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A classification of 1-well-covered graphs

Zakir DENİZ*
Department of Mathematics, Düzce University, Düzce, Turkey,

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Abstract: A graph is well-covered if all its maximal independent sets have the same size. If a graph is well-covered and remains well-covered upon removal of any vertex, then it is called 1-well-covered graph. It is well-known that $\lfloor \frac{n}{2} \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$ for any graph G with n vertices where $\alpha(G)$ and $\mu(G)$ are the independence and matching numbers of G, respectively. A graph G satisfying $\alpha(G) + \mu(G) = n$ is known as König-Egerváry graph, and such graphs are characterized by Levit and Mandrescu [14] under the assumption that G is 1-well-covered. In this paper, we investigate connected 1-well-covered graphs with respect to $\alpha(G) + \mu(G) = n - k$ for $k \geq 1$ and $|G| = n$. We further present some combinatorial properties of such graphs. In particular, we provide a tight upper bound on the size of those graphs in terms of k, namely $|G| \leq 10k - 2$, also we show that $\Delta(G) \leq 2k + 1$ and $\alpha(G) \leq \min\{4k - 1, n - 2k\}$. This particularly enables us to obtain a characterization of such graphs for $k = 1$, which settles a problem of Levit and Mandrescu [14].

Key words: Independent set, matching, well-covered

1. Introduction

A set of vertices in a graph is independent if no two vertices in the set are adjacent. An independent set of maximum cardinality is called a maximum independent set. The size of a maximum independent set in a graph G is called the independence number denoted by $\alpha(G)$. If every maximal independent set of vertices has the same cardinality, then the graph is called *well-covered*. These graphs have been introduced by Plummer in [22] and many kinds of research have been done related to them. In general, recognizing well-covered graphs is a co-NP-complete problem ([6, 24]) although some subclasses can be recognized in polynomial time such as well-covered line graphs [7], very well-covered graphs [9], well-covered claw-free graphs [26] and well-covered graphs that are 3-regular [5].

Well-covered graphs play a considerable role in graph theory as well as commutative algebra. For instance, let $R = \Bbbk[x_1, x_2, \ldots, x_n]$ be a polynomial ring of n variables over the field $\Bbbk$ (see [27] for terminology). Given a simple graph G on the vertex set $x_1, x_2, \ldots, x_n$. The edge ideal of G is defined by $I(G) = (x_i x_j \ : \ x_i x_j \in E(G)) \subseteq R$. We say that G is Cohen–Macaulay (resp. Gorenstein) graph over $\Bbbk$ if $I(G)$ is Cohen–Macaulay (resp. Gorenstein). It is known that every Cohen-Macaulay graph is well-covered, while a Gorenstein graph without isolated vertices is not only well-covered but also remains well-covered upon removal of any vertex, which are so-called $\mathbf{W}_2$ graphs (see [25]). Hoang and Trung proved in [11] that a triangle-free graph G is

*Correspondence: zakirdeniz@duzce.edu.tr
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Gorenstein if and only if every nontrivial connected component of G belongs to $\mathbf{W}_2$. In a recent work, Oboudi and Nikseresht [16] gave a characterization of Gorenstein graphs with $\alpha(G) = 2$ as well as a characterization of all triangle-free Gorenstein graphs with $\alpha(G) = 3$.

In 1979, Staples introduced the class of $\mathbf{W}_2$ graphs where a graph G belongs to the class $\mathbf{W}_2$ if any two disjoint independent sets are contained in two disjoint maximum independent sets [25]. Those graphs are also known as 1-well-covered graphs without isolated vertices which are well-covered graphs that remain well-covered after removal of any vertex. So, a graph G is in $\mathbf{W}_2$ if and only if G is 1-well-covered and has no isolated vertices. After the study of some basic properties of 1-well-covered graphs in [25], several papers focused on subclasses of 1-well-covered graphs [8, 10, 14, 18, 20, 21].

We recall that a vertex x of a graph G is said to be a shedding vertex if no independent set in $G - N_G[x]$ is maximal in G. The notion of the shedding vertex goes back to the work of Provan and Bilera [23], and plays a prominent role on defining the class of vertex-decomposable graphs [2]. Furthermore, this notion allows us to provide another characterization of $\mathbf{W}_2$ graphs, namely a graph G belongs to $\mathbf{W}_2$ if and only if every vertex of it is a shedding vertex. Indeed, Levit and Mandrescu showed [14] that a vertex v in a well-covered graph G without isolated vertices is shedding if and only if $G - v$ is well-covered.

A matching in a graph is a subset of edges no two of which share a vertex. The maximum cardinality of a matching of G is called the matching number of G and denoted by $\mu(G)$. It is well known that $\lfloor \frac{n}{2} \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$ for any graph G with n vertices (see [13]). In particular, the graphs satisfying $\alpha(G) + \mu(G) = n$ are defined as König-Egerváry graphs, and these graphs have been extensively studied in [3, 4, 12, 17]. In a recent work, Levit and Mandrescu proved in [15] that if G is an almost bipartite graph, which is the graph having only one odd cycle, with n vertices then $n - 1 \leq \alpha(G) + \mu(G) \leq n$. For the graphs in $\mathbf{W}_2$, the case $\alpha(G) + \mu(G) = n$ clearly corresponds to bipartite graphs. Levit and Mandrescu [14] proved that K_2 is the unique graph in $\mathbf{W}_2$ satisfying this equality, and in particular, they posed the problem of finding all graphs satisfying $\alpha(G) + \mu(G) = n - 1$. Such an observation naturally raises a simple question: What $\mathbf{W}_2$ graphs belong to the class consisting of the graphs satisfying $\alpha(G) + \mu(G) = n - k$ for a fixed $k \geq 1$? This brings a new idea to find the structural properties of $\mathbf{W}_2$ graphs on regarding to their independence and matching numbers, since we intend to classify $\mathbf{W}_2$ graphs in respect to $\alpha(G) + \mu(G) = n - k$.

Remark 1.1. *For each $n \geq 2$, $G = K_n$ is in $\mathbf{W}_2$, and $\alpha(G) + \mu(G) = n - \left(\lceil \frac{n}{2} \rceil - 1\right)$.*

Following above remark, we can say that, for each $k \geq 0$, there exists a graph G in $\mathbf{W}_2$ with n vertices such that $\alpha(G) + \mu(G) = n - k$. Due to $0 \leq k \leq \lceil \frac{n}{2} \rceil - 1$, we have $\lceil \frac{n}{2} \rceil$ nonempty subclasses of $\mathbf{W}_2$ graphs with n vertices. The case $k = 0$ clearly corresponds to the class of bipartite graphs in $\mathbf{W}_2$, and $G = K_2$ is the unique graph in $\mathbf{W}_2$ satisfying the equality $\alpha(G) + \mu(G) = n$ by [14]. So, for each $k \geq 1$, any graph $G \in \mathbf{W}_2$ corresponds to a nonbipartite graph. On the other hand, it is known that C_3 and C_5 are only cycles in $\mathbf{W}_2$, and those satisfy $\alpha(G) + \mu(G) = n - 1$.

In this work, we investigate $\mathbf{W}_2$ graphs in respect to $\alpha(G) + \mu(G) = n - k$ for $k \geq 1$ and $|G| = n$. First, some combinatorial properties of the graphs in $\mathbf{W}_2$ are presented. Particularly, we give a characterization of the graphs in $\mathbf{W}_2$ with independence number 2, and we show that for every $k \geq 1$, there exists a connected graph G in $\mathbf{W}_2$ with independence number 2 such that $\alpha(G) + \mu(G) = n - k$.

For a connected graph G in $\mathbf{W}_2$ with $\alpha(G) + \mu(G) = n - k$ for $k \geq 1$ and $|G| = n$, we provide a tight upper bound on the size of G in terms of k as well as the maximum degree of G for $k \geq 0$.

Theorem 1.2. *If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k \geq 0$, then $\Delta(G) \leq 2k+1$.*

Theorem 1.3. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$. Then $n \leq 10k-2$ and $\alpha(G) \leq 4k-1$.*

These results allows us to obtain a characterization of $\mathbf{W}_2$ graphs satisfying $\alpha(G)+\mu(G)=|G|-1$, which settles a question of Levit and Mandrescu in [14].

Theorem 1.4. *A connected graph G with $\alpha(G)+\mu(G)=|G|-1$ belongs to $\mathbf{W}_2$ if and only if G is C_3 or C_5 or K_4 or one of those five graphs in Figure 1.*

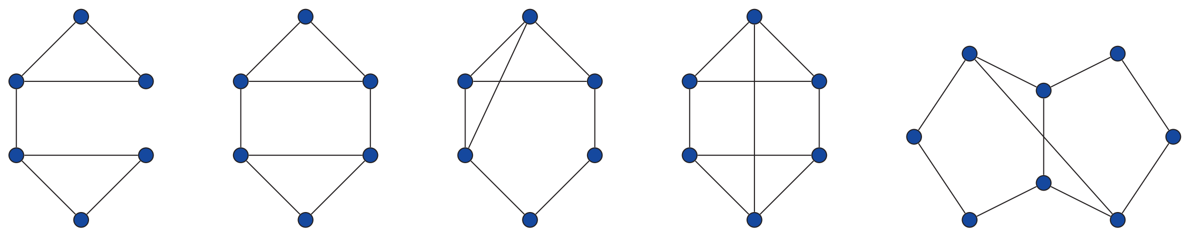

Figure 1. The connected $\mathbf{W}_2$ graphs satisfying $\alpha(G)+\mu(G)=|G|-1$ for $|G| \geq 6$.

On the other hand, we address the $\mathbf{W}_2$ graphs satisfying $\alpha(G)+\mu(G)=n-k$ for large k and show that only complete graphs satisfy $\alpha(G)+\mu(G)=\lfloor \frac{n}{2} \rfloor + 1$. In particular, we bound the independence number of $\mathbf{W}_2$ graphs satisfying $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$ in terms of k.

Theorem 1.5. *If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$, then $\alpha(G) \leq \min\{4k-1, n-2k\}$.*

Our paper is organized as follows. We start in Section 2 with some definitions and preliminary results on 1-well-covered graphs. In Section 3, we present some structural properties on those graphs. Section 4 is devoted to $\mathbf{W}_2$ graphs under the parameter $\alpha(G)+\mu(G)$ in which we study the combinatorial properties of those graphs with $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$. We finish the paper with Section 5 in which we discuss the results that we obtain.

2. Preliminaries

All graphs in this paper are assumed to be simple i.e. finite and undirected, with no loops or multiple edges. We refer to [28] for terminology and notation not defined here. Given a graph $G=(V,E)$ and a subset of vertices S, $G[S]$ denotes the subgraph of G induced by S, and $G-S=G[V-S]$. We denote $G-S$ by $G-v$ when S consists of a single vertex v. For a vertex v, the open neighbourhood of v in a subgraph H is denoted by $N_H(v)$ while the closed neighbourhood of v is $N_H(v) \cup \{v\}$, denoted by $N_H[v]$. We omit the subscript H whenever there is no ambiguity on H. For a subset $S \subseteq V$, $N_H(S)$ (resp. $N_H[S]$) is the union of the open (resp. closed) neighbourhoods of the vertices in S. We say that S is complete to T for $S,T \subset V(G)$ if all vertices of S are adjacent to all vertices of T. We use the notation $[k]$ to denote the set of integers $1,2,\ldots,k$.

We denote by K_n, C_n and P_n, the complete graph, the cycle, and the path on n vertices, respectively. Also, we denote by $K_{r,s}$, the complete bipartite for any $r,s \geq 1$. In addition, rK_2 corresponds to the graph consisting of r copies of K_2.

A subset $S \subset V(G)$ is called a clique of G if $G[S]$ is isomorphic to a complete graph. We say that G is F-free if no induced subgraph of G is isomorphic to F. The degree of a vertex x, the maximum and the minimum degrees of a graph G are denoted by $d_G(x)$, $\Delta(G)$ and $\delta(G)$, respectively. A graph is called k-regular if every vertex in the graph is of degree k. A leaf is a vertex with degree one while an isolated vertex is a vertex with degree zero.

A matching M saturates a vertex v if it is the endvertex of an edge in M, otherwise M leaves the vertex v unsaturated. A vertex u of a graph G is said to be dominated by a vertex $v \in V(G) - u$ if $N_G[u] \subseteq N_G[v]$. A subset $S \subseteq V(G)$ dominates a set of vertices T if every vertex in T is adjacent to at least one vertex of S.

We start with some known results concerning well-covered graphs, which we shall use in the rest of the paper.

Theorem 2.1. *[1] In a graph G, an independent set S is maximum if and only if every independent set disjoint from S can be matched into S.*

Since a graph $G \in \mathbf{W}_2$ has two disjoint maximum independent sets I_1, I_2, we have a perfect matching on the graph $G[I_1 \cup I_2]$ by Theorem 2.1. Therefore, we immediately have the following.

Corollary 2.2. *If G is a graph belonging to $\mathbf{W}_2$, then $\mu(G) \geq \alpha(G)$.*

The following implies that $\mathbf{W}_2$ graphs and 1-well-covered graphs without isolated vertices coincide.

Theorem 2.3. *[25] A graph G is in $\mathbf{W}_2$ if and only if $\alpha(G - v) = \alpha(G)$ and $G - v$ is well-covered, for every $v \in V(G)$.*

Proposition 2.4. *[14]*

(i) If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G) + \mu(G) = n$, then $G = K_2$

(ii) K_2 is the only connected bipartite graph belonging to $\mathbf{W}_2$.

The following lemma follows directly from the definition of $\mathbf{W}_2$ graphs, and we inductively apply it in our several proofs.

Lemma 2.5. *[19] If G is in $\mathbf{W}_2$, then for every independent set S in G, the graph $G - N_G[S]$ is in the class $\mathbf{W}_2$ as well. In particular, $\alpha(G) = \alpha(G - N_G[S]) + |S|$.*

Theorem 2.6. *[14] Let v be a nonisolated vertex of a well-covered graph G. Then $G - v$ is well-covered if and only if v is a shedding vertex.*

It simply follows from Theorem 2.6 that every vertex of a graph $G \in \mathbf{W}_2$ is a shedding vertex.

If v is a shedding vertex in a graph G, then, by definition, there is no independent set S in $G - N_G[v]$ which dominates $N_G(v)$. This particularly implies that $\mathbf{W}_2$ graphs have no dominated vertex.

Corollary 2.7. *For a connected graph G of size at least 3, any shedding vertex cannot be leaf vertex of G. In particular, if $G \in \mathbf{W}_2$, then $\delta(G) \geq 2$.*

3. 1-well-covered graphs

A graph G is 1-well-covered if both G and $G-v$ are well-covered for each $v \in V(G)$. As we have already noticed, 1-well-covered graphs without isolated vertices are equivalent to $\mathbf{W}_2$ graphs, and so connected 1-well-covered graphs correspond to connected $\mathbf{W}_2$ graphs. We therefore use the $\mathbf{W}_2$ notation instead of referring to connected 1-well-covered graphs in the remainder of this paper.

On the other hand, a graph G is in $\mathbf{W}_2$ if every two disjoint independent sets of G are included in two disjoint maximum independent sets. Therefore, any pair of disjoint independent sets in a graph $G \in \mathbf{W}_2$ can be extended to two disjoint maximum independent sets in G. We often use this property of $\mathbf{W}_2$ graphs in order to show that a graph belongs to the class $\mathbf{W}_2$.

Obviously, if a disconnected graph is in $\mathbf{W}_2$, then its every connected component is in $\mathbf{W}_2$ as well. Here, we show that the removal of a cut vertex does not change it being in $\mathbf{W}_2$.

Proposition 3.1. *If v is a cut vertex of a graph G in $\mathbf{W}_2$, then each component of $G-v$ is in $\mathbf{W}_2$.*

Proof. Let G be a graph in $\mathbf{W}_2$ with a cut vertex v, and let $H_1, H_2, \ldots, H_k$ be the components of $G-v$. Since $G-v$ is well-covered with $\alpha(G) = \alpha(G-v)$, each component H_i of $G-v$ is a well-covered graph. This implies that there exists a maximum independent set in G which exposes v. Thus, $\alpha(G) = \alpha(H_1) + \alpha(H_2) + \ldots + \alpha(H_k)$.

Now, we claim that the removal of any vertex from H_i does not change its well-covered property. Indeed, assume by contradiction that there exists a vertex $u \in V(H_i)$ such that $H_i - u$ is not well-covered. Then there exist two maximal independent sets I_1, I_2 in $H_i - u$ with $|I_1| < |I_2|$. Let $w \in H_j \cap N_G(v)$ with $j \neq i$. We extend $I_1 \cup \{w\}$ and $I_2 \cup \{w\}$ to maximal independent sets S_1 and S_2 in $G-u$, respectively. Obviously, $v \notin S_1 \cup S_2$. Since $G-u$ is well-covered, the sets S_1, S_2 have to be maximum independent sets with $|S_1| = |S_2| = \alpha(G)$. However, this implies that there exists a component H_ℓ for $\ell \in [k] \setminus \{i,j\}$ such that $|H_\ell \cap S_1| > |H_\ell \cap S_2|$ since $|I_1| < |I_2|$ and $v \notin S_1 \cup S_2$. This contradicts that H_ℓ is a well-covered graph. $\square$

Notice that there are exactly two graphs of size at most 5 in the class $\mathbf{W}_2$ with independence number 2, and those are $2K_2$ and C_5. For the graphs having more vertices, we have the following result.

Lemma 3.2. *Let G be a connected graph in $\mathbf{W}_2$ with $\alpha(G) = 2$. Then G has a matching that leaves at most one unsaturated vertex.*

Proof. Let $|G| = n$. Obviously, G has a matching saturating at least $n-2$ vertices since $\alpha(G) = 2$. Therefore we only need to prove that G has a perfect matching when $n = 2r$ for some $r \geq 3$. Let $I_1 = \{x_1, x_2\}$ and $I_2 = \{y_1, y_2\}$ be two disjoint maximum independent sets in G with $x_1y_1, x_2y_2 \in E(G)$. We set $S = V(G) - (I_1 \cup I_2)$. Notice that if S induces a clique in G, then we add the edges x_1y_1, x_2y_2 into any perfect matching of $G[S]$ (there exists such a matching since $|G|$ is even), and so the claim follows. Thus, we may assume that S has a pair of non-adjacent vertices u, v. Let M be a perfect matching on $S - \{u,v\}$, indeed there exists such a matching M since $\alpha(G) = 2$. By the same reason, every vertex of $I_1 \cup I_2$ is adjacent to u or v. Moreover, if there exists $i \in \{1,2\}$ such that $u, v \in N(\{x_i, y_i\})$, then G clearly has a perfect matching $M' = M \cup \{ux_i, vy_i, x_{3-i}y_{3-i}\}$ or $M' = M \cup \{vx_i, uy_i, x_{3-i}y_{3-i}\}$. We may therefore assume that u (resp. v) is adjacent to only x_1 and y_1 (resp. x_2 and y_2) on $I_1 \cup I_2$. On the other hand, if there exists $x_iy_j \in E(G)$ for some $i, j \in \{1,2\}$ with $i \neq j$, say $x_1y_2 \in E(G)$, then G has a perfect matching $M' = M \cup \{uy_1, vx_2, x_1y_2\}$. Thus, we assume that $I_1 \cup I_2$ induces $2K_2$ in G, and so $I_1 \cup I_2 \cup \{u,v\}$ induces $2C_3$ in G. Since G is connected, we have $S-\{u,v\}\neq\emptyset$. It then follows that u and v are adjacent to some vertices of $S-\{u,v\}$. Since $\alpha(G)=2$, we have $N_G[\{u,v\}]=V(G)$. In addition, both u and v must be complete to $S-\{u,v\}$ since $\alpha(G)=2$ and $I_1\cup I_2\cup\{u,v\}$ induces $2C_3$ in G. Hence, G has a perfect matching $M'=(M\setminus\{s_1s_2\})\cup\{x_1y_1,x_2y_2,us_1,vs_2\}$ for some $s_1s_2\in M$. □

In [14], the authors posed the problem of characterizing connected $\mathbf{W}_2$ graphs with independence number 2. In what follows, we show that the complements of such graphs correspond to a subclass of triangle-free graphs.

Theorem 3.3. *A graph G is a connected $\mathbf{W}_2$ graph with $\alpha(G)=2$ if and only if $\overline{G}$ is a triangle-free graph with $\delta(\overline{G})\geq 2$ and $\overline{G}\neq K_{r,s}$ for $r,s\geq 1$.*

Proof. Suppose that G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)=2$. Clearly, $\overline{G}$ has no triangle. Since G is in $\mathbf{W}_2$, the graph $G-v$ is well-covered and $\alpha(G)=\alpha(G-v)=2$ for each $v\in V(G)$. This implies that every vertex of G is adjacent to at most $n-3$ vertices in G. Thus, $\delta(\overline{G})\geq 2$. Moreover, $\overline{G}\neq K_{r,s}$ for $r,s\geq 1$, since G is connected.

Conversely, given a graph G and we assume that $\overline{G}$ is a triangle-free graph with $\delta(\overline{G})\geq 2$ and $\overline{G}\neq K_{r,s}$ for $r,s\geq 1$. Clearly, G has independence number 2. Also, G is connected since $\overline{G}\neq K_{r,s}$ for $r,s\geq 1$. Since $\delta(\overline{G})\geq 2$, the graph G has no vertex of degree at least $n-2$, and so $\Delta(G)\leq n-3$. We therefore deduce that G is well-covered since $\alpha(G)=2$. Let $u\in V(G)$ be given. It remains to show that $G-u$ remains well-covered. Since $\Delta(G)\leq n-3$, for each $v\in G-u$, there exists a vertex $w\in G-u$ with $v\neq w$ such that v and w are nonadjacent. Thus, $G-u$ is well-covered, and so G is in $\mathbf{W}_2$. □

4. 1-well-covered graphs with parameter $\alpha+\mu$

Recall that for each $k\geq 0$, there exists a graph G in $\mathbf{W}_2$ with n vertices such that $\alpha(G)+\mu(G)=n-k$ by Remark 1.1. Therefore, we have $\lceil\frac{n}{2}\rceil$ nonempty subclasses of $\mathbf{W}_2$ graphs due to $0\leq k\leq\lceil\frac{n}{2}\rceil-1$. Actually, this statement holds even for graphs in $\mathbf{W}_2$ with independence number 2. In order to show this claim, we consider the complement of a cycle C_n for $n\geq 5$. Clearly, the graph $G=\overline{C_n}$ has independence number 2, also it is in $\mathbf{W}_2$ by Theorem 3.3. In addition, we can easily compute the matching number of G so that $\mu(G)=\lfloor\frac{n}{2}\rfloor$. It follows that

$$\alpha(G)+\mu(G)=n-\left(\left\lceil\frac{n}{2}\right\rceil-2\right)$$

We then immediately have the following.

Corollary 4.1. *For every $k\geq 1$, there exists a connected graph G in $\mathbf{W}_2$ with $\alpha(G)=2$ such that $\alpha(G)+\mu(G)=|G|-k$.*

It was shown in [14] that if a disconnected graph G with n vertices is in $\mathbf{W}_2$, and $\alpha(G)+\mu(G)=n-1$, then all its components but one are K_2. Here, we extend this statement to all $k\geq 0$ for which $\alpha(G)+\mu(G)=n-k$.

Proposition 4.2. *If G is a disconnected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 0$, then all components of G but k of them are K_2.*

Proof. Suppose that G is a disconnected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 0$. Let m be the number of K_2 components in G. Assume by contradiction that G has more than k components which are different from K_2. Let $C_1, C_2, \ldots, C_r$ be those components for $r\geq k+1$. Then $|C_1|+|C_2|+\ldots+|C_r|=n-2m$. Note that $H=K_2$ is the unique connected graph satisfying $\alpha(H)+\mu(H)=|H|$ by Proposition 2.4. Then for each C_i, we have $\alpha(C_i)+\mu(C_i)=|C_i|-k_i$ for $k_i\geq 1$, and so

$$\alpha(G)+\mu(G)=\sum_{i=1}^{r}\alpha(C_i)+\sum_{i=1}^{r}\mu(C_i)+2m=\sum_{i=1}^{r}|C_i|+2m-\sum_{i=1}^{r}k_i=n-\sum_{i=1}^{r}k_i$$

where it is clear that $k\neq\sum_{i=1}^{r}k_i$ since $k_i\geq 1$ and $r\geq k+1$, a contradiction. □

Every graph G belonging to $\mathbf{W}_2$ has a partition into three sets I_1, I_2, S where I_1 and I_2 are two disjoint independent sets in G and $S=V(G)-(I_1\cup I_2)$. Since every maximum independent set of G has the same size due to $G\in\mathbf{W}_2$, the size of S is constant for the graph G.

Proposition 4.3. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 0$. Then*

$$2\alpha(G)+k\leq n\leq 2\alpha(G)+2k$$

Proof. Assume that I_1 and I_2 are two disjoint maximum independent sets in G and let $S=V(G)-(I_1\cup I_2)$. Since $\alpha(G)+\mu(G)=n-k$ and $\mu(G)\geq\alpha(G)$ by Corollary 2.2, we deduce $k\leq|S|\leq 2k$. Since $|S|=n-2\alpha(G)$, it follows that $2\alpha(G)+k\leq n\leq 2\alpha(G)+2k$. □

The following corollaries can be easily obtained from Proposition 4.3.

Corollary 4.4. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 0$. Suppose $S=V(G)-(I_1\cup I_2)$ for disjoint maximum independent sets I_1 and I_2. Then $k\leq|S|\leq 2k$.*

Corollary 4.5. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 0$. Suppose $S=V(G)-(I_1\cup I_2)$ for disjoint maximum independent sets I_1 and I_2. If $|S|=k+p$ for $0\leq p\leq k$, then G has a matching of size $\alpha(G)+p$. In particular, $\mu(G)=\alpha(G)+p$.*

In the case where the set $S=V(G)-(I_1\cup I_2)$ induces a clique in a graph $G\in\mathbf{W}_2$, we can strengthen the bounds in Corollary 4.4.

Proposition 4.6. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 1$. If $S=V(G)-(I_1\cup I_2)$ induces a clique in G for disjoint maximum independent sets I_1 and I_2, then $|S|$ is equal to $2k-1$ or $2k$.*

Proof. Suppose that $S=V(G)-(I_1\cup I_2)$ induces a clique of size $|S|=p$ in G for disjoint maximum independent sets I_1 and I_2. Then $G[S]$ has a matching of size $\lfloor\frac{p}{2}\rfloor$. Let $\alpha(G)=r$, and so $n=2r+p$. It follows that $\mu(G)=r+\lfloor\frac{p}{2}\rfloor$, and thus we have $\alpha(G)+\mu(G)=r+r+\lfloor\frac{p}{2}\rfloor=n-p+\lfloor\frac{p}{2}\rfloor=n-\lceil\frac{p}{2}\rceil$. Consequently, $p=|S|$ is equal to $2k-1$ or $2k$. □

We now show that the maximum degree of a connected graph G in $\mathbf{W}_2$ for which $\alpha(G)+\mu(G)=|G|-k$ is bounded in terms of k. For the sake of convenience we restate it here.

Theorem 1.2. *If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k\geq 0$, then $\Delta(G)\leq 2k+1$.*

Proof. Let I_1 and I_2 be two disjoint maximum independent sets in G with $\alpha(G)=r$ and $S=V(G)-(I_1\cup I_2)$. Assume for a contradiction that there exists a vertex $v\in V(G)$ with $d_G(v)\geq 2k+2$. Note that the graph $G-N_G[v]$ is in $\mathbf{W}_2$ by Lemma 2.5, and $\alpha(G-N_G[v])=r-1$. Also, we have $|S|\leq 2k$ by Corollary 4.4. It then follows that $|V(G)-N_G[v]|=n-|N_G[v]|\leq 2r+2k-(2k+3)\leq 2r-3$. However, this contradicts Proposition 4.3, since $\alpha(G-N_G[v])=r-1$ and $G-N_G[v]\in\mathbf{W}_2$. □

The provided bound in Theorem 1.2 is best possible since complete graphs attain the bound for each $k\geq 0$ by Remark 1.1.

By [14, Corollary 2.12], the connected graphs in $\mathbf{W}_2$ of order $2\alpha(G)+1$ are only C_3 and C_5. We then deduce the following.

Corollary 4.7. *Let G be a connected graph in $\mathbf{W}_2$. Then $G-w$ is a bipartite well-covered graph for some $w\in V(G)$ if and only if G is C_3 or C_5.*

Let $G\in\mathbf{W}_2$ be given with two disjoint maximum independent sets I_1 and I_2 and $S=V(G)-(I_1\cup I_2)$. Our next aim is to determine the number of neighbours of S in $I_1\cup I_2$ so that we bound the size of G in term of $|S|$. Eventually, it turns out that $|V(G)|\leq 5|S|-2$ (see Theorem 4.13).

Proposition 4.8. *Let $G\in\mathbf{W}_2$, and let $S=V(G)-(I_1\cup I_2)$ for disjoint maximum independent sets I_1 and I_2. Then every independent set $T\subseteq S$ has at least $|T|$ neighbours in I_i for each $i\in\{1,2\}$.*

Proof. If there exists an independent set $T\subseteq S$ such that T has less than $|T|$ neighbours in I_i for some $i\in\{1,2\}$, then $T\cup(I_i-N_G(T))$ would be an independent set of size at least $|I_i|+1=\alpha(G)+1$, a contradiction. □

Remark 4.9. *Let $G\in\mathbf{W}_2$, and let $S=V(G)-(I_1\cup I_2)$ for disjoint maximum independent sets I_1 and I_2. Then every vertex in S has a neighbour in each of I_1, I_2.*

We next give a useful lemma for the number of neighbours of a maximal independent set in $S=V(G)-(I_1\cup I_2)$ where I_1 and I_2 are disjoint maximum independent sets in the graph $G\in\mathbf{W}_2$.

Lemma 4.10. *Let $G\in\mathbf{W}_2$, and let $S=V(G)-(I_1\cup I_2)$ for disjoint maximum independent sets I_1 and I_2. Then every maximal independent set T in $G[S]$ has exactly $|T|$ neighbours in I_i for each $i\in\{1,2\}$.*

Proof. Let $|I_1|=|I_2|=r$. By Proposition 4.8, $|N_G(T)\cap I_i|\geq|T|$ for each independent set $T\subseteq S$ and for $i\in\{1,2\}$.

Thus we only need to show that $|N_G(T)\cap I_i|\leq|T|$ for each maximal independent set $T\subset S$ and for $i\in\{1,2\}$. Assume by contradiction that there exists a maximal independent set $T\subseteq S$ such that it has more than $|T|$ neighbours in I_1 (or I_2), say $|N_G(T)\cap I_1|=t>|T|$ and set $H:=G-N_G[T]$. It follows that $|I_1-N_G[T]|\leq r-(|T|+1)$. Since T is a maximal independent set in $G[S]$, we have $\alpha(H)=r-|T|$ by Lemma 2.5, and $V(H)=(I_1\cup I_2)-N_G[T]$. Note also that we have $|N_G(T)\cap I_2|\geq|T|$ by Proposition 4.8. Thus, H has at most $2r-(2|T|+1)$ vertices. However, this contradicts that $\alpha(H)=r-|T|$ and $H\in\mathbf{W}_2$ by Lemma 2.5. Hence $|N_G(T)\cap I_i|\leq|T|$. □

Corollary 4.11. *Let $G \in \mathbf{W}_2$. If $S = V(G) - (I_1 \cup I_2)$ is an independent set for disjoint maximum independent sets I_1 and I_2, then $|N_G(S) \cap I_1| = |N_G(S) \cap I_2| = |S|$.*

By Lemma 4.10, if $G \in \mathbf{W}_2$, and $S = V(G) - (I_1 \cup I_2)$ for disjoint maximum independent sets I_1 and I_2, then every maximal independent set $T \subseteq S$ has exactly $|T|$ neighbours in each of I_1, I_2. We next show that the whole set S (not necessary independent) in such a graph has at most $|S|$ neighbours in each of I_1, I_2.

Lemma 4.12. *Let G be a connected graph in $\mathbf{W}_2$, and let $S = V(G) - (I_1 \cup I_2)$ for disjoint maximum independent sets I_1 and I_2. Then $|N_G(S) \cap I_i| \leq |S|$ for each $i \in \{1, 2\}$.*

Proof. First, if S is an independent set, then the claim follows from Corollary 4.11. Thus, we may assume that S is not independent. Consider the graph $H = G[S]$, and suppose that S has a partition into ℓ disjoint independent sets $S_1, S_2 \ldots, S_\ell$ for $\ell \geq 2$ such that S_j is maximal in $H - (S_1 \cup S_2 \cup \ldots \cup S_{j-1})$ for $j \geq 1$. Remark that such an integer ℓ corresponds to the number of color classes of H (see [28] for details).

We now show that S has at most $|S|$ neighbours in I_i for each $i = 1, 2$. By symmetry, it suffices to show only the case $i = 1$. We proceed by the induction on ℓ. Consider S_1, it is clear that S_1 has exactly $|S_1|$ neighbours in I_1 by Lemma 4.10. From the inductive hypothesis, each S_j for $2 \leq j < \ell$ has at most $|S_j|$ new neighbours in I_1 apart from the vertices of $I_1 \cap N_G(S_1 \cup S_2 \cup \ldots \cup S_{j-1})$. Next, we consider S_ℓ and claim that it has at most $|S_\ell|$ new neighbours in I_1 apart form $I_1 \cap N_G(S_1 \cup S_2 \cup \ldots \cup S_{\ell-1})$. Assume to the contrary that S_ℓ has more than $|S_\ell|$ new neighbours in I_1. Then, we extend S_ℓ to a maximal independent set in H. In this manner, there is a set $S' \subset (S_1 \cup S_2 \cup \ldots \cup S_{\ell-1})$ such that $S_\ell \cup S'$ is a maximal independent set in H, and $|N_G(S_\ell \cup S') \cap I_1| = |S_\ell \cup S'|$ by Lemma 4.10. This implies that S' has less than $|S'|$ neighbours in I_1 since $S_1 \cup S_2 \cup \ldots \cup S_{\ell-1}$ has no neighbour in $I_1 - N_G(S_1 \cup S_2 \cup \ldots \cup S_{\ell-1})$. However, it is a contradiction by Proposition 4.8. Hence, we conclude that S_ℓ has at most $|S_\ell|$ new neighbours in I_1 apart from the vertices in $I_1 \cap N_G(S_1 \cup S_2 \cup \ldots \cup S_{\ell-1})$. This completes the inductive proof, and thus S has at most $|S|$ neighbours in I_1. □

We are now ready to prove one of our main results which limits the size of any graph in $\mathbf{W}_2$.

Theorem 4.13. *Let G be a connected graph in $\mathbf{W}_2$ and let $S = V(G) - (I_1 \cup I_2)$ for disjoint maximum independent sets I_1 and I_2. If $|S| \geq 2$, then G has at most $5|S| - 2$ vertices.*

Proof. Suppose that I_1, I_2 are disjoint maximum independent sets in G, and let $S = V(G) - (I_1 \cup I_2)$ and $|S| \geq 2$. Assume by contradiction that $|G| = n > 5|S| - 2$. Then G has at least $5|S| \geq 10$ vertices since $S = V(G) - (I_1 \cup I_2)$ with $|I_1| = |I_2| = r$ and $n = 2r + |S|$. It follows that $r \geq 2|S|$. Let $I_1 = \{x_1, x_2 \ldots, x_r\}$, $I_2 = \{y_1, y_2 \ldots, y_r\}$. By Theorem 2.1, we may assume $\{x_1y_1, x_2y_2, \ldots, x_ry_r\} \subset E(G)$. Let $F = \{x_1y_1, x_2y_2, \ldots, x_ry_r\}$. Note that every vertex in S has a neighbour in both I_1 and I_2 by Remark 4.9.

Claim: There exists an index $j \in [r]$ such that $N_G(S) \cap \{x_j, y_j\} = \emptyset$.

Proof of the claim. First, if $r > 2|S|$, there exists such index j by Lemma 4.12. Besides, we have such index j when $|N_G(S) \cap I_i| < |S|$ for some $i = 1, 2$ due to $r \geq 2|S|$. Thus, we conclude that $|N_G(S) \cap I_i| = |S|$ for $i \in \{1, 2\}$ by Lemma 4.12, and so $r = 2|S|$. Let $|S| = m$.

Assume for a contradiction that there is no such index j. Then at least one endpoint of each edge $x_iy_i \in F$ belongs to $N_G(S)$. Since $|N_G(S) \cap I_i| = |S|$ for $i \in \{1, 2\}$, we may then assume, without loss of generality, that $N_G(S) \cap I_1 = \{x_1, x_2, \ldots, x_m\}$ and $N_G(S) \cap I_2 = \{y_{m+1}, y_{m+2}, \ldots, y_{2m}\}$ where $2m = r$. We set $R_i = I_i - N_G(S)$ for $i \in \{1,2\}$. Recall that every vertex of S has a neighbour in each of I_1, I_2 by Remark 4.9, and so $I_i - R_i$ dominates all vertices of S for each $i \in \{1,2\}$. Since $G - N_G[S] \in \mathbf{W}_2$ by Lemma 2.5, and $\alpha(G - N_G[S]) = m$, we observe that the graph $G - N_G[S] = G[R_1 \cup R_2]$ is bipartite, and so it is isomorphic to mK_2 by Proposition 2.4. We may therefore assume $E(G - N_G[S]) = \{x_{m+1}y_1, x_{m+2}y_2, \ldots, x_{2m}y_m\}$ (possibly after relabelling), see Figure 2. On the other hand, $G - N_G[R_i] \in \mathbf{W}_2$ for each $i \in \{1,2\}$ and $\alpha(G - N_G[R_i]) = m$ by Lemma 2.5. It then follows from Proposition 2.4 that the graph $G - N_G[R_i]$ is isomorphic to mK_2 since $m = |R_i| \geq |S|$. This implies that S is an independent set of size m, also each vertex of S has a unique neighbour in I_i for each $i \in \{1,2\}$ since $|N_G(S) \cap I_i| = |S|$.

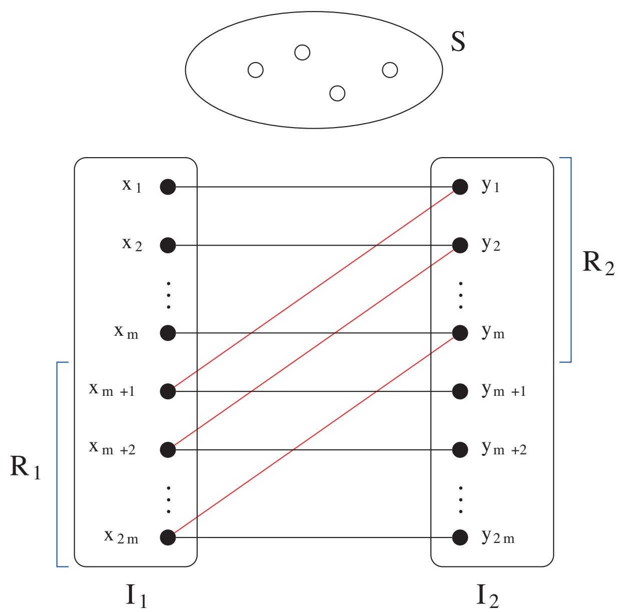

Figure 2. Illustration of the graph G with I_1, I_2, and S.

Notice that $G - N_G[I_i - R_i] \in \mathbf{W}_2$ for each $i \in \{1,2\}$ by Lemma 2.5, and $|R_i| = m$, so the graph $G - N_G[I_i - R_i]$ is isomorphic to mK_2 by Proposition 2.4. This implies that each of the sets $R_1 \cup (I_2 - R_2)$ and $R_2 \cup (I_1 - R_1)$ induces a graph which is isomorphic to mK_2 by Proposition 2.4. Additionally, any vertex of $I_1 - R_1$ cannot be adjacent to $I_2 - R_2$. Thus every vertex in $N_G(S)$ has a unique neighbour in $G[I_1 \cup I_2]$ while every vertex in $R_1 \cup R_2$ has degree 2 in G. We therefore conclude $G[I_1 \cup I_2]$ is isomorphic to mP_4.

Since each vertex of S has a unique neighbour in I_i for each $i \in \{1,2\}$, and $|N_G(S) \cap I_i| = |S|$, we deduce that every vertex in $I_1 \cup I_2$ has degree 2 in G, and consequently, G is a 2-regular graph. However, G cannot be a cycle of size greater than 5 since C_3 and C_5 are only cycles in $\mathbf{W}_2$. Hence, G consists of some C_3 and C_5 cycles due to $|S| \geq 2$. But it contradicts the connectivity of G. This completes the proof of the claim. $\diamondsuit$

By above claim, there exists an index $j \in [r]$ such that $N_G(S) \cap \{x_j, y_j\} = \emptyset$. This implies that $N_G(x_j) \subseteq I_2$ and $N_G(y_j) \subseteq I_1$. Remark that each vertex of a graph belonging to $\mathbf{W}_2$ has degree at least 2 by Corollary 2.7, so $|N_G(y_j) \cap I_1| \geq 2$. It then follows that $N_G(x_j)$ is dominated by the independent set $I_1 - x_j$. However, this contradicts that x_j is a shedding vertex. Thus $r \leq 2|S| - 1$, and so G has at most $5|S| - 2$

vertices. □

As a consequence of Theorem 4.13 together with Corollaries 4.4 and 4.7, we conclude the following.

Theorem 1.3. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$. Then $n \leq 10k-2$ and $\alpha(G) \leq 4k-1$.*

By Theorem 1.3 together with Theorem 1.2, we have the following.

Corollary 4.14. *Let G be a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-1$. Then $\Delta(G) \leq 3$ and $n \leq 8$.*

Corollary 4.14 provides that any connected graph G in $\mathbf{W}_2$, which satisfies $\alpha(G)+\mu(G)=n-1$, can have at most 8 vertices. We have detected all these graphs of size at most 8 vertices by using computer programs written in Python-Sage. There are only 8 such graphs: C_3, C_5, K_4, and those five graphs in Figure 1.

Theorem 1.4. *A connected graph G with $\alpha(G)+\mu(G)=|G|-1$ belongs to $\mathbf{W}_2$ if and only if G is C_3 or C_5 or K_4 or one of those five graphs in Figure 1.*

We now turn our attention to the graphs in $\mathbf{W}_2$ satisfying $\alpha(G)+\mu(G)=|G|-k$ for large k. Recall that for each graph G in $\mathbf{W}_2$ with n vertices, there exists an integer k with $0 \leq k \leq \lceil \frac{n}{2} \rceil - 1$ such that G satisfies $\alpha(G)+\mu(G)=n-k$. Consider the case $k=\lceil \frac{n}{2} \rceil - 1$, we shall show that only complete graphs enjoy this case.

Theorem 4.15. *A connected graph G with n vertices belongs to $\mathbf{W}_2$ such that $\alpha(G)+\mu(G)=n-\left(\lceil \frac{n}{2} \rceil - 1\right)$ if and only if G is a complete graph.*

Proof. It is easy to see that a complete graph G satisfies $\alpha(G)+\mu(G)=\lfloor \frac{n}{2} \rfloor + 1$, also G is in $\mathbf{W}_2$ by Remark 1.1. So we only need to verify the necessity of the claim. Suppose that G is a connected graph in $\mathbf{W}_2$ with n vertices such that $\alpha(G)+\mu(G)=\lfloor \frac{n}{2} \rfloor + 1$. Obviously, the claim holds if we show that $\alpha(G)=1$. Assume to the contrary that $\alpha(G) \geq 2$. If $\alpha(G)=2$, then G has a matching that leaves at most one unsaturated vertex by Lemma 3.2. Thus $\mu(G)=\lfloor \frac{n}{2} \rfloor$, and so $\alpha(G)+\mu(G)=\lfloor \frac{n}{2} \rfloor + 2$, a contradiction. Thus, we further suppose that $\alpha(G)=r \geq 3$. Let I_1, I_2 be two disjoint maximum independent sets in G, and write $S=V(G)-(I_1 \cup I_2)$. Notice that $\mu(G) \geq \frac{n-r}{2}$ due to $\alpha(G)=r$. It follows that $\alpha(G)+\mu(G) \geq r+\frac{n-r}{2} > \lfloor \frac{n}{2} \rfloor + 1$ since $r \geq 3$, a contradiction. Hence $\alpha(G)=1$, and so G is isomorphic to K_n. □

We next bound the independence number of a graph in $\mathbf{W}_2$ for a large value k in the equation $\alpha(G)+\mu(G)=n-k$ so that we improve Theorem 1.3.

Lemma 4.16. *If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k \geq 0$, then $\alpha(G) \leq n-2k$.*

Proof. Suppose that G is a connected graph with n vertices belonging to $\mathbf{W}_2$ such that $\alpha(G)+\mu(G)=n-k$. Let I_1, I_2 be two disjoint maximum independent sets of size r in G, and write $S=V(G)-(I_1 \cup I_2)$. Notice that $\mu(G) \geq \frac{n-r}{2}$ due to $\alpha(G)=r$. It follows that $n-k=\alpha(G)+\mu(G) \geq r+\frac{n-r}{2} \geq \frac{n+r}{2}$, and so we obtain $r \leq n-2k$ as claimed. □

The following can be obtained from Lemma 4.16.

Corollary 4.17. *If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=\lfloor \frac{n}{2} \rfloor + t$ for $t \geq 1$, then $\alpha(G) \leq 2t$.*

When Theorem 1.3 and Lemma 4.16 are combined, we obtain the following.

Theorem 1.5. *If G is a connected graph in $\mathbf{W}_2$ with n vertices and $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$, then $\alpha(G) \leq \min\{4k-1, n-2k\}$.*

5. Conclusion

In this paper, we studied on $\mathbf{W}_2$ graphs, namely 1-well-covered graphs without isolated vertices. We classified those graphs with respect to $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$ and $|G|=n$. Recall that the inequality $\lfloor \frac{n}{2} \rfloor + 1 \leq \alpha(G)+\mu(G) \leq n$ holds for any graph G with n vertices. Since a complete graph $G=K_n$ belongs to $\mathbf{W}_2$ such that $\alpha(G)+\mu(G)=n-\left(\lceil \frac{n}{2} \rceil - 1\right)$, there exist $\lceil \frac{n}{2} \rceil$ nonempty subclasses of $\mathbf{W}_2$ graphs.

We first stated some combinatorial properties of graphs in $\mathbf{W}_2$ with $\alpha(G)+\mu(G)=n-k$ for $k \geq 1$. In particular, we proved that such a graph G has at most $10k-2$ vertices, $\Delta(G) \leq 2k+1$ and $\alpha(G) \leq \min\{4k-1, n-2k\}$. We also showed that the presented bounds on the number of vertices and the maximum degree of G are tight. We particularity obtained all graphs in $\mathbf{W}_2$ with $\alpha(G)+\mu(G)=n-k$ for $k=1$, which was proposed in [14]. Even though we only gave a complete characterization of graphs in $\mathbf{W}_2$ with $\alpha(G)+\mu(G)=n-k$ for $k=1$, the presented structural properties may lead to finding those graphs for a larger value of k. Furthermore it would be interesting to obtain a full characterization of $\mathbf{W}_2$ graphs for $k \geq 2$ which sheds light on 1-well-covered graphs from a new perspective.

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