tag{2.30}$$

Then,

$$\begin{aligned}&\widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_1)=-\beta\gamma,\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_2)=0,\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_3)=0,\\&\widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_2)=-\alpha\gamma,\quad \widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_3)=0,\quad \widetilde{\rho}^B(\widehat{e}_3,\widehat{e}_3)=0.\end{aligned}\tag{2.31}$$

By (2.7), we have

$$\begin{aligned}&(L^B_Vg)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L^B_Vg)(\widehat{e}_1,\widehat{e}_2)=0,\quad (L^B_Vg)(\widehat{e}_1,\widehat{e}_3)=-\mu_2\alpha,\\&(L^B_Vg)(\widehat{e}_2,\widehat{e}_2)=0,\quad (L^B_Vg)(\widehat{e}_2,\widehat{e}_3)=\mu_1\beta,\quad (L^B_Vg)(\widehat{e}_3,\widehat{e}_3)=0.\end{aligned}\tag{2.32}$$

Then, if (G_3, g, V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following five equations:

$$\begin{cases} \mu - \beta\gamma = 0 \\ \mu_2\alpha = 0 \\ \mu - \alpha\gamma = 0 \\ \mu_1\beta = 0 \\ \mu = 0 \end{cases} \tag{2.33}$$

By solving (2.33), we get

Theorem 2.13 *(G_3, g, V) is an affine Ricci soliton associated to the Bott connection ∇^B if and only if*

$$(1)\mu = \gamma = \alpha\mu_2 = \mu_1\beta = 0;$$
$$(2)\mu = \alpha = \beta = 0, \quad \gamma \neq 0.$$

2.4 Affine Ricci solitons of G_4

By [3], we have the following Lie algebra of G_4 satisfies

$$[\widehat{e}_1, \widehat{e}_2] = -\widehat{e}_2 + (2\eta - \beta)\widehat{e}_3, \ \ \eta = \pm 1, \quad [\widehat{e}_1, \widehat{e}_3] = -\beta\widehat{e}_2 + \widehat{e}_3, \quad [\widehat{e}_2, \widehat{e}_3] = \alpha\widehat{e}_1, \tag{2.34}$$

where $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 2.14 *([7],[3]) The Levi–Civita connection ∇^L of G_4 is given by*

$$\begin{aligned} &\nabla^L_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_2 = n_1\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_3 = n_1\widehat{e}_2, \\ &\nabla^L_{\widehat{e}_2}\widehat{e}_1 = \widehat{e}_2 + n_2\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_2 = -\widehat{e}_1, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_3 = n_2\widehat{e}_1, \\ &\nabla^L_{\widehat{e}_3}\widehat{e}_1 = n_3\widehat{e}_2 - \widehat{e}_3, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_2 = -n_3\widehat{e}_1, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_3 = -\widehat{e}_1, \end{aligned} \tag{2.35}$$

where

$$n_1 = \frac{\alpha}{2} + \eta - \beta, \quad n_2 = \frac{\alpha}{2} - \gamma, \quad n_3 = \frac{\alpha}{2} + \gamma. \tag{2.36}$$

Lemma 2.15 *The Bott connection ∇^B of G_4 is given by*

$$\begin{aligned} &\nabla^B_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_3 = \widehat{e}_3, \\ &\nabla^B_{\widehat{e}_2}\widehat{e}_1 = \widehat{e}_2, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_2 = -\widehat{e}_1, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_3 = 0, \\ &\nabla^B_{\widehat{e}_3}\widehat{e}_1 = \beta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_2 = -\alpha\widehat{e}_1, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_3 = 0. \end{aligned} \tag{2.37}$$

Lemma 2.16 *The curvature R^B of the Bott connection ∇^B of (G_4, g) is given by*

$$\begin{aligned} &R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = (\beta - \eta)^2\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = (2\alpha\eta - \alpha\beta - 1)\widehat{e}_1, \quad R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = 0, \\ &R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = 0, \quad R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = (\alpha - \beta)\widehat{e}_1, \quad R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = 0, \\ &R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = (\alpha - \beta)\widehat{e}_1, \quad R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = (\beta - \alpha)\widehat{e}_2, \quad R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = -\alpha\widehat{e}_3. \end{aligned} \tag{2.38}$$

By (2.5), we have

$$\rho^B(\widehat{e}_1,\widehat{e}_1)=-(\beta-\eta)^2,\quad \rho^B(\widehat{e}_1,\widehat{e}_2)=0,\quad \rho^B(\widehat{e}_1,\widehat{e}_3)=0,$$
$$\rho^B(\widehat{e}_2,\widehat{e}_1)=0,\quad \rho^B(\widehat{e}_2,\widehat{e}_2)=(2\alpha\eta-\alpha\beta-1),\quad \rho^B(\widehat{e}_2,\widehat{e}_3)=\alpha,$$
$$\rho^B(\widehat{e}_3,\widehat{e}_1)=\rho^B(\widehat{e}_3,\widehat{e}_2)=\rho^B(\widehat{e}_3,\widehat{e}_3)=0. \tag{2.39}$$

Then,

$$\widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_1)=-(\beta-\eta)^2,\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_2)=0,\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_3)=0,$$
$$\widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_2)=(2\alpha\eta-\alpha\beta-1),\quad \widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_3)=\frac{\alpha}{2},\quad \widetilde{\rho}^B(\widehat{e}_3,\widehat{e}_3)=0. \tag{2.40}$$

By (2.7), we have

$$(L^B_Vg)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L^B_Vg)(\widehat{e}_1,\widehat{e}_2)=-\mu_2,\quad (L^B_Vg)(\widehat{e}_1,\widehat{e}_3)=-\mu_3-\mu_2\alpha,$$
$$(L^B_Vg)(\widehat{e}_2,\widehat{e}_2)=2\mu_1,\quad (L^B_Vg)(\widehat{e}_2,\widehat{e}_3)=\mu_1\beta,\quad (L^B_Vg)(\widehat{e}_3,\widehat{e}_3)=0. \tag{2.41}$$

Then, if (G_4,g,V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following six equations:

$$\begin{cases}(\beta-\eta)^2-\mu=0\\ \mu_2=0\\ \mu_2\alpha+\mu_3=0\\ \mu_1+2\alpha\eta-\alpha\beta-1+\mu=0\\ \mu_1\beta+\alpha=0\\ \mu=0\end{cases} \tag{2.42}$$

By solving (2.42), we get

Theorem 2.17 *(G_4,g,V) is not an affine Ricci soliton associated to the Bott connection ∇^B.*

3. Affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian nonunimodular Lie groups with the first distribution

3.1 Affine Ricci solitons of G_5

By [3], we have the following Lie algebra of G_5 satisfies

$$[\widehat{e}_1,\widehat{e}_2]=0,\quad [\widehat{e}_1,\widehat{e}_3]=\alpha\widehat{e}_1+\beta\widehat{e}_2,\quad [\widehat{e}_2,\widehat{e}_3]=\gamma\widehat{e}_1+\delta\widehat{e}_2,\quad \alpha+\delta\neq 0,\quad \alpha\gamma+\beta\delta=0, \tag{3.1}$$

where $\widehat{e}_1,\widehat{e}_2,\widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 3.1 *([7],[3]) The Levi–Civita connection ∇^L of G_5 is given by*

$$\nabla^L_{\widehat{e}_1}\widehat{e}_1=\alpha\widehat{e}_3,\quad \nabla^L_{\widehat{e}_1}\widehat{e}_2=\frac{\beta+\gamma}{2}\widehat{e}_3,\quad \nabla^L_{\widehat{e}_1}\widehat{e}_3=\alpha\widehat{e}_1+\frac{\beta+\gamma}{2}\widehat{e}_2,$$
$$\nabla^L_{\widehat{e}_2}\widehat{e}_1=\frac{\beta+\gamma}{2}\widehat{e}_3,\quad \nabla^L_{\widehat{e}_2}\widehat{e}_2=\delta\widehat{e}_3,\quad \nabla^L_{\widehat{e}_2}\widehat{e}_3=\frac{\beta+\gamma}{2}\widehat{e}_1+\delta\widehat{e}_2,$$
$$\nabla^L_{\widehat{e}_3}\widehat{e}_1=\frac{\gamma-\beta}{2}\widehat{e}_2,\quad \nabla^L_{\widehat{e}_3}\widehat{e}_2=\frac{\beta-\gamma}{2}\widehat{e}_1,\quad \nabla^L_{\widehat{e}_3}\widehat{e}_3=0. \tag{3.2}$$

Lemma 3.2 *The Bott connection* ∇^B *of* G_5 *is given by*

$$\begin{aligned}
&\nabla^B_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_3 = 0,\\
&\nabla^B_{\widehat{e}_2}\widehat{e}_1 = 0, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_3 = 0,\\
&\nabla^B_{\widehat{e}_3}\widehat{e}_1 = -\alpha\widehat{e}_1 - \beta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_2 = -\gamma\widehat{e}_1 - \delta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_3 = 0.
\end{aligned} \tag{3.3}$$

Lemma 3.3 *The curvature* R^B *of the Bott connection* ∇^B *of* (G_5, g) *is given by*

$$R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = 0, \tag{3.4}$$

for any (s,t,p)*.*

By (2.5), we have

$$\rho^B(\widehat{e}_s, \widehat{e}_t) = 0, \tag{3.5}$$

Then,

$$\widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t) = 0, \tag{3.6}$$

for any pairs (s,t).
By (2.7), we have

$$\begin{aligned}
&(L^B_V g)(\widehat{e}_1, \widehat{e}_1) = 0, \quad (L^B_V g)(\widehat{e}_1, \widehat{e}_2) = -\mu_2, \quad (L^B_V g)(\widehat{e}_1, \widehat{e}_3) = -\mu_1\alpha - \mu_2\gamma,\\
&(L^B_V g)(\widehat{e}_2, \widehat{e}_2) = 2\mu_1, \quad (L^B_V g)(\widehat{e}_2, \widehat{e}_3) = \mu_1\beta - \mu_2\delta, \quad (L^B_V g)(\widehat{e}_3, \widehat{e}_3) = 0.
\end{aligned} \tag{3.7}$$

Then, if (G_5, g, V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following three equations:

$$\begin{cases} \mu = 0 \\ \mu_1\alpha + \mu_2\gamma = 0 \\ \mu_1\beta + \mu_2\delta = 0 \end{cases} \tag{3.8}$$

By solving (3.8), we get

Theorem 3.4 (G_5, g, V) *is an affine Ricci soliton associated to the Bott connection* ∇^B *if and only if*

$$\begin{aligned}
&(1)\mu = \mu_1 = \mu_2 = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma + \beta\delta = 0;\\
&(2)\mu = \mu_2 = \alpha = \beta = 0, \quad \mu_1 \neq 0, \quad \delta \neq 0;\\
&(3)\mu = \mu_1 = \gamma = \delta = 0, \quad \mu_2 \neq 0, \quad \alpha \neq 0.
\end{aligned}$$

3.2 Affine Ricci solitons of G_6

By [3], we have the following Lie algebra of G_6 satisfies

$$[\widehat{e}_1, \widehat{e}_2] = \alpha\widehat{e}_2 + \beta\widehat{e}_3, \quad [\widehat{e}_1, \widehat{e}_3] = \gamma\widehat{e}_2 + \delta\widehat{e}_3, \quad [\widehat{e}_2, \widehat{e}_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma - \beta\delta = 0, \tag{3.9}$$

where $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 3.5 *([7],[3]) The Levi–Civita connection ∇^L of G_6 is given by*

$$\nabla^L_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_2 = \frac{\beta+\gamma}{2}\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_3 = \frac{\beta+\gamma}{2}\widehat{e}_2,$$

$$\nabla^L_{\widehat{e}_2}\widehat{e}_1 = -\alpha\widehat{e}_2 - \frac{\beta-\gamma}{2}\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_2 = \alpha\widehat{e}_1, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_3 = \frac{\gamma-\beta}{2}\widehat{e}_1,$$

$$\nabla^L_{\widehat{e}_3}\widehat{e}_1 = \frac{\beta-\gamma}{2}\widehat{e}_2 - \delta\widehat{e}_3, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_2 = -\frac{\gamma-\beta}{2}\widehat{e}_1, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_3 = -\delta\widehat{e}_1. \tag{3.10}$$

Lemma 3.6 *The Bott connection ∇^B of G_6 is given by*

$$\nabla^B_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_3 = \delta\widehat{e}_3,$$

$$\nabla^B_{\widehat{e}_2}\widehat{e}_1 = -\alpha\widehat{e}_2, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_2 = \alpha\widehat{e}_1, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_3 = 0,$$

$$\nabla^B_{\widehat{e}_3}\widehat{e}_1 = -\gamma\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_2 = 0, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_3 = 0. \tag{3.11}$$

Lemma 3.7 *The curvature R^B of the Bott connection ∇^B of (G_6, g) is given by*

$$R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1 = (\alpha^2+\beta\gamma)\widehat{e}_2, \quad R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = -\alpha^2\widehat{e}_1, \quad R^B(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3 = 0,$$

$$R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1 = \gamma(\alpha+\delta)\widehat{e}_2, \quad R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = -\alpha\gamma\widehat{e}_1, \quad R^B(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3 = 0,$$

$$R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1 = -\alpha\gamma\widehat{e}_1, \quad R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = \alpha\gamma\widehat{e}_2, \quad R^B(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3 = 0. \tag{3.12}$$

By (2.5), we have

$$\rho^B(\widehat{e}_1,\widehat{e}_1) = -(\alpha^2+\beta\gamma), \quad \rho^B(\widehat{e}_1,\widehat{e}_2) = \rho^B(\widehat{e}_1,\widehat{e}_3) = 0,$$

$$\rho^B(\widehat{e}_2,\widehat{e}_1) = 0, \quad \rho^B(\widehat{e}_2,\widehat{e}_2) = -\alpha^2, \quad \rho^B(\widehat{e}_2,\widehat{e}_3) = 0,$$

$$\rho^B(\widehat{e}_3,\widehat{e}_1) = \rho^B(\widehat{e}_3,\widehat{e}_2) = \rho^B(\widehat{e}_3,\widehat{e}_3) = 0. \tag{3.13}$$

Then,

$$\widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_1) = -(\alpha^2+\beta\gamma), \quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_2) = \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_3) = 0,$$

$$\widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_2) = -\alpha^2, \quad \widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_3) = 0, \quad \widetilde{\rho}^B(\widehat{e}_3,\widehat{e}_3) = 0. \tag{3.14}$$

By (2.7), we have

$$(L^B_V g)(\widehat{e}_1,\widehat{e}_1) = 0, \quad (L^B_V g)(\widehat{e}_1,\widehat{e}_2) = \mu_2\alpha, \quad (L^B_V g)(\widehat{e}_1,\widehat{e}_3) = -\mu_3\delta,$$

$$(L^B_V g)(\widehat{e}_2,\widehat{e}_2) = -2\mu_1\alpha, \quad (L^B_V g)(\widehat{e}_2,\widehat{e}_3) = -\mu_1\gamma, \quad (L^B_V g)(\widehat{e}_3,\widehat{e}_3) = 0. \tag{3.15}$$

Then, if (G_6, g, V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following six equations:

$$\begin{cases} \alpha^2+\beta\gamma-\mu = 0 \\ \mu_2\alpha = 0 \\ \mu_3\delta = 0 \\ \mu_1\alpha+\alpha^2-\mu = 0 \\ \mu_1\gamma = 0 \\ \mu = 0 \end{cases} \tag{3.16}$$

By solving (3.16), we get

Theorem 3.8 (G_6, g, V) *is an affine Ricci soliton associated to the Bott connection* ∇^B *if and only if*

$$(1)\mu = \mu_1 = \mu_3 = \alpha = \beta = 0, \quad \delta \neq 0;$$
$$(2)\mu = \mu_3 = \alpha = \beta = \gamma == 0, \quad \mu_1 \neq 0, \quad \delta \neq 0.$$

3.3 Affine Ricci solitons of G_7

By [3], we have the following Lie algebra of G_7 satisfies

$$[\widehat{e}_1, \widehat{e}_2] = -\alpha\widehat{e}_1 - \beta\widehat{e}_2 - \beta\widehat{e}_3, \quad [\widehat{e}_1, \widehat{e}_3] = \alpha\widehat{e}_1 + \beta\widehat{e}_2 + \beta\widehat{e}_3, \quad [\widehat{e}_2, \widehat{e}_3] = \gamma\widehat{e}_1 + \delta\widehat{e}_2 + \delta\widehat{e}_3, \quad \alpha + \delta \neq 0, \quad \alpha\gamma = 0, \tag{3.17}$$

where $\widehat{e}_1, \widehat{e}_2, \widehat{e}_3$ is a pseudo-orthonormal basis, with $\widehat{e}_3$ timelike.

Lemma 3.9 *([7],[3]) The Levi–Civita connection* ∇^L *of* G_7 *is given by*

$$\begin{aligned}
&\nabla^L_{\widehat{e}_1}\widehat{e}_1 = \alpha\widehat{e}_2 + \alpha\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_2 = -\alpha\widehat{e}_1 + \frac{\gamma}{2}\widehat{e}_3, \quad \nabla^L_{\widehat{e}_1}\widehat{e}_3 = \alpha\widehat{e}_1 + \frac{\gamma}{2}\widehat{e}_2,\\
&\nabla^L_{\widehat{e}_2}\widehat{e}_1 = \beta\widehat{e}_2 + (\beta + \frac{\gamma}{2})\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_2 = -\beta\widehat{e}_1 + \delta\widehat{e}_3, \quad \nabla^L_{\widehat{e}_2}\widehat{e}_3 = (\beta + \frac{\gamma}{2})\widehat{e}_1 + \delta\widehat{e}_2,\\
&\nabla^L_{\widehat{e}_3}\widehat{e}_1 = (\frac{\gamma}{2} - \beta)\widehat{e}_2 - \beta\widehat{e}_3, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_2 = (\beta - \frac{\gamma}{2})\widehat{e}_1 - \delta\widehat{e}_3, \quad \nabla^L_{\widehat{e}_3}\widehat{e}_3 = -\beta\widehat{e}_1 - \delta\widehat{e}_2.
\end{aligned} \tag{3.18}$$

Lemma 3.10 *The Bott connection* ∇^B *of* G_7 *is given by*

$$\begin{aligned}
&\nabla^B_{\widehat{e}_1}\widehat{e}_1 = \alpha\widehat{e}_2, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_2 = -\alpha\widehat{e}_1, \quad \nabla^B_{\widehat{e}_1}\widehat{e}_3 = \beta\widehat{e}_3,\\
&\nabla^B_{\widehat{e}_2}\widehat{e}_1 = \beta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_2 = -\beta\widehat{e}_1, \quad \nabla^B_{\widehat{e}_2}\widehat{e}_3 = \delta\widehat{e}_3,\\
&\nabla^B_{\widehat{e}_3}\widehat{e}_1 = -\alpha\widehat{e}_1 - \beta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_2 = -\gamma\widehat{e}_1 - \delta\widehat{e}_2, \quad \nabla^B_{\widehat{e}_3}\widehat{e}_3 = 0.
\end{aligned} \tag{3.19}$$

Lemma 3.11 *The curvature* R^B *of the Bott connection* ∇^B *of* (G_7, g) *is given by*

$$\begin{aligned}
&R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = -\alpha\beta\widehat{e}_1 + \alpha^2\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = -(\alpha^2 + \beta^2 + \beta\gamma)\widehat{e}_1 - \beta\delta\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = \beta(\alpha - \delta)\widehat{e}_3,\\
&R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = \alpha(2\beta + \gamma)\widehat{e}_1 + (\alpha\delta - 2\alpha^2)\widehat{e}_2, \quad R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = (\alpha\delta + \beta^2 + \beta\gamma)\widehat{e}_1 + (\beta\delta - \alpha\beta - \alpha\gamma)\widehat{e}_2,\\
&R^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = -\beta(\alpha + \delta)\widehat{e}_3, \quad R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = (\beta^2 + \beta\gamma + \alpha\delta)\widehat{e}_1 + (\beta\delta - \alpha\beta - \alpha\gamma)\widehat{e}_2,\\
&R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = (2\beta\delta + \delta\gamma + \alpha\gamma - \alpha\beta)\widehat{e}_1 + (\delta^2 - \beta^2 - \beta\gamma)\widehat{e}_2, \quad R^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = -(\beta\gamma + \delta^2)\widehat{e}_3.
\end{aligned} \tag{3.20}$$

By (2.5), we have

$$\begin{aligned}
&\rho^B(\widehat{e}_1, \widehat{e}_1) = -\alpha^2, \quad \rho^B(\widehat{e}_1, \widehat{e}_2) = \beta\delta, \quad \rho^B(\widehat{e}_1, \widehat{e}_3) = \beta(\alpha + \delta),\\
&\rho^B(\widehat{e}_2, \widehat{e}_1) = -\alpha\beta, \quad \rho^B(\widehat{e}_2, \widehat{e}_2) = -(\alpha^2 + \beta^2 + \beta\gamma), \quad \rho^B(\widehat{e}_2, \widehat{e}_3) = (\beta\gamma + \delta^2),\\
&\rho^B(\widehat{e}_3, \widehat{e}_1) = \beta(\alpha + \delta), \quad \rho^B(\widehat{e}_3, \widehat{e}_2) = \delta(\alpha + \delta), \quad \rho^B(\widehat{e}_3, \widehat{e}_3) = 0.
\end{aligned} \tag{3.21}$$

Then,

$$\widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_1)=-\alpha^2,\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_2)=\frac{\beta(\delta-\alpha)}{2},\quad \widetilde{\rho}^B(\widehat{e}_1,\widehat{e}_3)=\beta(\alpha+\delta),$$
$$\widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_2)=-(\alpha^2+\beta^2+\beta\gamma),\quad \widetilde{\rho}^B(\widehat{e}_2,\widehat{e}_3)=\delta^2+\frac{\beta\gamma+\alpha\delta}{2},\quad \widetilde{\rho}^B(\widehat{e}_3,\widehat{e}_3)=0. \tag{3.22}$$

By (2.7), we have

$$(L^B_Vg)(\widehat{e}_1,\widehat{e}_1)=-2\mu_2\alpha,\quad (L^B_Vg)(\widehat{e}_1,\widehat{e}_2)=\mu_1\alpha-\mu_2\beta,\quad (L^B_Vg)(\widehat{e}_1,\widehat{e}_3)=-\mu_1\alpha-\mu_2\gamma-\mu_3\beta,$$
$$(L^B_Vg)(\widehat{e}_2,\widehat{e}_2)=2\mu_1\beta,\quad (L^B_Vg)(\widehat{e}_2,\widehat{e}_3)=-\mu_1\beta-\mu_2\delta-\mu_3\delta,\quad (L^B_Vg)(\widehat{e}_3,\widehat{e}_3)=0. \tag{3.23}$$

Then, if (G_7,g,V) is an affine Ricci soliton associated to the Bott connection ∇^B, by (2.8), we have the following six equations:

$$\begin{cases}\mu_2\alpha+\alpha^2-\mu=0\\ \mu_1\alpha-\mu_2\beta+\beta\delta-\alpha\beta=0\\ \mu_3\beta+\mu_1\alpha+\mu_2\gamma-2\alpha\beta-2\delta\beta=0\\ \mu_1\beta-\alpha^2-\beta^2-\beta\gamma+\mu=0\\ \mu_3\delta+\mu_1\beta+\mu_2\delta-2\delta^2-\beta\gamma-\alpha\delta=0\\ \mu=0\end{cases} \tag{3.24}$$

By solving (3.24), we get

Theorem 3.12 *(G_7,g,V) is an affine Ricci soliton associated to the Bott connection ∇^B if and only if*

$$(1)\mu=\alpha=\beta=\gamma=0,\quad \delta\neq0,\quad \mu_2+\mu_3-2\delta=0;$$
$$(2)\mu=\mu_2=\alpha=\beta=0,\quad \gamma\neq0,\quad \delta\neq0,\quad \mu_3-2\delta=0;$$
$$(3)\mu=\alpha=0,\quad \beta\neq0,\quad \delta\neq0,\quad \mu_1=\gamma+\beta,\quad \mu_2=\delta,\quad \mu_3=\frac{\delta(2\beta-\gamma)}{\beta},\quad \gamma=\frac{\beta(\beta^2+\delta^2)}{\delta^2}.$$

Specially, let $V=0$, we get the following corollary:

Corollary 3.13 *(I) (G_1,g,V) is not an affine Einstein associated to the Bott connection ∇^B;*
(II) (G_2,g,V) is not an affine Einstein associated to the Bott connection ∇^B;
(III) (G_3,g,V) is an affine Einstein associated to the Bott connection ∇^B if and only if $(1)\mu=\alpha=\beta=0,\quad \gamma\neq0$; $(2)\mu=\gamma=0$*;*
(IV) (G_4,g,V) is not an affine Einstein associated to the Bott connection ∇^B;
(V) (G_5,g,V) is an affine Einstein associated to the Bott connection ∇^B if and only if $\mu=0,\quad \alpha+\delta\neq0,\quad \alpha\gamma+\beta\delta=0$;
(VI) (G_6,g,V) is an affine Einstein associated to the Bott connection ∇^B if and only if $\mu=\alpha=\beta=0,\quad \delta\neq0$;
(VII) (G_7,g,V) is not an affine Einstein associated to the Bott connection ∇^B.

4. Affine Ricci solitons associated to the perturbed Bott connection on three-dimensional Lorentzian Lie groups with the first distribution

By the above calculations, we always obtain $\mu = 0$. In order to get the affine Ricci soliton with nonzero μ, we introduce the perturbed Bott connection $\widetilde{\nabla}^B$ in the following. Let $\widehat{e}_3^*$ be the dual base of e_3. We define on $G_{i=1,\cdots,7}$

$$\widetilde{\nabla}^B_X Y = \nabla^B_X Y + a_0 \widehat{e}_3^*(X) \widehat{e}_3^*(Y) e_3, \tag{4.1}$$

where a_0 is a nonzero real number. Then

$$\widetilde{\nabla}^B_{\widehat{e}_3} \widehat{e}_3 = a_0 \widehat{e}_3, \quad \widetilde{\nabla}^B_{\widehat{e}_s} \widehat{e}_t = \nabla^B_{\widehat{e}_s} \widehat{e}_t, \tag{4.2}$$

where s and t does not equal 3. We define

$$(\widetilde{L}^B_V g)(X,Y) := g(\widetilde{\nabla}^B_X V, Y) + g(X, \widetilde{\nabla}^B_Y V), \tag{4.3}$$

for vector fields X, Y, V. Then we have for $G_{i=1,\cdots,7}$

$$(\widetilde{L}^B_V g)(\widehat{e}_3, \widehat{e}_3) = -2a_0\mu_3, \quad (\widetilde{L}^B_V g)(\widehat{e}_s, \widehat{e}_t) = (L^B_V g)(\widehat{e}_s, \widehat{e}_t), \tag{4.4}$$

where s and t does not equal 3.

Definition 4.1 *(G_i, V, g) is called the affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$ if it satisfies*

$$(\widetilde{L}^B_V g)(X,Y) + 2\widetilde{\widetilde{\rho}}^B(X,Y) + 2\mu g(X,Y) = 0. \tag{4.5}$$

For $(G_1, \widetilde{\nabla}^B)$, we have

$$\widetilde{R}^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = a_0\beta\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = -\alpha(a_0+\alpha)\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{4.6}$$

for $(s,t,p) \neq (1,2,3), (2,3,3)$.

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_2, \widehat{e}_3) = \frac{\alpha(a_0+\alpha)}{2}, \quad \widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.7}$$

for the pair $(s,t) \neq (2,3)$. If (G_1, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then by (4.5), we have

$$\begin{cases} \mu_2\alpha - \alpha^2 - \beta^2 + \mu = 0 \\ 2\alpha\beta - \mu_1\alpha = 0 \\ \mu_1\alpha - \mu_2\beta - \alpha\beta = 0 \\ \alpha^2 + \beta^2 - \mu = 0 \\ (\mu_2+\mu_3)\alpha - \mu_1\beta - a_0\alpha - \alpha^2 = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.8}$$

Solve (4.8), we get $a_0 = \alpha = 0$, there is a contradiction. So

Theorem 4.2 *(G_1, V, g) is not an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$.*

For $(G_2, \widetilde{\nabla}^B)$, we have

$$\widetilde{R}^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = a_0\beta\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = a_0\gamma\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{4.9}$$

for $(s, t, p) \neq (1, 2, 3), (1, 3, 3)$.

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_1, \widehat{e}_3) = \frac{a_0\gamma}{2}, \quad \widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.10}$$

for the pair $(s, t) \neq (1, 3)$. If (G_2, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then by (4.5), we have

$$\begin{cases} \gamma^2 + \beta^2 - \mu = 0 \\ \mu_2\gamma = 0 \\ \mu_2\alpha - \mu_3\gamma + a_0\gamma = 0 \\ \gamma^2 + \mu_1\gamma + \alpha\beta - \mu = 0 \\ \mu_1\beta - \alpha\gamma = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.11}$$

Solve (4.11), we get $a_0 = \beta = \gamma = 0$, there is a contradiction. So

Theorem 4.3 *(G_2, V, g) is not an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$.*

For $(G_3, \widetilde{\nabla}^B)$, we have

$$\widetilde{R}^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = -a_0\beta\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{4.12}$$

for $(s, t, p) \neq (1, 2, 3)$.

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.13}$$

for any pairs (s, t). If (G_3, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then by (4.5), we have

$$\begin{cases} \beta\gamma - \mu = 0 \\ \mu_2\alpha = 0 \\ \mu - \alpha\gamma = 0 \\ \mu_1\beta = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.14}$$

Solve (4.14), we get

Theorem 4.4 *(G_3, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$ if and only if*

$$(1)\mu = \gamma = \mu_3 = \mu_2\alpha = \mu_1\beta = 0;$$
$$(2)\gamma \neq 0, \quad \alpha = \beta = \mu = \mu_3 = 0;$$
$$(3)\gamma \neq 0, \quad \alpha = \beta \neq 0, \quad \mu_1 = \mu_2 = 0, \quad \mu = \alpha\gamma, \quad \mu_3 = -\frac{\alpha\gamma}{a_0}.$$

For $(G_4, \widetilde{\nabla}^B)$, we have

$$\widetilde{R}^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = a_0(\beta - 2\eta)\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = -a_0\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{4.15}$$

for $(s, t, p) \neq (1, 2, 3), (1, 3, 3)$.

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_1, \widehat{e}_3) = \frac{a_0}{2}, \quad \widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.16}$$

for the pair $(s, t) \neq (1, 3)$. If (G_4, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then by (4.5), we have

$$\begin{cases} \beta^2 - 2\beta\eta + 1 - \mu = 0 \\ \mu_2 = 0 \\ \mu_2\alpha + \mu_3 - a_0 = 0 \\ \mu_1 + 2\alpha\eta - \alpha\beta - 1 + \mu = 0 \\ \mu_1\beta + \alpha = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.17}$$

Solve (4.17), we get $a_0 = 0$, there is a contradiction. So

Theorem 4.5 *(G_4, V, g) is not an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$.*

For $(G_5, \widetilde{\nabla}^B)$, we have

$$\widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{4.18}$$

for any (s, t, p).

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.19}$$

for any pairs (s, t). If (G_5, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then by (4.5), we have

$$\begin{cases} \mu = 0 \\ \mu_1\alpha + \mu_2\gamma = 0 \\ \mu_1\beta + \mu_2\delta = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.20}$$

Solve (4.20), we get

Theorem 4.6 *(G_5, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$ if and only if*

$$(1)\mu = \mu_1 = \mu_2 = \mu_3 = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma + \beta\delta = 0;$$
$$(2)\mu = \mu_2 = \mu_3 = \alpha = \beta = 0, \quad \mu_1 \neq 0, \quad \delta \neq 0;$$
$$(3)\mu = \mu_1 = \mu_3 = \delta = \gamma = 0 = 0, \quad \mu_2 \neq 0, \quad \alpha \neq 0.$$

For $(G_6, \widetilde{\nabla}^B)$, we have

$$\widetilde{R}^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = a_0\gamma\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = -a_0\delta\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{4.21}$$

for $(s, t, p) \neq (1, 2, 3), (1, 3, 3)$.

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_1, \widehat{e}_3) = \frac{a_0\delta}{2}, \quad \widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.22}$$

for the pair $(s, t) \neq (1, 3)$. If (G_6, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then by (4.5), we have

$$\begin{cases} \alpha^2 + \beta\gamma - \mu = 0 \\ \mu_2\alpha = 0 \\ \mu_3\delta - a_0\delta = 0 \\ \mu - \mu_1\alpha - \alpha^2 = 0 \\ \mu_1\gamma = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.23}$$

Solve (4.23), we get

Theorem 4.7 *(G_6, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$ if and only if $\alpha \neq 0, \quad \mu_1 = \mu_2 = \gamma = \delta = 0, \quad \mu = \alpha^2, \quad \mu_3 = -\frac{\alpha^2}{a_0}$.*

For $(G_7, \widetilde{\nabla}^B)$, we have

$$\begin{aligned} &\widetilde{R}^B(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = \beta(\alpha - \delta + a_0)\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = -\beta(a_0 + \alpha + \delta)\widehat{e}_3, \\ &\widetilde{R}^B(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = -(a_0\delta + \delta^2 + \beta\gamma)\widehat{e}_3, \quad \widetilde{R}^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \end{aligned} \tag{4.24}$$

for $(s, t, p) \neq (1, 2, 3), (1, 3, 3), (2, 3, 3)$.

$$\widetilde{\widetilde{\rho}}^B(\widehat{e}_1, \widehat{e}_3) = \frac{\beta(a_0 + 2\alpha + 2\delta)}{2}, \quad \widetilde{\widetilde{\rho}}^B(\widehat{e}_2, \widehat{e}_3) = \frac{a_0\delta + 2\delta^2 + \alpha\delta + \beta\gamma}{2}, \quad \widetilde{\widetilde{\rho}}^B(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^B(\widehat{e}_s, \widehat{e}_t), \tag{4.25}$$

for the pair $(s, t) \neq (1, 3), (2, 3)$. If (G_7, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$, then

by (4.5), we have

$$\begin{cases} \mu_2\alpha + \alpha^2 - \mu = 0 \\ \mu_2\beta - \mu_1\alpha - \beta\delta + \alpha\beta = 0 \\ \mu_1\alpha + \mu_2\gamma + \mu_3\beta - (2\alpha\beta + 2\beta\delta + a_0\beta) = 0 \\ \mu_1\beta - \alpha^2 - \beta^2 - \beta\gamma - \mu = 0 \\ (\mu_2+\mu_3)\delta + \mu_1\beta - (\beta\gamma + \alpha\delta + 2\delta^2 + a_0\delta) = 0 \\ a_0\mu_3 + \mu = 0 \end{cases} \tag{4.26}$$

Solve (4.26), we get

Theorem 4.8 *(G_7, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^B$ if and only if*

(1)$\mu = \alpha = \beta = \gamma = \mu_3 = 0, \quad \delta \neq 0, \quad \mu_2 = 2\delta + a_0;$

(2)$\alpha = \beta = \mu = \mu_2 = \mu_3 = 0, \quad \gamma \neq 0, \quad \delta \neq 0, \quad a_0 = -2\delta;$

(3)$\alpha = \mu = \mu_3 = 0, \quad \beta \neq 0, \quad \delta \neq 0, \quad \mu_1 = \beta + \gamma, \quad \mu_2 = \delta, \quad a_0 = \dfrac{\delta(\gamma - 2\beta)}{\beta}, \quad \gamma = \dfrac{\beta(\beta^2+\delta^2)}{\delta^2};$

(4)$\alpha \neq 0, \quad \mu_1 = \mu_2 = \beta = \gamma = \delta = 0, \quad \mu = \alpha^2, \quad \mu_3 = -\dfrac{\alpha^2}{a_0};$

(5)$\alpha \neq 0, \quad \mu_1 = \mu_2 = \beta = \gamma = 0, \quad \mu = \alpha^2, \quad \mu_3 = \alpha + 2\delta + a_0, \quad a_0^2 + a_0\alpha + 2a_0\delta + \alpha^2 = 0.$

5. Affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups with the second distribution

Let $TM = span\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$, then took the distribution $D_1 = span\{\widehat{e}_1, \widehat{e}_3\}$ and $D_1^{\perp} = span\{\widehat{e}_2\}$. Similar to (2.3), we have

$$\nabla^{B_1}_X Y = \begin{cases} \pi_{D_1}(\nabla^L_X Y), & X, Y \in \Gamma^\infty(D_1) \\ \pi_{D_1}([X,Y]), & X \in \Gamma^\infty({D_1}^{\perp}), Y \in \Gamma^\infty(D_1) \\ \pi_{{D_1}^{\perp}}([X,Y]), & X \in \Gamma^\infty(D_1), Y \in \Gamma^\infty({D_1}^{\perp}) \\ \pi_{{D_1}^{\perp}}(\nabla^L_X Y), & X, Y \in \Gamma^\infty({D_1}^{\perp}) \end{cases} \tag{5.1}$$

where π_{D_1} (resp. $\pi^{\perp}_{D_1}$) the projection on D_1 (resp. ${D_1}^{\perp}$).

5.1 Affine Ricci solitons of G_1

Lemma 5.1 *The Bott connection ∇^{B_1} of G_1 is given by*

$$\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1 = -\alpha\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2 = 0, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3 = -\alpha\widehat{e}_1,$$
$$\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1 = -\alpha\widehat{e}_1 + \beta\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2 = 0, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3 = \beta\widehat{e}_1 + \alpha\widehat{e}_3,$$
$$\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1 = 0, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2 = -\alpha\widehat{e}_2, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3 = 0. \tag{5.2}$$

Lemma 5.2 *The curvature R^{B_1} of the Bott connection ∇^{B_1} of (G_1,g) is given by*

$$R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=3\alpha^2\widehat{e}_3,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=-\alpha\beta\widehat{e}_2,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=-\alpha^2\widehat{e}_1,$$
$$R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=-\alpha\beta\widehat{e}_1+(\beta^2-\alpha^2)\widehat{e}_3,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=0,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=(\beta^2-\alpha^2)\widehat{e}_1+\alpha\beta\widehat{e}_3,$$
$$R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=\alpha^2\widehat{e}_1,\quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=\alpha^2\widehat{e}_2,\quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=-\alpha^2\widehat{e}_3. \tag{5.3}$$

By (2.5), we have

$$\rho^{B_1}(\widehat{e}_1,\widehat{e}_1)=\alpha^2-\beta^2,\quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_2)=\alpha\beta,\quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_3)=-\alpha\beta,$$
$$\rho^{B_1}(\widehat{e}_2,\widehat{e}_1)=\rho^{B_1}(\widehat{e}_2,\widehat{e}_2)=\rho^{B_1}(\widehat{e}_2,\widehat{e}_3)=0,$$
$$\rho^{B_1}(\widehat{e}_3,\widehat{e}_1)=-\alpha\beta,\quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_2)=\alpha^2,\quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_3)=(\beta^2-\alpha^2). \tag{5.4}$$

Then,

$$\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_1)=\alpha^2-\beta^2,\quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_2)=\frac{\alpha\beta}{2},\quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_3)=-\alpha\beta,$$
$$\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_2)=0,\quad \widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_3)=\frac{\alpha^2}{2},\quad \widetilde{\rho}^{B_1}(\widehat{e}_3,\widehat{e}_3)=(\beta^2-\alpha^2). \tag{5.5}$$

By (2.7), we have

$$(L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_1)=-2\mu_3\alpha,\quad (L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_2)=-\mu_1\alpha+\mu_3\beta,\quad (L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_3)=\mu_1\alpha,$$
$$(L_V^{B_1}g)(\widehat{e}_2,\widehat{e}_2)=0,\quad (L_V^{B_1}g)(\widehat{e}_2,\widehat{e}_2)=-(\mu_2+\mu_3)\alpha-\mu_1\beta,\quad (L_V^{B_1}g)(\widehat{e}_3,\widehat{e}_3)=0. \tag{5.6}$$

Then, if (G_1,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following six equations:

$$\begin{cases}\mu_3\alpha-\alpha^2+\beta^2-\mu=0\\ \mu_3\beta-\mu_1\alpha+\alpha\beta=0\\ \mu_1\alpha-2\alpha\beta=0\\ \mu=0\\ \mu_1\beta+\mu_3\alpha+\mu_2\alpha-\alpha^2=0\\ \beta^2-\alpha^2-\mu=0\end{cases} \tag{5.7}$$

By solving (5.7), we get $\alpha=0$, there is a contradiction. So

Theorem 5.3 (G_1,g,V) *is not an affine Ricci soliton associated to the Bott connection ∇^{B_1}.*

5.2 Affine Ricci solitons of G_2

Lemma 5.4 *The Bott connection ∇^{B_1} of G_2 is given by*

$$\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1=0,\quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2=\gamma\widehat{e}_2,\quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3=0,$$
$$\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1=\beta\widehat{e}_3,\quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2=0,\quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3=\alpha\widehat{e}_1,$$
$$\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1=\gamma\widehat{e}_3,\quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2=0,\quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3=\gamma\widehat{e}_1. \tag{5.8}$$

Lemma 5.5 *The curvature* R^{B_1} *of the Bott connection* ∇^{B_1} *of* (G_2, g) *is given by*

$$
\begin{aligned}
&R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=0,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=0,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=\gamma(\beta-\alpha)\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=(\beta^2+\gamma^2)\widehat{e}_3,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=0,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=(\alpha\beta+\gamma^2)\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=\gamma(\alpha-\beta)\widehat{e}_1,\quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=-\alpha\gamma\widehat{e}_2,\quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=\gamma(\beta-\alpha)\widehat{e}_3.
\end{aligned}
\tag{5.9}
$$

By (2.5), we have

$$
\begin{aligned}
&\rho^{B_1}(\widehat{e}_1,\widehat{e}_1)=-(\beta^2+\gamma^2),\quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_2)=0,\quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_3)=0,\\
&\rho^{B_1}(\widehat{e}_2,\widehat{e}_1)=\rho^{B_1}(\widehat{e}_2,\widehat{e}_2)=\rho^{B_1}(\widehat{e}_2,\widehat{e}_3)=0,\\
&\rho^{B_1}(\widehat{e}_3,\widehat{e}_1)=0,\quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_2)=-\alpha\gamma,\quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_3)=\alpha\beta+\gamma^2.
\end{aligned}
\tag{5.10}
$$

Then,

$$
\begin{aligned}
&\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_1)=-(\beta^2+\gamma^2),\quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_2)=0,\quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_3)=0,\\
&\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_2)=0,\quad \widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_3)=-\frac{\alpha\gamma}{2},\quad \widetilde{\rho}^{B_1}(\widehat{e}_3,\widehat{e}_3)=\alpha\beta+\gamma^2.
\end{aligned}
\tag{5.11}
$$

By (2.7), we have

$$
\begin{aligned}
&(L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_2)=\mu_2\gamma+\mu_3\alpha,\quad (L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_3)=\mu_3\gamma,\\
&(L_V^{B_1}g)(\widehat{e}_2,\widehat{e}_2)=0,\quad (L_V^{B_1}g)(\widehat{e}_2,\widehat{e}_3)=-\mu_1\beta,\quad (L_V^{B_1}g)(\widehat{e}_3,\widehat{e}_3)=-2\mu_1\gamma.
\end{aligned}
\tag{5.12}
$$

Then, if (G_2, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following six equations:

$$
\begin{cases}
\beta^2+\gamma^2-\mu=0\\
\mu_2\gamma+\mu_3\alpha=0\\
\mu_3\gamma=0\\
\mu=0\\
\mu_1\beta+\alpha\gamma=0\\
\mu_1\gamma-\alpha\beta-\gamma^2+\mu=0
\end{cases}
\tag{5.13}
$$

By solving (5.13), we get $\beta=\gamma=0$, there is a contradiction. So

Theorem 5.6 (G_2, g, V) *is not an affine Ricci soliton associated to the Bott connection* ∇^{B_1}.

5.3 Affine Ricci solitons of G_3

Lemma 5.7 *The Bott connection* ∇^{B_1} *of* G_3 *is given by*

$$
\begin{aligned}
&\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1=\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2=\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3=-\gamma\widehat{e}_3,\\
&\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1=\gamma\widehat{e}_3,\quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2=0,\quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3=\alpha\widehat{e}_1,\\
&\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1=\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2=\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3=0.
\end{aligned}
\tag{5.14}
$$

Lemma 5.8 *The curvature* R^{B_1} *of the Bott connection* ∇^{B_1} *of* (G_3, g) *is given by*

$$\begin{aligned}
&R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1 = R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3 = 0,\\
&R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1 = \beta\gamma\widehat{e}_3, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = 0, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3 = \alpha\beta\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1 = R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3 = 0.
\end{aligned} \tag{5.15}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_1}(\widehat{e}_1,\widehat{e}_1) = -\beta\gamma, \quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_2) = 0, \quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_3) = 0,\\
&\rho^{B_1}(\widehat{e}_2,\widehat{e}_1) = \rho^{B_1}(\widehat{e}_2,\widehat{e}_2) = \rho^{B_1}(\widehat{e}_2,\widehat{e}_3) = 0,\\
&\rho^{B_1}(\widehat{e}_3,\widehat{e}_1) = \rho^{B_1}(\widehat{e}_3,\widehat{e}_2) = 0, \quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_3) = \alpha\beta.
\end{aligned} \tag{5.16}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_1) = -\beta\gamma, \quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_2) = \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_3) = 0,\\
&\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_2) = \widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_3) = 0, \quad \widetilde{\rho}^{B_1}(\widehat{e}_3,\widehat{e}_3) = \alpha\beta.
\end{aligned} \tag{5.17}$$

By (2.7), we have

$$\begin{aligned}
&(L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_1) = 0, \quad (L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_2) = \mu_3\alpha, \quad (L_V^{B_1}g)(\widehat{e}_1,\widehat{e}_3) = 0,\\
&(L_V^{B_1}g)(\widehat{e}_2,\widehat{e}_2) = 0, \quad (L_V^{B_1}g)(\widehat{e}_2,\widehat{e}_3) = -\mu_1\gamma, \quad (L_V^{B_1}g)(\widehat{e}_3,\widehat{e}_3) = 0.
\end{aligned} \tag{5.18}$$

Then, if (G_3, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following five equations:

$$\begin{cases}
\mu - \beta\gamma = 0\\
\mu_3\alpha = 0\\
\mu = 0\\
\mu_1\gamma = 0\\
\alpha\beta - \mu = 0
\end{cases} \tag{5.19}$$

By solving (5.19), we get

Theorem 5.9 (G_3, g, V) *is an affine Ricci soliton associated to the Bott connection* ∇^{B_1} *if and only if*

$$\begin{aligned}
&(1)\mu = \beta = \mu_3\alpha = \mu_1\gamma = 0;\\
&(2)\mu = \alpha = \gamma = 0, \quad \beta \neq 0.
\end{aligned}$$

5.4 Affine Ricci solitons of G_4

Lemma 5.10 *The Bott connection ∇^{B_1} of G_4 is given by*

$$\begin{aligned}
&\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2 = -\widehat{e}_2, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3 = 0,\\
&\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1 = (\beta - 2\eta)\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2 = 0, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3 = \alpha\widehat{e}_1,\\
&\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1 = -\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2 = 0, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3 = -\widehat{e}_1.
\end{aligned} \tag{5.20}$$

Lemma 5.11 *The curvature R^{B_1} of the Bott connection ∇^{B_1} of (G_4, g) is given by*

$$\begin{aligned}
&R^{B_1}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = 0, \quad R^{B_1}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = 0, \quad R^{B_1}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = (2\eta + \alpha - \beta)\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = (\beta - \eta)^2\widehat{e}_3, \quad R^{B_1}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = 0, \quad R^{B_1}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = (1 + \alpha\beta)\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = (\beta - 2\eta - \alpha)\widehat{e}_1, \quad R^{B_1}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = \alpha\widehat{e}_2, \quad R^{B_1}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = (2\eta + \alpha - \beta)\widehat{e}_3.
\end{aligned} \tag{5.21}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_1}(\widehat{e}_1, \widehat{e}_1) = -(\beta - \eta)^2, \quad \rho^{B_1}(\widehat{e}_1, \widehat{e}_2) = \rho^{B_1}(\widehat{e}_1, \widehat{e}_3) = 0,\\
&\rho^{B_1}(\widehat{e}_2, \widehat{e}_1) = \rho^{B_1}(\widehat{e}_2, \widehat{e}_2) = \rho^{B_1}(\widehat{e}_2, \widehat{e}_3) = 0,\\
&\rho^{B_1}(\widehat{e}_3, \widehat{e}_1) = 0, \quad \rho^{B_1}(\widehat{e}_3, \widehat{e}_2) = \alpha, \quad \rho^{B_1}(\widehat{e}_3, \widehat{e}_3) = \alpha\beta + 1.
\end{aligned} \tag{5.22}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_1}(\widehat{e}_1, \widehat{e}_1) = -(\beta - \eta)^2, \quad \widetilde{\rho}^{B_1}(\widehat{e}_1, \widehat{e}_2) = 0, \quad \widetilde{\rho}^{B_1}(\widehat{e}_1, \widehat{e}_3) = 0,\\
&\widetilde{\rho}^{B_1}(\widehat{e}_2, \widehat{e}_2) = 0, \quad \widetilde{\rho}^{B_1}(\widehat{e}_2, \widehat{e}_3) = \frac{\alpha}{2}, \quad \widetilde{\rho}^{B_1}(\widehat{e}_3, \widehat{e}_3) = \alpha\beta + 1.
\end{aligned} \tag{5.23}$$

By (2.7), we have

$$\begin{aligned}
&(L^{B_1}_V g)(\widehat{e}_1, \widehat{e}_1) = 0, \quad (L^{B_1}_V g)(\widehat{e}_1, \widehat{e}_2) = -\mu_2 + \mu_3\alpha, \quad (L^{B_1}_V g)(\widehat{e}_1, \widehat{e}_3) = -\mu_3,\\
&(L^{B_1}_V g)(\widehat{e}_2, \widehat{e}_2) = 0, \quad (L^{B_1}_V g)(\widehat{e}_2, \widehat{e}_3) = \mu_1(2\eta - \beta), \quad (L^{B_1}_V g)(\widehat{e}_3, \widehat{e}_3) = 2\mu_1.
\end{aligned} \tag{5.24}$$

Then, if (G_4, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following six equations:

$$\begin{cases}
(\beta - \eta)^2 - \mu = 0\\
\mu_2 - \mu_3\alpha = 0\\
\mu_3 = 0\\
\mu = 0\\
\mu_1(2\eta - \beta) + \alpha = 0\\
\mu_1 + \alpha\beta + 1 - \mu = 0
\end{cases} \tag{5.25}$$

By solving (5.25), we get

Theorem 5.12 *(G_4, g, V) is not an affine Ricci soliton associated to the Bott connection ∇^{B_1}.*

5.5 Affine Ricci solitons of G_5

Lemma 5.13 *The Bott connection ∇^{B_1} of G_5 is given by*

$$\begin{aligned}
&\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1=\alpha\widehat{e}_3,\quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2=0,\quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3=\alpha\widehat{e}_1,\\
&\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1=0,\quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2=0,\quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3=\gamma\widehat{e}_1,\\
&\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1=0,\quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2=-\delta\widehat{e}_2,\quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3=0.
\end{aligned}\tag{5.26}$$

Lemma 5.14 *The curvature R^{B_1} of the Bott connection ∇^{B_1} of (G_5,g) is given by*

$$\begin{aligned}
&R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=-\alpha\gamma\widehat{e}_1,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=0,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=\alpha\gamma\widehat{e}_3,\\
&R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=-\alpha^2\widehat{e}_3,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=0,\quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=-(\beta\gamma+\alpha^2)\widehat{e}_2,\\
&R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=-\alpha\gamma\widehat{e}_3,\quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=0,\quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=-\gamma(\alpha+\delta)\widehat{e}_1.
\end{aligned}\tag{5.27}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_1}(\widehat{e}_1,\widehat{e}_1)=\alpha^2,\quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_2)=\rho^{B_1}(\widehat{e}_1,\widehat{e}_3)=0,\\
&\rho^{B_1}(\widehat{e}_2,\widehat{e}_1)=\rho^{B_1}(\widehat{e}_2,\widehat{e}_2)=\rho^{B_1}(\widehat{e}_2,\widehat{e}_3)=0,\\
&\rho^{B_1}(\widehat{e}_3,\widehat{e}_1)=\rho^{B_1}(\widehat{e}_3,\widehat{e}_2)=0,\quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_3)=-(\beta\gamma+\alpha^2).
\end{aligned}\tag{5.28}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_1)=\alpha^2,\quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_2)=\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_3)=0,\\
&\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_2)=\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_3)=0,\quad \widetilde{\rho}^{B_1}(\widehat{e}_3,\widehat{e}_3)=-(\beta\gamma+\alpha^2).
\end{aligned}\tag{5.29}$$

By (2.7), we have

$$\begin{aligned}
&(L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_1)=2\mu_3\alpha,\quad (L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_2)=\mu_3\gamma,\quad (L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_3)=-\mu_1\alpha,\\
&(L^{B_1}_V g)(\widehat{e}_2,\widehat{e}_2)=0,\quad (L^{B_1}_V g)(\widehat{e}_2,\widehat{e}_3)=-\mu_2\delta,\quad (L^{B_1}_V g)(\widehat{e}_3,\widehat{e}_3)=0.
\end{aligned}\tag{5.30}$$

Then, if (G_5,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following six equations:

$$\begin{cases}
\mu_3\alpha+\alpha^2+\mu=0\\
\mu_3\gamma=0\\
\mu_1\alpha=0\\
\mu=0\\
\mu_2\delta=0\\
\alpha^2+\beta\gamma+\mu=0
\end{cases}\tag{5.31}$$

By solving (5.31), we get

Theorem 5.15 *(G_5, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1} if and only if* $\mu = \alpha = \beta = \mu_2 = \mu_3\gamma = 0, \quad \delta \neq 0.$

5.6 Affine Ricci solitons of G_6

Lemma 5.16 *The Bott connection ∇^{B_1} of G_6 is given by*

$$\begin{aligned}
&\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2 = \alpha\widehat{e}_2, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3 = 0,\\
&\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1 = -\beta\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2 = \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3 = 0,\\
&\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1 = -\delta\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2 = 0, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3 = -\delta\widehat{e}_1.
\end{aligned} \tag{5.32}$$

Lemma 5.17 *The curvature R^{B_1} of the Bott connection ∇^{B_1} of (G_6, g) is given by*

$$\begin{aligned}
&R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1 = \beta(\alpha+\delta)\widehat{e}_3, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = 0, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3 = \beta\delta\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1 = (\beta\gamma+\delta^2)\widehat{e}_3, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = 0, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3 = \delta^2\widehat{e}_1,\\
&R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1 = -\beta\delta\widehat{e}_1, \quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = 0, \quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3 = \beta\delta\widehat{e}_3.
\end{aligned} \tag{5.33}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_1}(\widehat{e}_1,\widehat{e}_1) = -(\beta\gamma+\delta^2), \quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_2) = \rho^{B_1}(\widehat{e}_1,\widehat{e}_3) = 0,\\
&\rho^{B_1}(\widehat{e}_2,\widehat{e}_1) = \rho^{B_1}(\widehat{e}_2,\widehat{e}_2) = \rho^{B_1}(\widehat{e}_2,\widehat{e}_3) = 0,\\
&\rho^{B_1}(\widehat{e}_3,\widehat{e}_1) = \rho^{B_1}(\widehat{e}_3,\widehat{e}_2) = 0, \quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_3) = \delta^2.
\end{aligned} \tag{5.34}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_1) = -(\delta^2+\beta\gamma), \quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_2) = \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_3) = 0,\\
&\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_2) = \widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_3) = 0, \quad \widetilde{\rho}^{B_1}(\widehat{e}_3,\widehat{e}_3) = \delta^2.
\end{aligned} \tag{5.35}$$

By (2.7), we have

$$\begin{aligned}
&(L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_1) = 0, \quad (L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_2) = \mu_2\alpha, \quad (L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_3) = -\mu_3\delta,\\
&(L^{B_1}_V g)(\widehat{e}_2,\widehat{e}_2) = 0, \quad (L^{B_1}_V g)(\widehat{e}_2,\widehat{e}_3) = \mu_1\beta, \quad (L^{B_1}_V g)(\widehat{e}_3,\widehat{e}_3) = 2\mu_1\delta.
\end{aligned} \tag{5.36}$$

Then, if (G_6, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following six equations:

$$\begin{cases}
\delta^2 + \beta\gamma - \mu = 0\\
\mu_2\alpha = 0\\
\mu_3\delta = 0\\
\mu = 0\\
\mu_1\beta = 0\\
\mu_1\delta + \delta^2 - \mu = 0
\end{cases} \tag{5.37}$$

By solving (5.37), we get

Theorem 5.18 *(G_6, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1} if and only if* $\mu = \mu_2 = \delta = \gamma = \mu_1\beta = 0, \quad \alpha \neq 0.$

5.7 Affine Ricci solitons of G_7

Lemma 5.19 *The Bott connection ∇^{B_1} of G_7 is given by*

$$\begin{aligned}
&\nabla^{B_1}_{\widehat{e}_1}\widehat{e}_1 = \alpha\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_2 = -\beta\widehat{e}_2, \quad \nabla^{B_1}_{\widehat{e}_1}\widehat{e}_3 = \alpha\widehat{e}_1,\\
&\nabla^{B_1}_{\widehat{e}_2}\widehat{e}_1 = \alpha\widehat{e}_1 + \beta\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_2 = 0, \quad \nabla^{B_1}_{\widehat{e}_2}\widehat{e}_3 = \gamma\widehat{e}_1 + \delta\widehat{e}_3,\\
&\nabla^{B_1}_{\widehat{e}_3}\widehat{e}_1 = -\beta\widehat{e}_3, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_2 = -\delta\widehat{e}_2, \quad \nabla^{B_1}_{\widehat{e}_3}\widehat{e}_3 = -\beta\widehat{e}_1.
\end{aligned} \tag{5.38}$$

Lemma 5.20 *The curvature R^{B_1} of the Bott connection ∇^{B_1} of (G_7, g) is given by*

$$\begin{aligned}
&R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1 = \alpha(2\beta-\gamma)\widehat{e}_1 + \alpha(2\alpha-\delta)\widehat{e}_3, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = -\beta(\alpha+\delta)\widehat{e}_2,\\
&R^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3 = (\alpha\delta+\beta\gamma-\beta^2)\widehat{e}_1 + (\alpha\gamma+\beta\delta-\alpha\beta)\widehat{e}_3, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1 = -\alpha\beta\widehat{e}_1 - \alpha^2\widehat{e}_3,\\
&R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = \beta(\alpha+\delta)\widehat{e}_2, \quad R^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3 = (\beta^2-\alpha^2-\beta\gamma)\widehat{e}_1 - \beta\delta\widehat{e}_3,\\
&R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1 = (\beta^2-\beta\gamma-\alpha\delta)\widehat{e}_1 + (\alpha\beta-\beta\delta-\alpha\gamma)\widehat{e}_3, \quad R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = (\beta\gamma+\delta^2)\widehat{e}_2,\\
&R^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3 = (2\beta\delta-\delta\gamma-\alpha\gamma-\alpha\beta)\widehat{e}_1 + (\beta\gamma-\beta^2-\delta^2)\widehat{e}_3.
\end{aligned} \tag{5.39}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_1}(\widehat{e}_1,\widehat{e}_1) = \alpha^2, \quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_2) = \beta(\alpha+\delta), \quad \rho^{B_1}(\widehat{e}_1,\widehat{e}_3) = \beta\delta,\\
&\rho^{B_1}(\widehat{e}_2,\widehat{e}_1) = \beta(\alpha+\delta), \quad \rho^{B_1}(\widehat{e}_2,\widehat{e}_2) = 0, \quad \rho^{B_1}(\widehat{e}_2,\widehat{e}_3) = \delta(\alpha+\delta),\\
&\rho^{B_1}(\widehat{e}_3,\widehat{e}_1) = -\alpha\beta, \quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_2) = \beta\gamma+\delta^2, \quad \rho^{B_1}(\widehat{e}_3,\widehat{e}_3) = \beta^2-\alpha^2-\beta\gamma.
\end{aligned} \tag{5.40}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_1) = \alpha^2, \quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_2) = \beta(\alpha+\delta), \quad \widetilde{\rho}^{B_1}(\widehat{e}_1,\widehat{e}_3) = \frac{\beta(\delta-\alpha)}{2},\\
&\widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_2) = 0, \quad \widetilde{\rho}^{B_1}(\widehat{e}_2,\widehat{e}_3) = \delta^2 + \frac{\beta\gamma+\alpha\delta}{2}, \quad \widetilde{\rho}^{B_1}(\widehat{e}_3,\widehat{e}_3) = \beta^2-\alpha^2-\beta\gamma.
\end{aligned} \tag{5.41}$$

By (2.7), we have

$$\begin{aligned}
&(L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_1) = 2\mu_3\alpha, \quad (L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_2) = \mu_1\alpha - \mu_2\beta + \mu_3\gamma, \quad (L^{B_1}_V g)(\widehat{e}_1,\widehat{e}_3) = -\mu_1\alpha - \mu_3\beta,\\
&(L^{B_1}_V g)(\widehat{e}_2,\widehat{e}_2) = 0, \quad (L^{B_1}_V g)(\widehat{e}_2,\widehat{e}_3) = -\mu_1\beta - \mu_2\delta - \mu_3\delta, \quad (L^{B_1}_V g)(\widehat{e}_3,\widehat{e}_3) = 2\mu_1\beta.
\end{aligned} \tag{5.42}$$

Then, if (G_7, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1}, by (2.8), we have the following six equations:

$$\begin{cases} \mu_3\alpha + \alpha^2 + \mu = 0 \\ \mu_1\alpha - \mu_2\beta + \mu_3\gamma + 2\beta(\delta+\alpha) = 0 \\ \mu_3\beta + \mu_1\alpha + \beta(\alpha-\delta) = 0 \\ \mu = 0 \\ \mu_3\delta + \mu_1\beta + \mu_2\delta - 2\delta^2 - \beta\gamma - \alpha\delta = 0 \\ \mu_1\beta + \beta^2 - \alpha^2 - \beta\gamma - \mu = 0 \end{cases} \tag{5.43}$$

By solving (5.43), we get

Theorem 5.21 *(G_7, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_1} if and only if*

$$(1)\mu = \alpha = \beta = \gamma = 0, \quad \delta \neq 0, \quad \mu_2 + \mu_3 - 2\delta = 0;$$
$$(2)\mu = \mu_3 = \alpha = \beta = 0, \quad \gamma \neq 0, \quad \delta \neq 0, \quad \mu_2 - 2\delta = 0;$$
$$(3)\mu = \alpha = 0, \quad \beta \neq 0, \quad \delta \neq 0, \quad \mu_1 = \gamma - \beta, \quad \mu_3 = \delta, \quad \mu_2 = \frac{\gamma\delta + 2\beta\delta}{\beta}, \quad \gamma = \frac{\beta(\beta^2 - \beta\delta)}{\delta};$$

Specially, let $V = 0$, we get the following corollary:

Corollary 5.22 *(I) (G_1, g, V) is not an affine Einstein associated to the Bott connection ∇^{B_1};*
(II) (G_2, g, V) is not an affine Einstein associated to the Bott connection ∇^{B_1};
(III) (G_3, g, V) is an affine Einstein associated to the Bott connection ∇^{B_1} if and only if $(1)\mu = \alpha = \gamma = 0, \quad \beta \neq 0;$ $(2)\mu = \beta = 0;$
(IV) (G_4, g, V) is not an affine Einstein associated to the Bott connection ∇^{B_1};
(V) (G_5, g, V) is an affine Einstein associated to the Bott connection ∇^{B_1} if and only if $\mu = \alpha = \beta = 0, \quad \delta \neq 0$;
(VI) (G_6, g, V) is an affine Einstein associated to the Bott connection ∇^{B_1} if and only if $\mu = \delta = \gamma = 0, \quad \alpha \neq 0;$
(VII) (G_7, g, V) is not an affine Einstein associated to the Bott connection ∇^{B_1}.

6. Affine Ricci solitons associated to the perturbed Bott connection on three-dimensional Lorentzian Lie groups with the second distribution

Similarly, by the above calculations, we always obtain $\mu = 0$. In order to get the affine Ricci soliton with nonzero μ, we introduce the perturbed Bott connection $\widetilde{\nabla}^{B_1}$ in the following. Let $\widehat{e}_2^*$ be the dual base of e_2. We define on $G_{i=1,\cdots,7}$

$$\widetilde{\nabla}^{B_1}_X Y = \nabla^{B_1}_X Y + a_0 \widehat{e}_2^*(X)\widehat{e}_2^*(Y)e_2, \tag{6.1}$$

where a_0 is a nonzero real number. Then

$$\widetilde{\nabla}^{B_1}_{\widehat{e}_2}\widehat{e}_2 = a_0\widehat{e}_2, \quad \widetilde{\nabla}^{B_1}_{\widehat{e}_s}\widehat{e}_t = \nabla^{B_1}_{\widehat{e}_s}\widehat{e}_t, \tag{6.2}$$

where s and t does not equal 2. We define

$$(\widetilde{L}_V^{B_1}g)(X,Y) := g(\widetilde{\nabla}_X^{B_1}V,Y) + g(X,\widetilde{\nabla}_Y^{B_1}V), \tag{6.3}$$

for vector fields X,Y,V. Then we have for $G_{i=1,\cdots,7}$

$$(\widetilde{L}_V^{B_1}g)(\widehat{e}_2,\widehat{e}_2) = 2a_0\mu_2, \quad (\widetilde{L}_V^{B_1}g)(\widehat{e}_s,\widehat{e}_t) = (L_V^{B_1}g)(\widehat{e}_s,\widehat{e}_t), \tag{6.4}$$

where s and t does not equal 2.

Definition 6.1 (G_i,V,g) *is called the affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_1}$ *if it satisfies*

$$(\widetilde{L}_V^{B_1}g)(X,Y) + 2\widetilde{\widetilde{\rho}}^{B_1}(X,Y) + 2\mu g(X,Y) = 0. \tag{6.5}$$

For $(G_1,\widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = a_0\beta\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = (\alpha^2 - a_0\alpha)\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p = R^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p, \tag{6.6}$$

for $(s,t,p) \neq (1,3,2),(2,3,2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_2,\widehat{e}_3) = \frac{\alpha(\alpha-a_0)}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s,\widehat{e}_t) = \widetilde{\rho}^{B_1}(\widehat{e}_s,\widehat{e}_t), \tag{6.7}$$

for the pair $(s,t) \neq (2,3)$. If (G_1,g,V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases} \mu_3\alpha - \alpha^2 + \beta^2 - \mu = 0 \\ \alpha\beta - \mu_1\alpha + \mu_3\beta = 0 \\ \mu_1\alpha - 2\alpha\beta = 0 \\ a_0\mu_2 + \mu = 0 \\ (\mu_2+\mu_3)\alpha + \mu_1\beta - \alpha^2 + a_0\alpha = 0 \\ \beta^2 - \alpha^2 - \mu = 0 \end{cases} \tag{6.8}$$

Solve (6.8), we get $a_0 = \alpha = 0$, there is a contradiction. So

Theorem 6.2 (G_1,V,g) *is not an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_1}$.

For $(G_2,\widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = -a_0\gamma\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = a_0\beta\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p = R^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p, \tag{6.9}$$

for $(s,t,p) \neq (1,2,2),(1,3,2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_1,\widehat{e}_2) = \frac{a_0\gamma}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s,\widehat{e}_t) = \widetilde{\rho}^{B_1}(\widehat{e}_s,\widehat{e}_t), \tag{6.10}$$

for the pair $(s,t)\neq(1,2)$. If (G_2,g,V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases}\gamma^2+\beta^2-\mu=0\\ \mu_2\gamma+\mu_3\alpha+a_0\gamma=0\\ \mu_3\gamma=0\\ a_0\mu_2+\mu=0\\ \mu_1\beta+\alpha\gamma=0\\ \mu_1\gamma-\alpha\beta-\gamma^2+\mu=0\end{cases} \tag{6.11}$$

Solve (6.11), we get

Theorem 6.3 *(G_2,V,g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$ if and only if*

$$(1)\beta=\mu_3=0,\quad \gamma\neq 0,\quad a_0=\pm\gamma,\quad \mu_2=\mp\gamma;$$
$$(2)\mu_3=0,\quad \gamma\neq 0,\quad \beta\neq 0,\quad \mu_1=-\frac{\alpha\gamma}{\beta},\quad \beta^3-\alpha\gamma^2-\alpha\beta^2=0.$$

For $(G_3,\widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=a_0\beta\widehat{e}_2,\quad \widetilde{R}^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p=R^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p, \tag{6.12}$$

for $(s,t,p)\neq(1,3,2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s,\widehat{e}_t)=\widetilde{\rho}^{B_1}(\widehat{e}_s,\widehat{e}_t), \tag{6.13}$$

for any pairs (s,t). If (G_3,g,V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases}\beta\gamma-\mu=0\\ \mu_3\alpha=0\\ a_0\mu_2+\mu=0\\ \mu_1\gamma=0\\ \alpha\beta-\mu=0\end{cases} \tag{6.14}$$

Solve (6.14), we get

Theorem 6.4 *(G_3,V,g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$ if and only if*

$$(1)\beta=\mu=\mu_2=\mu_3\alpha=\mu_1\gamma=0;$$
$$(2)\beta\neq 0,\quad \gamma=\alpha=\mu=\mu_2=0;$$
$$(3)\beta\neq 0,\quad \gamma=\alpha\neq 0,\quad \mu_1=\mu_3=0,\quad \mu_2=-\frac{\alpha\beta}{a_0},\quad \mu=\alpha\beta.$$

For $(G_4, \widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = a_0\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = a_0\beta\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p = R^B(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p, \tag{6.15}$$

for $(s,t,p) \neq (1,2,2),(1,3,2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_1,\widehat{e}_2) = -\frac{a_0}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s,\widehat{e}_t) = \widetilde{\rho}^{B_1}(\widehat{e}_s,\widehat{e}_t), \tag{6.16}$$

for the pair $(s,t) \neq (1,2)$. If (G_4, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases} (\beta-\eta)^2 - \mu = 0 \\ \mu_3\alpha - \mu_2 - a_0 = 0 \\ \mu_3 = 0 \\ a_0\mu_2 + \mu = 0 \\ \mu_1(2\eta-\beta) + \alpha = 0 \\ \mu_1 + \alpha\beta + 1 - \mu = 0 \end{cases} \tag{6.17}$$

Solve (6.17), we get

Theorem 6.5 *(G_4, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$ if and only if $\beta \neq \eta$, $a_0 = \pm(\beta-\eta)$, $\mu_3 = 0$, $\mu_2 = \mp(\beta-\eta)$, $\mu_1 = 1 - \frac{1}{(\beta-\eta)^2}$, $\mu = (\beta-\eta)^2$.*

For $(G_5, \widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = -a_0\beta\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = -a_0\delta\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p = R^{B_1}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p, \tag{6.18}$$

for $(s,t,p) \neq (1,3,2),(2,3,2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_2,\widehat{e}_3) = -\frac{a_0\delta}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s,\widehat{e}_t) = \widetilde{\rho}^{B_1}(\widehat{e}_s,\widehat{e}_t), \tag{6.19}$$

for the pair $(s,t) \neq (2,3)$. If (G_5, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases} \mu_3\alpha + \alpha^2 + \mu = 0 \\ \mu_3\gamma = 0 \\ \mu_1\alpha = 0 \\ a_0\mu_2 + \mu = 0 \\ \delta(\mu_2 + a_0) = 0 \\ \alpha^2 + \beta\gamma + \mu = 0 \end{cases} \tag{6.20}$$

Solve (6.20), we get

Theorem 6.6 (G_5, V, g) *is an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_1}$ *if and only if*

$$(1)\beta = \alpha = \mu_2 = \mu = a_0 = \mu_3\gamma = 0, \qquad \delta \neq 0;$$
$$(2)\alpha \neq 0, \quad \delta = \mu_3 = \gamma = 0, \quad \mu = \mu_2 a_0 = -\alpha^2;$$
$$(3)\alpha \neq 0, \quad \delta \neq 0, \quad \alpha + \delta \neq 0, \quad \mu_3 = \beta = \gamma = 0, \quad \mu_2 = -a_0 = |\alpha|.$$

For $(G_6, \widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = -a_0\alpha\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = -a_0\gamma\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_1}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{6.21}$$

for $(s, t, p) \neq (1, 2, 2), (1, 3, 2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_1, \widehat{e}_2) = \frac{a_0\alpha}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_1}(\widehat{e}_s, \widehat{e}_t), \tag{6.22}$$

for the pair $(s, t) \neq (1, 2)$. If (G_6, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases} \delta^2 + \beta\gamma - \mu = 0 \\ \mu_2\alpha + a_0\alpha = 0 \\ \mu_3\delta = 0 \\ a_0\mu_2 + \mu = 0 \\ \mu_1\beta = 0 \\ \mu_1\delta + \delta^2 - \mu = 0 \end{cases} \tag{6.23}$$

Solve (6.23), we get

Theorem 6.7 (G_6, V, g) *is an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_1}$ *if and only if*

$$(1)\beta = \alpha = \mu_3 = \mu_1 = 0, \quad \delta \neq 0, \quad \mu_2 = -\frac{\delta^2}{a_0};$$
$$(2)\alpha \neq 0, \quad \delta \neq 0, \quad \mu_1 = 0, \quad a_0 = \pm\delta, \quad \mu_2 = \mp\delta.$$

For $(G_7, \widetilde{\nabla}^{B_1})$, we have

$$\widetilde{R}^{B_1}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = (a_0 - \alpha - \delta)\beta\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = \beta(\alpha + \delta - a_0)\widehat{e}_2,$$
$$\widetilde{R}^{B_1}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = (\delta^2 + \beta\gamma - a_0\delta)\widehat{e}_2, \quad \widetilde{R}^{B_1}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_1}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{6.24}$$

for $(s, t, p) \neq (1, 2, 2), (1, 3, 2), (2, 3, 2)$.

$$\widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_1, \widehat{e}_2) = \frac{2\beta(\alpha\delta) - a_0\beta}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_2, \widehat{e}_3) = \frac{2\delta^2 + \alpha\delta + \beta\gamma - a_0\delta}{2}, \quad \widetilde{\widetilde{\rho}}^{B_1}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_1}(\widehat{e}_s, \widehat{e}_t), \tag{6.25}$$

for the pair $(s,t) \neq (1,2),(2,3)$. If (G_7, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$, then by (6.5), we have

$$\begin{cases} \mu_3\alpha + \alpha^2 + \mu = 0 \\ \mu_2\beta - \mu_1\alpha - \mu_3\gamma - 2\beta\delta - 2\alpha\beta + a_0\beta = 0 \\ \mu_1\alpha + \mu_3\beta + \alpha\beta - \beta\delta = 0 \\ a_0\mu_2 + \mu = 0 \\ \mu_1\beta + \mu_2\delta + \mu_3\delta - 2\delta^2 - \alpha\delta - \beta\gamma + a_0\delta = 0 \\ \mu_1\beta + \beta^2 - \alpha^2 - \beta\gamma - \mu = 0 \end{cases} \tag{6.26}$$

Solve (6.26), we get

Theorem 6.8 *(G_7, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_1}$ if and only if*

$(1)\mu = \alpha = \beta = \gamma = \mu_2 = 0, \quad \delta \neq 0, \quad \mu_3 = 2\delta - a_0;$

$(2)\alpha = \beta = \mu = \mu_2 = \mu_3 = 0, \quad \gamma \neq 0, \quad \delta \neq 0, \quad a_0 = 2\delta;$

$(3)\alpha = \mu = \mu_2 = 0, \quad \beta \neq 0, \quad \delta \neq 0, \quad \mu_1 = \gamma - \beta, \quad \mu_3 = \delta, \quad a_0 = \dfrac{2\beta\delta + \gamma\delta}{\beta}, \quad \gamma = \dfrac{\beta^3 - \beta\delta^2}{\delta^2};$

$(4)\alpha \neq 0, \quad \beta = \gamma = \delta = \mu_1 = \mu_3 = 0, \quad \mu = -\alpha^2, \quad \mu_2 = \dfrac{\alpha^2}{a_0};$

$(5)\alpha \neq 0, \quad \beta = \gamma = \mu_1 = \mu_3 = 0, \quad \mu = -\alpha^2, \quad \mu_2 = \alpha + 2\delta - a_0, \quad a_0^2 - a_0\alpha - 2a_0\delta + \alpha^2 = 0;$

$(6)\alpha \neq 0, \quad \beta \neq 0, \quad \gamma = 0, \quad \alpha^2 \neq \beta^2, \quad \mu = \dfrac{\alpha\beta^2\delta}{\beta^2 - \alpha^2} - \alpha^2, \quad \mu = \dfrac{\alpha\beta\delta}{\beta^2 - \alpha^2} - \beta,$

$\mu_2 = \alpha + 2\delta + \dfrac{\alpha^2\delta}{\beta^2 - \alpha^2} - a_0, \quad \alpha\beta^2\delta + \alpha^2\delta^2 - \beta^2\delta^2 + \alpha^2\beta^2 - \beta^4 = 0.$

7. Affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian Lie groups with the third distribution

Let $TM = span\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$, then took the distribution $D_2 = span\{\widehat{e}_2, \widehat{e}_3\}$ and $D_2^\perp = span\{\widehat{e}_1\}$.
Similar to (2.3), we have

$$\nabla^{B_2}_X Y = \begin{cases} \pi_{D_2}(\nabla^L_X Y), & X, Y \in \Gamma^\infty(D_2) \\ \pi_{D_2}([X,Y]), & X \in \Gamma^\infty({D_2}^\perp), Y \in \Gamma^\infty(D_2) \\ \pi_{{D_2}^\perp}([X,Y]), & X \in \Gamma^\infty(D_2), Y \in \Gamma^\infty({D_2}^\perp) \\ \pi_{{D_2}^\perp}(\nabla^L_X Y), & X, Y \in \Gamma^\infty({D_2}^\perp) \end{cases} \tag{7.1}$$

where π_{D_2}(resp. $\pi^\perp_{D_2}$) the projection on D_2 (resp. ${D_2}^\perp$).

7.1 Affine Ricci solitons of G_1

Lemma 7.1 *The Bott connection* ∇^{B_2} *of* G_1 *is given by*

$$\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2 = -\beta\widehat{e}_3, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3 = -\beta\widehat{e}_2,$$
$$\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1 = -\alpha\widehat{e}_1, \quad \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2 = \alpha\widehat{e}_3, \quad \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3 = \alpha\widehat{e}_2,$$
$$\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1 = \alpha\widehat{e}_1, \quad \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2 = -\alpha\widehat{e}_3, \quad \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3 = -\alpha\widehat{e}_2. \tag{7.2}$$

Lemma 7.2 *The curvature* R^{B_2} *of the Bott connection* ∇^{B_2} *of* (G_1, g) *is given by*

$$R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = \alpha\beta\widehat{e}_1, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = 0, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = 0,$$
$$R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = -\alpha\beta\widehat{e}_1, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = 0, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = 0,$$
$$R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = 0, \quad R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = \beta^2\widehat{e}_3, \quad R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = \beta^2\widehat{e}_2. \tag{7.3}$$

By (2.5), we have

$$\rho^{B_2}(\widehat{e}_1, \widehat{e}_1) = \rho^{B_2}(\widehat{e}_1, \widehat{e}_2) = \rho^{B_2}(\widehat{e}_1, \widehat{e}_3) = 0,$$
$$\rho^{B_2}(\widehat{e}_2, \widehat{e}_1) = \alpha\beta, \quad \rho^{B_2}(\widehat{e}_2, \widehat{e}_2) = -\beta^2, \quad \rho^{B_2}(\widehat{e}_2, \widehat{e}_3) = 0,$$
$$\rho^{B_2}(\widehat{e}_3, \widehat{e}_1) = -\alpha\beta, \quad \rho^{B_2}(\widehat{e}_3, \widehat{e}_2) = 0, \quad \rho^{B_2}(\widehat{e}_3, \widehat{e}_3) = \beta^2. \tag{7.4}$$

Then,

$$\widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_1) = 0, \quad \widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_2) = \frac{\alpha\beta}{2}, \quad \widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_3) = -\frac{\alpha\beta}{2},$$
$$\widetilde{\rho}^{B_2}(\widehat{e}_2, \widehat{e}_2) = -\beta^2, \quad \widetilde{\rho}^{B_2}(\widehat{e}_2, \widehat{e}_3) = 0, \quad \widetilde{\rho}^{B_2}(\widehat{e}_3, \widehat{e}_3) = \beta^2. \tag{7.5}$$

By (2.7), we have

$$(L^{B_2}_V g)(\widehat{e}_1, \widehat{e}_1) = 0, \quad (L^{B_2}_V g)(\widehat{e}_1, \widehat{e}_2) = -\mu_1\alpha - \mu_3\beta, \quad (L^{B_2}_V g)(\widehat{e}_1, \widehat{e}_3) = \mu_1\alpha + \mu_2\beta,$$
$$(L^{B_2}_V g)(\widehat{e}_2, \widehat{e}_2) = 2\mu_3\alpha, \quad (L^{B_2}_V g)(\widehat{e}_2, \widehat{e}_3) = -(\mu_2 + \mu_3)\alpha, \quad (L^{B_2}_V g)(\widehat{e}_3, \widehat{e}_3) = 2\mu_2\alpha. \tag{7.6}$$

Then, if (G_1, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following six equations:

$$\begin{cases} \mu = 0 \\ \mu_3\beta + \mu_1\alpha - \alpha\beta = 0 \\ \mu_1\alpha + \mu_2\beta - \alpha\beta = 0 \\ (\mu_2 + \mu_3)\alpha = 0 \\ \mu_3\alpha - \beta^2 + \mu = 0 \\ \beta^2 + \mu_2\alpha - \mu = 0 \end{cases} \tag{7.7}$$

By solving (7.7) , we get

Theorem 7.3 (G_1, g, V) *is an affine Ricci soliton associated to the Bott connection* ∇^{B_2} *if and only if* $\mu = \beta = \mu_2 = \mu_3 = 0, \quad \alpha \neq 0$.

7.2 Affine Ricci solitons of G_2

Lemma 7.4 *The Bott connection* ∇^{B_2} *of* G_2 *is given by*

$$\begin{aligned}&\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1=0,\quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2=\gamma\widehat{e}_2-\beta\widehat{e}_3,\quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3=-\beta\widehat{e}_2-\gamma\widehat{e}_3,\\&\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1=\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2=\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3=0,\\&\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1=\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2=\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3=0.\end{aligned}\tag{7.8}$$

Lemma 7.5 *The curvature* R^{B_2} *of the Bott connection* ∇^{B_2} *of* (G_2,g) *is given by*

$$\begin{aligned}&R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=0,\\&R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=0,\\&R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=0,\quad R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=-\alpha\gamma\widehat{e}_2+\alpha\beta\widehat{e}_3,\quad R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=\alpha\beta\widehat{e}_2+\alpha\gamma\widehat{e}_3.\end{aligned}\tag{7.9}$$

By (2.5), we have

$$\begin{aligned}&\rho^{B_2}(\widehat{e}_1,\widehat{e}_1)=\rho^{B_2}(\widehat{e}_1,\widehat{e}_2)=\rho^{B_2}(\widehat{e}_1,\widehat{e}_3)=0,\\&\rho^{B_2}(\widehat{e}_2,\widehat{e}_1)=0,\quad \rho^{B_2}(\widehat{e}_2,\widehat{e}_2)=-\alpha\beta,\quad \rho^{B_2}(\widehat{e}_2,\widehat{e}_3)=-\alpha\gamma,\\&\rho^{B_2}(\widehat{e}_3,\widehat{e}_1)=0,\quad \rho^{B_2}(\widehat{e}_3,\widehat{e}_2)=-\alpha\gamma,\quad \rho^{B_2}(\widehat{e}_3,\widehat{e}_3)=\alpha\beta.\end{aligned}\tag{7.10}$$

Then,

$$\begin{aligned}&\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_1)=\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_2)=\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_3)=0,\\&\widetilde{\rho}^{B_2}(\widehat{e}_2,\widehat{e}_2)=-\alpha\beta,\quad \widetilde{\rho}^{B_2}(\widehat{e}_2,\widehat{e}_3)=-\alpha\gamma,\quad \widetilde{\rho}^{B_2}(\widehat{e}_3,\widehat{e}_3)=\alpha\beta.\end{aligned}\tag{7.11}$$

By (2.7), we have

$$\begin{aligned}&(L^{B_2}_V g)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L^{B_2}_V g)(\widehat{e}_1,\widehat{e}_2)=\mu_2\gamma-\mu_3\beta,\quad (L^{B_2}_V g)(\widehat{e}_1,\widehat{e}_3)=\mu_2\beta+\mu_3\gamma,\\&(L^{B_2}_V g)(\widehat{e}_2,\widehat{e}_2)=(L^{B_2}_V g)(\widehat{e}_2,\widehat{e}_3)=(L^{B_2}_V g)(\widehat{e}_3,\widehat{e}_3)=0.\end{aligned}\tag{7.12}$$

Then, if (G_2,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following five equations:

$$\begin{cases}\mu=0\\ \mu_2\gamma-\mu_3\beta=0\\ \mu_2\beta+\mu_3\gamma=0\\ \alpha\beta-\mu=0\\ \alpha\gamma=0\end{cases}\tag{7.13}$$

By solving (7.13), we get

Theorem 7.6 *(G_2, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2} if and only if $\mu = \alpha = \mu_2 = \mu_3 = 0, \quad \gamma \neq 0$.*

7.3 Affine Ricci solitons of G_3

Lemma 7.7 *The Bott connection ∇^{B_2} of G_3 is given by*

$$\begin{aligned}
&\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2 = -\gamma\widehat{e}_3, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3 = -\beta\widehat{e}_2,\\
&\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1 = \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2 = \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3 = 0,\\
&\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1 = \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2 = \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3 = 0.
\end{aligned} \tag{7.14}$$

Lemma 7.8 *The curvature R^{B_2} of the Bott connection ∇^{B_2} of (G_3, g) is given by*

$$\begin{aligned}
&R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1 = R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2 = R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3 = 0,\\
&R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1 = R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2 = R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3 = 0,\\
&R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1 0, \quad = R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2 = \alpha\gamma\widehat{e}_3, \quad R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3 = \alpha\beta\widehat{e}_2.
\end{aligned} \tag{7.15}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_2}(\widehat{e}_1,\widehat{e}_1) = \rho^{B_2}(\widehat{e}_1,\widehat{e}_2) = \rho^{B_2}(\widehat{e}_1,\widehat{e}_3) = 0,\\
&\rho^{B_2}(\widehat{e}_2,\widehat{e}_1) = \rho^{B_2}(\widehat{e}_2,\widehat{e}_2) = -\alpha\gamma, \quad \rho^{B_2}(\widehat{e}_2,\widehat{e}_3) = 0,\\
&\rho^{B_2}(\widehat{e}_3,\widehat{e}_1) = \rho^{B_2}(\widehat{e}_3,\widehat{e}_2) = 0, \quad \rho^{B_2}(\widehat{e}_3,\widehat{e}_3) = \alpha\beta.
\end{aligned} \tag{7.16}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_1) = \widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_2) = \widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_3) = 0,\\
&\widetilde{\rho}^{B_2}(\widehat{e}_2,\widehat{e}_2) = -\alpha\gamma, \quad \widetilde{\rho}^{B_2}(\widehat{e}_2,\widehat{e}_3) = 0, \quad \widetilde{\rho}^{B_2}(\widehat{e}_3,\widehat{e}_3) = \alpha\beta.
\end{aligned} \tag{7.17}$$

By (2.7), we have

$$\begin{aligned}
&(L^{B_2}_V g)(\widehat{e}_1,\widehat{e}_1) = 0, \quad (L^{B_2}_V g)(\widehat{e}_1,\widehat{e}_2) = -\mu_3\beta, \quad (L^{B_2}_V g)(\widehat{e}_1,\widehat{e}_3) = \mu_2\gamma,\\
&(L^{B_2}_V g)(\widehat{e}_2,\widehat{e}_2) = (L^{B_2}_V g)(\widehat{e}_2,\widehat{e}_3) = (L^{B_2}_V g)(\widehat{e}_3,\widehat{e}_3) = 0.
\end{aligned} \tag{7.18}$$

Then, if (G_3, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following five equations:

$$\begin{cases}
\mu = 0\\
\mu_3\beta = 0\\
\mu_2\gamma = 0\\
\alpha\beta - \mu = 0\\
\alpha\gamma - \mu = 0
\end{cases} \tag{7.19}$$

By solving (7.19), we get

Theorem 7.9 *(G_3, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2} if and only if*

$$(1)\alpha \neq 0, \quad \mu = \beta = \gamma = 0;$$
$$(2)\alpha = \mu = \mu_3\beta = \mu_2\gamma = 0.$$

7.4 Affine Ricci solitons of G_4

Lemma 7.10 *The Bott connection ∇^{B_2} of G_4 is given by*

$$\begin{aligned}
&\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2 = -\widehat{e}_2 + (2\eta - \beta)\widehat{e}_3, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3 = -\beta\widehat{e}_2 + \widehat{e}_3,\\
&\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1 = \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2 = \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3 = 0,\\
&\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1 = \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2 = \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3 = 0.
\end{aligned} \tag{7.20}$$

Lemma 7.11 *The curvature R^{B_2} of the Bott connection ∇^{B_2} of (G_4, g) is given by*

$$\begin{aligned}
&R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = 0,\\
&R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = 0,\\
&R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = 0, \quad R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = \alpha\widehat{e}_2 + \alpha(\beta - 2\eta)\widehat{e}_3, \quad R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = \alpha\beta\widehat{e}_2 - \alpha\widehat{e}_3.
\end{aligned} \tag{7.21}$$

By (2.5), we have

$$\begin{aligned}
&\rho^{B_2}(\widehat{e}_1, \widehat{e}_1) = \rho^{B_2}(\widehat{e}_1, \widehat{e}_2) = \rho^{B_2}(\widehat{e}_1, \widehat{e}_3) = 0,\\
&\rho^{B_2}(\widehat{e}_2, \widehat{e}_1) = 0, \quad \rho^{B_2}(\widehat{e}_2, \widehat{e}_2) = \alpha(2\eta - \beta), \quad \rho^{B_2}(\widehat{e}_2, \widehat{e}_3) = \alpha,\\
&\rho^{B_2}(\widehat{e}_3, \widehat{e}_1) = 0, \quad \rho^{B_2}(\widehat{e}_3, \widehat{e}_2) = \alpha, \quad \rho^{B_2}(\widehat{e}_3, \widehat{e}_3) = \alpha\beta.
\end{aligned} \tag{7.22}$$

Then,

$$\begin{aligned}
&\widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_1) = \widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_2) = \widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_3) = 0,\\
&\widetilde{\rho}^{B_2}(\widehat{e}_2, \widehat{e}_2) = \alpha(2\eta - \beta), \quad \widetilde{\rho}^{B_2}(\widehat{e}_2, \widehat{e}_3) = \alpha, \quad \widetilde{\rho}^{B_2}(\widehat{e}_3, \widehat{e}_3) = \alpha\beta.
\end{aligned} \tag{7.23}$$

By (2.7), we have

$$\begin{aligned}
&(L^{B_2}_V g)(\widehat{e}_1, \widehat{e}_1) = 0, \quad (L^{B_2}_V g)(\widehat{e}_1, \widehat{e}_2) = -\mu_2 - \mu_3\beta, \quad (L^{B_2}_V g)(\widehat{e}_1, \widehat{e}_3) = \mu_2(\beta - 2\eta) - \mu_3,\\
&(L^{B_2}_V g)(\widehat{e}_2, \widehat{e}_2) = (L^{B_2}_V g)(\widehat{e}_2, \widehat{e}_3) = (L^{B_2}_V g)(\widehat{e}_3, \widehat{e}_3) = 0.
\end{aligned} \tag{7.24}$$

Then, if (G_4, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following six equations:

$$\begin{cases} \mu = 0 \\ \mu_2 + \mu_3\beta = 0 \\ \mu_2(\beta - 2\eta) - \mu_3 = 0 \\ \alpha = 0 \\ \alpha(2\eta - \beta) + \mu = 0 \\ \alpha\beta - \mu = 0 \end{cases} \tag{7.25}$$

By solving (7.25), we get

Theorem 7.12 *(G_4, g, V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2} if and only if*

$$(1)\mu = \alpha = \mu_2 + \mu_3\eta = 0, \quad \beta = \eta;$$
$$(2)\mu = \alpha = \mu_3 = \mu_2 = 0, \quad \beta \neq \eta.$$

7.5 Affine Ricci solitons of G_5

Lemma 7.13 *The Bott connection ∇^{B_2} of G_5 is given by*

$$\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1 = 0, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2 = 0, \quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3 = \beta\widehat{e}_2,$$
$$\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1 = 0, \quad \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2 = \delta\widehat{e}_3, \quad \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3 = \delta\widehat{e}_2,$$
$$\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1 = -\alpha\widehat{e}_1, \quad \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2 = 0, \quad \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3 = 0. \tag{7.26}$$

Lemma 7.14 *The curvature R^{B_2} of the Bott connection ∇^{B_2} of (G_5, g) is given by*

$$R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = 0, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_2 = \beta\delta\widehat{e}_2, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_3 = -\beta\delta\widehat{e}_3,$$
$$R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = 0, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_2 = -\beta\delta\widehat{e}_3, \quad R^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_3 = -\beta(\alpha + \delta)\widehat{e}_2,$$
$$R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = 0, \quad R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_2 = -\delta^2\widehat{e}_3, \quad R^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_3 = -(\beta\gamma + \delta^2)\widehat{e}_2. \tag{7.27}$$

By (2.5), we have

$$\rho^{B_2}(\widehat{e}_1, \widehat{e}_1) = \rho^{B_2}(\widehat{e}_1, \widehat{e}_2) = \rho^{B_2}(\widehat{e}_1, \widehat{e}_3) = 0,$$
$$\rho^{B_2}(\widehat{e}_2, \widehat{e}_1) = 0, \quad \rho^{B_2}(\widehat{e}_2, \widehat{e}_2) = \delta^2, \quad \rho^{B_2}(\widehat{e}_2, \widehat{e}_3) = 0,$$
$$\rho^{B_2}(\widehat{e}_3, \widehat{e}_1) = \rho^{B_2}(\widehat{e}_3, \widehat{e}_2) = 0, \quad \rho^{B_2}(\widehat{e}_3, \widehat{e}_3) = -(\beta\gamma + \delta^2). \tag{7.28}$$

Then,

$$\widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_1) = \widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_2) = \widetilde{\rho}^{B_2}(\widehat{e}_1, \widehat{e}_3) = 0,$$
$$\widetilde{\rho}^{B_2}(\widehat{e}_2, \widehat{e}_2) = \delta^2, \quad \widetilde{\rho}^{B_2}(\widehat{e}_2, \widehat{e}_3) = 0, \quad \widetilde{\rho}^{B_2}(\widehat{e}_3, \widehat{e}_3) = -(\beta\gamma + \delta^2). \tag{7.29}$$

By (2.7), we have

$$(L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_2)=\mu_3\beta,\quad (L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_3)=-\mu_1\alpha,$$
$$(L_V^{B_2}g)(\widehat{e}_2,\widehat{e}_2)=2\mu_3\delta,\quad (L_V^{B_2}g)(\widehat{e}_2,\widehat{e}_3)=-\mu_2\delta,\quad (L_V^{B_2}g)(\widehat{e}_3,\widehat{e}_3)=0. \tag{7.30}$$

Then, if (G_5,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following six equations:

$$\begin{cases}\mu=0\\ \mu_3\beta=0\\ \mu_1\alpha=0\\ \mu_3\delta+\delta^2+\mu=0\\ \mu_2\delta=0\\ \delta^2+\beta\gamma+\mu=0\end{cases} \tag{7.31}$$

By solving (7.31), we get

Theorem 7.15 *(G_5,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2} if and only if* $\mu=\delta=\gamma=\mu_1=\mu_3\beta=0,\quad \alpha\neq 0.$

7.6 Affine Ricci solitons of G_6

Lemma 7.16 *The Bott connection ∇^{B_2} of G_6 is given by*

$$\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1=0,\quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2=\alpha\widehat{e}_2+\beta\widehat{e}_3,\quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3=\gamma\widehat{e}_2+\delta\widehat{e}_3,$$
$$\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1=\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2=\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3=0,$$
$$\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1=\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2=\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3=0. \tag{7.32}$$

Lemma 7.17 *The curvature R^{B_2} of the Bott connection ∇^{B_2} of (G_6,g) is given by*

$$R^{B_2}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p=0, \tag{7.33}$$

for any (s,t,p).

By (2.5), we have

$$\rho^{B_2}(\widehat{e}_s,\widehat{e}_t)=0, \tag{7.34}$$

for any pairs (s,t). Similarly,

$$\widetilde{\rho}^{B_2}(\widehat{e}_s,\widehat{e}_t)=0, \tag{7.35}$$

for any pairs (s,t). By (2.7), we have

$$(L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_2)=\mu_2\alpha+\mu_3\gamma,\quad (L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_3)=-\mu_2\beta-\mu_3\delta,$$
$$(L_V^{B_2}g)(\widehat{e}_2,\widehat{e}_2)=(L_V^{B_2}g)(\widehat{e}_2,\widehat{e}_3)=(L_V^{B_2}g)(\widehat{e}_3,\widehat{e}_3)=0. \tag{7.36}$$

Then, if (G_6,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following three equations:

$$\begin{cases}\mu=0\\ \mu_2\alpha+\mu_3\gamma=0\\ \mu_2\beta+\mu_3\delta=0\end{cases} \tag{7.37}$$

By solving (7.37), we get

Theorem 7.18 *(G_6,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2} if and only if*

$$(1)\mu=\mu_2=\mu_3=0,\quad \alpha\gamma-\beta\delta=0,\quad \alpha+\delta\neq 0;$$
$$(2)\mu=0,\quad \alpha=\beta,\quad \delta=\gamma,\quad \mu_2\neq 0,\quad \mu_3\neq 0;$$
$$(3)\mu=\alpha+\beta=\delta+\gamma=0,\quad \mu_2\neq 0,\quad \mu_3\neq 0.$$

7.7 Affine Ricci solitons of G_7

Lemma 7.19 *The Bott connection ∇^{B_2} of G_7 is given by*

$$\nabla^{B_2}_{\widehat{e}_1}\widehat{e}_1=0,\quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_2=-\beta\widehat{e}_2-\beta\widehat{e}_3,\quad \nabla^{B_2}_{\widehat{e}_1}\widehat{e}_3=\beta\widehat{e}_2+\beta\widehat{e}_3,$$
$$\nabla^{B_2}_{\widehat{e}_2}\widehat{e}_1=\alpha\widehat{e}_1,\quad \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_2=\delta\widehat{e}_3,\quad \nabla^{B_2}_{\widehat{e}_2}\widehat{e}_3=\delta\widehat{e}_2,$$
$$\nabla^{B_2}_{\widehat{e}_3}\widehat{e}_1=-\alpha\widehat{e}_1,\quad \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_2=-\delta\widehat{e}_3,\quad \nabla^{B_2}_{\widehat{e}_3}\widehat{e}_3=-\delta\widehat{e}_2. \tag{7.38}$$

Lemma 7.20 *The curvature R^{B_2} of the Bott connection ∇^{B_2} of (G_7,g) is given by*

$$R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1=0,\quad R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_2=\beta(2\delta-\alpha)(\widehat{e}_2+\widehat{e}_3),$$
$$R^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_3=\beta(\alpha-2\delta)(\widehat{e}_2+\widehat{e}_3),\quad R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1=0,$$
$$R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_2=\beta(\alpha-2\delta)(\widehat{e}_2+\widehat{e}_3),\quad R^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_3=\beta(2\delta-\alpha)(\widehat{e}_2+\widehat{e}_3),$$
$$R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1=0,\quad R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_2=\beta\gamma(\widehat{e}_2+\widehat{e}_3),$$
$$R^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_3=-\beta\gamma(\widehat{e}_2+\widehat{e}_3). \tag{7.39}$$

By (2.5), we have

$$\rho^{B_2}(\widehat{e}_1,\widehat{e}_1)=\rho^{B_2}(\widehat{e}_1,\widehat{e}_2)=\rho^{B_2}(\widehat{e}_1,\widehat{e}_3)=0,$$
$$\rho^{B_2}(\widehat{e}_2,\widehat{e}_1)=0,\quad \rho^{B_2}(\widehat{e}_2,\widehat{e}_2)=-\beta\gamma,\quad \rho^{B_2}(\widehat{e}_2,\widehat{e}_3)=\beta\gamma,$$
$$\rho^{B_2}(\widehat{e}_3,\widehat{e}_1)=0,\quad \rho^{B_2}(\widehat{e}_3,\widehat{e}_2)=\beta\gamma,\quad \rho^{B_2}(\widehat{e}_3,\widehat{e}_3)=-\beta\gamma. \tag{7.40}$$

Then,

$$\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_1)=\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_2)=\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_3)=0,$$
$$\widetilde{\rho}^{B_2}(\widehat{e}_2,\widehat{e}_2)=-\beta\gamma,\quad \widetilde{\rho}^{B_2}(\widehat{e}_2,\widehat{e}_3)=\beta\gamma,\quad \widetilde{\rho}^{B_2}(\widehat{e}_3,\widehat{e}_3)=-\beta\gamma. \tag{7.41}$$

By (2.7), we have

$$(L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_1)=0,\quad (L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_2)=\mu_1\alpha-\mu_2\beta+\mu_3\beta,\quad (L_V^{B_2}g)(\widehat{e}_1,\widehat{e}_3)=\mu_2\beta-\mu_1\alpha-\mu_3\beta,$$
$$(L_V^{B_2}g)(\widehat{e}_2,\widehat{e}_2)=2\mu_3\delta,\quad (L_V^{B_2}g)(\widehat{e}_2,\widehat{e}_3)=-(\mu_2+\mu_3)\delta,\quad (L_V^{B_2}g)(\widehat{e}_3,\widehat{e}_3)=2\mu_2\delta. \tag{7.42}$$

Then, if (G_7,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2}, by (2.8), we have the following five equations:

$$\begin{cases}\mu=0\\ \mu_1\alpha-\mu_2\beta+\mu_3\beta=0\\ \mu_3\delta-\beta\gamma+\mu=0\\ (\mu_3+\mu_2)\delta=0\\ \mu_2\delta=0\end{cases} \tag{7.43}$$

By solving (7.43), we get

Theorem 7.21 *(G_7,g,V) is an affine Ricci soliton associated to the Bott connection ∇^{B_2} if and only if*

$$(1)\mu=\delta=\gamma=\mu_1\alpha-\mu_2\beta+\mu_3\beta=0,\quad \alpha\neq 0;$$
$$(2)\mu=\alpha=\beta\gamma=\mu_2=\mu_3=0,\quad \delta\neq 0;$$
$$(3)\mu=\gamma=\mu_1=\mu_2=\mu_3=0,\quad \alpha\neq 0,\quad \delta\neq 0,\quad \alpha+\delta\neq 0.$$

Specially, let $V=0$, we get the following corollary:

Corollary 7.22 *(I) (G_1,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=\beta=0$;*
(II) (G_2,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=\alpha=0$;
(III) (G_3,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=\alpha\beta=0$;
(IV) (G_4,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=\alpha=0$;
(V) (G_5,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=\delta=\gamma=0,\quad \alpha\neq 0$;
(VI) (G_6,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=0,\quad \alpha\gamma-\beta\delta=0,\quad \alpha+\delta\neq 0$;
(VII) (G_7,g,V) is an affine Einstein associated to the Bott connection ∇^{B_2} if and only if $\mu=\gamma=\alpha+2\delta=0,\quad \alpha+\delta\neq 0$.

8. Affine Ricci solitons associated to the perturbed Bott connection on three-dimensional Lorentzian Lie groups with the third distribution

Similarly, by the above calculations, we always obtain $\mu = 0$. In order to get the affine Ricci soliton with nonzero μ, we introduce the perturbed Bott connection $\widetilde{\nabla}^{B_2}$ in the following. Let $\widehat{e}_1^*$ be the dual base of e_1. We define on $G_{i=1,\cdots,7}$

$$\widetilde{\nabla}^{B_2}_X Y = \nabla^{B_2}_X Y + a_0 \widehat{e}_1^*(X)\widehat{e}_1^*(Y)e_1, \tag{8.1}$$

where a_0 is a nonzero real number. Then

$$\widetilde{\nabla}^{B_2}_{\widehat{e}_1}\widehat{e}_1 = a_0\widehat{e}_1, \quad \widetilde{\nabla}^{B_2}_{\widehat{e}_s}\widehat{e}_t = \nabla^{B_2}_{\widehat{e}_s}\widehat{e}_t, \tag{8.2}$$

where s and t does not equal 1. We define

$$(\widetilde{L}^{B_2}_V g)(X,Y) := g(\widetilde{\nabla}^{B_2}_X V, Y) + g(X, \widetilde{\nabla}^{B_2}_Y V), \tag{8.3}$$

for vector fields X, Y, V. Then we have for $G_{i=1,\cdots,7}$

$$(\widetilde{L}^{B_2}_V g)(\widehat{e}_1,\widehat{e}_1) = 2a_0\mu_1, \quad (\widetilde{L}^{B_2}_V g)(\widehat{e}_s,\widehat{e}_t) = (L^{B_2}_V g)(\widehat{e}_s,\widehat{e}_t), \tag{8.4}$$

where s and t does not equal 1.

Definition 8.1 *(G_i, V, g) is called the affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$ if it satisfies*

$$(\widetilde{L}^{B_2}_V g)(X,Y) + 2\widetilde{\rho}^{B_2}(X,Y) + 2\mu g(X,Y) = 0. \tag{8.5}$$

For $(G_1, \widetilde{\nabla}^{B_2})$, we have

$$\begin{aligned} &\widetilde{R}^{B_2}(\widehat{e}_1,\widehat{e}_2)\widehat{e}_1 = \alpha(\beta - a_0)\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_1,\widehat{e}_3)\widehat{e}_1 = \alpha(a_0 - \beta)\widehat{e}_1, \\ &\widetilde{R}^{B_2}(\widehat{e}_2,\widehat{e}_3)\widehat{e}_1 = -a_0\beta\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s,\widehat{e}_t)\widehat{e}_p, \end{aligned} \tag{8.6}$$

for $(s,t,p) \neq (1,2,1), (1,3,1), (2,3,1)$.

$$\widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_2) = \frac{\alpha(\beta - a_0)}{2}, \quad \widetilde{\rho}^{B_2}(\widehat{e}_1,\widehat{e}_3) = \frac{\alpha(a_0 - \beta)}{2}, \quad \widetilde{\rho}^{B_2}(\widehat{e}_s,\widehat{e}_t) = \widetilde{\rho}^{B_2}(\widehat{e}_s,\widehat{e}_t), \tag{8.7}$$

for the pair $(s,t) \neq (1,2), (1,3)$. If (G_1, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$, then by (8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \alpha\beta - \mu_1\alpha - \mu_3\beta - a_0\alpha = 0 \\ \mu_2\beta + \mu_1\alpha + \alpha(a_0 - \beta) = 0 \\ \mu_3\alpha - \beta^2 + \mu = 0 \\ \mu_3\alpha + \mu_2\alpha = 0 \\ \beta^2 + \mu_2\alpha - \mu = 0 \end{cases} \tag{8.8}$$

Solve (8.8), we get

Theorem 8.2 (G_1, V, g) *is an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_2}$ *if and only if*

$$(1)\alpha \neq 0, \quad \beta = 0, \quad \mu_1 = -a_0, \quad \mu_2 = -\mu_3 = \frac{a_0^2}{\alpha}, \quad \mu = a_0^2;$$

$$(2)\alpha \neq 0, \quad \beta \neq 0, \quad \mu_2 = \mu_3 = 0, \quad \mu = \beta^2, \quad \mu_1 a_0 + \beta^2 = 0, \quad \mu_1 + a_0 - \beta = 0.$$

For $(G_2, \widetilde{\nabla}^{B_2})$, we have

$$\widetilde{R}^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = -a_0\alpha\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{8.9}$$

for $(s, t, p) \neq (2, 3, 1)$.

$$\widetilde{\widetilde{\rho}}^{B_2}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_2}(\widehat{e}_s, \widehat{e}_t), \tag{8.10}$$

for any pairs (s, t). If (G_2, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$, then by (8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \mu_2\gamma - \mu_3\beta = 0 \\ \mu_2\beta + \mu_3\gamma = 0 \\ \alpha\beta - \mu = 0 \\ \alpha\gamma = 0 \end{cases} \tag{8.11}$$

Solve (8.11), we get

Theorem 8.3 (G_2, V, g) *is an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_2}$ *if and only if* $\mu = \alpha = \mu_1 = \mu_2 = \mu_3 = 0, \quad \gamma \neq 0$.

For $(G_3, \widetilde{\nabla}^{B_2})$, we have

$$\widetilde{R}^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = -a_0\alpha\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{8.12}$$

for $(s, t, p) \neq (2, 3, 1)$.

$$\widetilde{\widetilde{\rho}}^{B_2}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_2}(\widehat{e}_s, \widehat{e}_t), \tag{8.13}$$

for any pairs (s, t). If (G_3, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$, then by (8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \mu_3\beta = 0 \\ \mu_2\gamma = 0 \\ \alpha\gamma - \mu = 0 \\ \alpha\beta - \mu = 0 \end{cases} \tag{8.14}$$

Solve (8.14), we get

Theorem 8.4 (G_3, V, g) *is an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_2}$ *if and only if*

$$(1)\alpha = \mu = \mu_1 a_0 = \mu_2\gamma = \mu_3\beta = 0;$$
$$(2)\alpha \neq 0, \quad \beta = \gamma = \mu = \mu_1 a_0 = 0;$$
$$(3)\alpha \neq 0, \quad \beta = \gamma \neq 0, \quad \mu_2 = \mu_3 = 0, \quad \mu = \alpha\beta, \quad \mu_1 a_0 = -\alpha\beta.$$

For $(G_4, \widetilde{\nabla}^{B_2})$, we have

$$\widetilde{R}^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = -a_0\alpha\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{8.15}$$

for $(s, t, p) \neq (2, 3, 1)$.

$$\widetilde{\widetilde{\rho}}^{B_2}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_2}(\widehat{e}_s, \widehat{e}_t), \tag{8.16}$$

for any pairs (s, t). If (G_4, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$, then by (8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \mu_3\beta + \mu_2 = 0 \\ \mu_2(\beta - 2\eta) - \mu_3 = 0 \\ \alpha(2\eta - \beta) + \mu = 0 \\ \alpha = 0 \\ \alpha\beta - \mu = 0 \end{cases} \tag{8.17}$$

Solve (8.17), we get

Theorem 8.5 (G_4, V, g) *is an affine Ricci soliton associated to the connection* $\widetilde{\nabla}^{B_2}$ *if and only if*

$$(1)\alpha = \mu = \mu_1 = \mu_3 = \mu_2 = 0, \quad \beta \neq \eta;$$
$$(2)\alpha = \mu = \mu_1 = 0, \quad \mu_2 + \mu_3\eta = 0, \quad \beta = \eta.$$

For $(G_5, \widetilde{\nabla}^{B_2})$, we have

$$\widetilde{R}^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = -a_0\alpha\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = -a_0\gamma\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{8.18}$$

for $(s, t, p) \neq (1, 3, 1), (2, 3, 1)$.

$$\widetilde{\widetilde{\rho}}^{B_2}(\widehat{e}_2, \widehat{e}_1) = -\frac{a_0\alpha}{2}, \quad \widetilde{\widetilde{\rho}}^{B_2}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_2}(\widehat{e}_s, \widehat{e}_t), \tag{8.19}$$

for the pair $(s, t) \neq (2, 1)$. If (G_5, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$, then by

(8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \mu_3\beta = 0 \\ \mu_1\alpha + a_0\alpha = 0 \\ \mu_3\delta + \delta^2 + \mu = 0 \\ \mu_2\delta = 0 \\ \delta^2 + \beta\gamma + \mu = 0 \end{cases} \tag{8.20}$$

Solve (8.20), we get

Theorem 8.6 *(G_5, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$ if and only if $\beta = \alpha = \mu_2 = \mu_3 = 0, \quad \delta \neq 0, \quad \mu = -\delta^2, \quad \mu_1 = \frac{\delta^2}{a_0}$.*

For $(G_6, \widetilde{\nabla}^{B_2})$, we have

$$\widetilde{R}^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{8.21}$$

for any (s, t, p).

$$\widetilde{\widetilde{\rho}}^{B_2}(\widehat{e}_s, \widehat{e}_t) = \widetilde{\rho}^{B_2}(\widehat{e}_s, \widehat{e}_t), \tag{8.22}$$

for any pairs (s, t). If (G_6, g, V) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$, then by (8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \mu_2\alpha + \mu_3\gamma = 0 \\ \mu_2\beta + \mu_3\delta = 0 \\ \mu = 0 \end{cases} \tag{8.23}$$

Solve (8.23), we get

Theorem 8.7 *(G_6, V, g) is an affine Ricci soliton associated to the connection $\widetilde{\nabla}^{B_2}$ if and only if*

$$(1)\mu = \mu_1 = \mu_2 = \mu_3 = 0, \quad \alpha\gamma - \beta\delta = 0, \quad \alpha + \delta \neq 0;$$
$$(2)\mu = \mu_1 = 0, \quad \alpha = \beta, \quad \delta = \gamma, \quad \mu_2 \neq 0, \quad \mu_3 \neq 0;$$
$$(3)\mu = \mu_1 = \alpha + \beta = \delta + \gamma = 0, \quad \mu_2 \neq 0, \quad \mu_3 \neq 0.$$

For $(G_7, \widetilde{\nabla}^{B_2})$, we have

$$\widetilde{R}^{B_2}(\widehat{e}_1, \widehat{e}_2)\widehat{e}_1 = a_0\alpha\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_1, \widehat{e}_3)\widehat{e}_1 = -a_0\alpha\widehat{e}_1,$$
$$\widetilde{R}^{B_2}(\widehat{e}_2, \widehat{e}_3)\widehat{e}_1 = a_0\gamma\widehat{e}_1, \quad \widetilde{R}^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p = R^{B_2}(\widehat{e}_s, \widehat{e}_t)\widehat{e}_p, \tag{8.24}$$

for $(s,t,p) \neq (1,2,1),(1,3,1),(2,3,1)$.

$$\tilde{\tilde{\rho}}^{B_2}(\widehat{e}_1,\widehat{e}_2) = \frac{a_0\alpha}{2}, \quad \tilde{\tilde{\rho}}^{B_2}(\widehat{e}_2,\widehat{e}_3) = -\frac{a_0\alpha}{2}, \quad \tilde{\tilde{\rho}}^{B_2}(\widehat{e}_s,\widehat{e}_t) = \tilde{\rho}^{B_2}(\widehat{e}_s,\widehat{e}_t), \tag{8.25}$$

for the pair $(s,t) \neq (1,2),(2,3)$. If (G_7,g,V) is an affine Ricci soliton associated to the connection $\tilde{\tilde{\nabla}}^{B_2}$, then by (8.5), we have

$$\begin{cases} a_0\mu_1 + \mu = 0 \\ \mu_2\beta - \mu_1\alpha - \mu_3\beta - a_0\alpha = 0 \\ \mu_3\delta - \beta\gamma + \mu = 0 \\ (\mu_2+\mu_3)\delta - \beta\gamma = 0 \\ \mu_2\delta - \beta\gamma - \mu = 0 \end{cases} \tag{8.26}$$

Solve (8.26), we get

Theorem 8.8 *(G_7,V,g) is an affine Ricci soliton associated to the connection $\tilde{\tilde{\nabla}}^{B_2}$ if and only if*

(1)$\alpha = \beta = \mu_2 + \mu_3 = 0, \quad \delta \neq 0, \quad \mu = -\mu_3\delta, \quad \mu_1 = \frac{\mu_3\delta}{a_0};$

(2)$\alpha = \gamma = \mu = \mu_1 = \mu_2 = \mu_3 = 0, \quad \beta \neq 0;$

(3)$\alpha \neq 0, \quad \delta = \gamma = \mu = \mu_1 = 0, \quad (\mu_2+\mu_3)\beta - a_0\alpha = 0;$

(4)$\alpha \neq 0, \quad \delta \neq 0, \quad \gamma = 0, \quad \mu_1 = -\frac{\mu}{a_0}, \quad \mu_2 = \frac{\mu}{\delta}, \quad \mu_3 = -\frac{\mu}{\delta}, \quad 2a_0\beta\mu + \mu\alpha\delta - a_0^2\alpha\delta = 0.$

9. Conclusion

Firstly, the distributions that we take are $D = span\{\widehat{e}_1,\widehat{e}_2\}$, $D_1 = span\{\widehat{e}_1,\widehat{e}_3\}$, $D_2 = span\{\widehat{e}_2,\widehat{e}_3\}$. In fact, we can also take other distributions, but $D = span\{\widehat{e}_1,\widehat{e}_2\}$, $D_1 = span\{\widehat{e}_1,\widehat{e}_3\}$, $D_2 = span\{\widehat{e}_2,\widehat{e}_3\}$ are the simplest distributions on three-dimensional Lie groups $\{G_i\}_{i=1,\cdots,7}$. In addition, the distribution $D = span\{\widehat{e}_1,\widehat{e}_2\}$ is used in [19] and [2]. In the paper, we get affine Ricci solitons associated to the Bott connection on three-dimensional Lorentzian unimodular and nonunimodular Lie groups, and the results are all $\mu = 0$. To get the results of affine Ricci solitons with $\mu \neq 0$, we use perturbed Bott connections in (4.1), and we get affine Ricci solitons with $\mu \neq 0$ by this kind of perturbed Bott connection on three-dimensional Lie groups. However, for some other perturbed Bott connections, we cannot get affine Ricci solitons with $\mu \neq 0$.

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