Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 2393 – 2405
© TÜBİTAK
doi:10.3906/mat-2105-68

Finite groups with three nonabelian subgroups

Bijan TAERI, Fatemeh TAYANLOO-BEYG*
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran

Received: 24.05.2021 • **Accepted/Published Online:** 21.08.2021 • **Final Version:** 29.11.2021

Abstract: We characterize finite groups with exactly two nonabelian proper subgroups. When G is nilpotent, we show that G is either the direct product of a minimal nonabelian p-group and a cyclic q-group or a 2-group. When G is nonnilpotent supersolvable group, we obtain the presentation of G. Finally, when G is nonsupersolvable, we show that G is a semidirect product of a p-group and a cyclic group.

Key words: Finite groups, minimal nonabelian groups, minimal nonnilpotent groups, critical groups

1. Introduction and results

A nonabelian group whose all proper subgroups are abelian is called minimal nonabelian. Miller and Moreno [10] started the study of minimal nonabelian groups and Rédi [11] classified finite minimal nonabelian groups. A finite minimal nonabelian group is either a p-group, p a prime, of a certain type or a $\{p, q\}$-group, where p and q are distinct primes, with a certain presentation, see Theorem 1.1 below.

A nonnilpotent group whose all proper subgroups are nilpotent is called a minimal nonnilpotent. Such groups are also called Schmidt groups. Schmidt [16] started the study of minimal nonnilpotent groups and Rédei [12] classified completely finite minimal nonnilpotent groups. A finite minimal nonnilpotent group is a semidirect product of a p-group by a cyclic q-group, see Theorem 2.1 in section 2.

Itô [8] considered the minimal non-p-nilpotent groups, where p is a prime, which turn out to be just the Schmidt groups. Ballester-Bolinches and Esteban-Romero [2] studied the structure of minimal nonsupersolvable groups.

Let $\mathscr{X}$ be a class of groups. A group G is called an $\mathscr{X}$-critical group or a minimal non-$\mathscr{X}$-group, if G is not an $\mathscr{X}$-group but every proper subgroup of G is an $\mathscr{X}$-group. If $\mathscr{A}$ (resp. $\mathscr{N}$) stands for the class of abelian (resp. the class of nilpotent) groups, then a minimal nonabelian (resp. a minimal nonnilpotent) group is exactly an $\mathscr{A}$-critical (resp. $\mathscr{N}$-critical) group.

Let n be a positive integer. We call a group G an n-$\mathscr{X}$-critical group, if $G \notin \mathscr{X}$ and has exactly $n-1$ proper subgroups that are not belong to $\mathscr{X}$ and other subgroups belong to $\mathscr{X}$. Therefore a minimal non-$\mathscr{X}$-group is just a 1-$\mathscr{X}$-critical group. A generalization of the class n-$\mathscr{X}$-critical is the class $\beta_n^{\mathscr{X}}$. A group G is called a $\beta_n^{\mathscr{X}}$-group, if G has exactly n subgroups that are not belong to $\mathscr{X}$. It is clear that every n-$\mathscr{X}$-critical group is a $\beta_n^{\mathscr{X}}$-group, for a nonempty subgroup closed class $\mathscr{X}$ of groups. The structure of $\beta_n^{\mathscr{X}}$-groups have been studied for some classes of groups. For instance, Zarrin [21] studied the structure of

*Correspondence: f.tayanloo@gmail.com
2010 *AMS Mathematics Subject Classification:* 20D10, 20D15

finite nonsolvable $\beta_n^{\mathscr{C}}$-groups, where $\mathscr{C}$ is the class of cyclic groups, with $n \leq 27$; and Shi and Zhang in [17] studied the structure of $\beta_n^{\mathscr{A}}$-groups and proved that every $\beta_n^{\mathscr{A}}$-group with $n \leq 21$ is solvable. Hence we can see that every finite n-$\mathscr{A}$-critical group with $n \leq 21$ is solvable.

Russo [15] studied 2-$\mathscr{N}$-critical groups, in infinite case and obtained some results in finite case. Also from the main result in [20], it follows that every finite n-$\mathscr{N}$-critical group with $n \leq 21$ is solvable. In [19], we studied the structure of finite 2-$\mathscr{N}$-critical groups.

In [18], we studied the structure of finite 2-$\mathscr{A}$-critical groups. In this paper we characterize finite 3-$\mathscr{A}$-critical groups. We show that $|\pi(G)| \leq 3$, where $\pi(G)$ is the set of prime divisors of G. Meng [9], in a special case, gave a classification of such groups with $|\pi(G)| = 3$. In this paper we classify these groups when $|\pi(G)| = 1$ and $|\pi(G)| = 2$.

All groups in this paper are assumed to be finite. Our notation and terminology are standard taken mainly from [13]. In particular the size of a finite group G is shown by $|G|$. The center, the Frattini subgroup, the Fitting subgroup, the group of automorphisms, the set of Sylow p-subgroups of G are denoted by $Z(G)$, $\Phi(G)$, $F(G)$, $\mathrm{Aut}(G)$, $\mathrm{Syl}_p(G)$, respectively. A cyclic group of order n is denoted by $\mathbb{Z}_n$. If G is a finite p-group, the dimension of $G/\Phi(G)$ over $\mathrm{GF}(p)$ is denoted by $d(G)$, which is the size of a minimal generating set of G, so that $p^{d(G)} = |G/\Phi(G)|$.

The following Theorem states a classification of minimal nonableian groups.

Theorem 1.1 [10, 11]. Let G be a finite group. Then G is minimal nonabelian if and only if G is isomorphic to one of the following groups

1. $U = P \rtimes \langle y \rangle$, where P is an elementary abelian p-subgroup and a minimal normal subgroup of U, $\langle y \rangle$ is a nonnormal cyclic Sylow q-subgroup, p, q are distinct primes, $y^q \in Z(U)$. Also the action of $\langle y \rangle$ on P is irreducible with kernel $\langle y^q \rangle$. More precisely $P = \langle a_1, \ldots, a_n \rangle$, $a_i^p = y^{q^s} = 1$, $a_i^y = a_{i+1}$, $a_n^y = a_1^{d_1} \cdots a_n^{d_n}$.

2. The quaternion of order 8,

3. $M_{m,n,q} = \langle a, b \mid a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}} \rangle$, where q is a prime number, $m \geq 2$, $n \geq 1$, of order q^{m+n},

4. $N_{m,n,q} = \langle a, b \mid a^{q^m} = b^{q^n} = [a,b]^q = [a,b,a] = [a,b,b] = 1 \rangle$, where q is a prime number, $m \geq n \geq 1$, of order q^{m+n+1}.

Throughout the paper p, q and r are prime numbers. In this paper, we classify finite groups with exactly two nonabelian proper subgroups. We divide our arguments to supersolvable and nonsupersolvable cases. The supersolvable case is divided to nilpotent and nonnilpotent cases. A classification theorem of finite nilpotent groups with exactly two nonabelian proper subgroups is the following.

Theorem A Let G be a finite nilpotent group. Then G has exactly two nonabelian proper subgroups if and only if G is one of the following groups

(1) $P \times Q$, where P is a minimal nonabelian p-group, $Q = \langle b \rangle$ is a cyclic q-group of order q^2 and p, q are distinct primes.

(2) $G(n,\varepsilon,\eta) = \langle a, b \mid a^8 = 1, b^{2^n} = a^{4\varepsilon}, a^b = a^{-1+4\eta} \rangle$, where $n \geq 1$, $\varepsilon, \eta \in \{0,1\}$.

A classification theorem of finite nonnilpotent supersolvable groups with exactly two nonabelian proper subgroups is the following.

Theorem B Let G be a finite nonnilpotent supersolvable group. Then G has exactly two nonabelian proper subgroups if and only if G is one of the following groups

(1) $G = \langle a, b \mid a^p = b^{q^{m+2}} = 1,\ [a, b^{q^3}] = 1,\ a^b = a^i \rangle$, where $m \geq 1$, $0 \leq i \leq p-1$, $i^{q^3} \equiv 1 \pmod p$ and $q^3 \mid p-1$.

(2) $G = \langle a, b, c \mid a^p = b^{q^m} = c^{r^2} = 1,\ [a,c] = [b,c] = [a, b^q] = 1,\ a^b = a^i \rangle$, where $m \geq 1$, $q \mid p-1$, $0 \leq i \leq p-1$ and $i^{q^m} \equiv 1 \pmod p$.

(3) $G = \langle a, b, c \mid a^p = b^{p^2} = c^{q^m} = 1,\ [a,b] = [c,b] = [c^q, a] = 1,\ a^c = a^i b^j \rangle$, where $m \geq 1$, $q \mid p-1$, $0 \leq i \leq p-1$, $0 \leq j \leq p^2-1$, $i^{q^m} \equiv 1 \pmod p$, $1 + i + \cdots + i^{q^m-1} \equiv 0 \pmod{p^2}$ (if $j \neq 0$) and $(i,j) \neq (1,0)$.

(4) $G = \langle a, b, c \mid a^p = b^{2^m} = c^2 = 1,\ [c,a] = [c,b] = [a,b^2] = 1,\ a^b = a^i \rangle$, where $m \geq 1$, $0 \leq i \leq p-1$ and $i^{2^m} \equiv 1 \pmod p$.

Finally, the following Theorem characterizes finite nonsupersolvable groups with exactly two nonabelian proper subgroups:

Theorem C Let G be a finite nonsupersolvable group. Then G has exactly two nonabelian proper subgroups if and only if G is one of the following groups

(1) $G = P \rtimes Q$, where P is abelian of type $(p, \ldots, p, p^2)$ of order p^{n+2} and Q is cyclic of order q^m that $q \nmid p-1$. Also $H = P_1 \rtimes Q$ and $K = P_2 \rtimes Q$ are nonabelian subgroups of G that $H \subset K$, H is minimal nonabelian, $|P_1| = p^n$ and $|P_2| = p^{n+1}$. Furthermore P_1 is an irreducible Q-module and P_2 is elementary abelian.

(2) $G = P \rtimes Q$, where P is elementary abelian of order p^n and $Q = \langle c \rangle$ is cyclic of order q^{m+1} that $q \nmid p-1$. Also $H = P \rtimes Q_1$ and $K = P \rtimes Q_2$ are nonabelian subgroups of G that $H \subset K$, H is minimal nonabelian, $|Q_1| = q^m$ and $|Q_2| = q^{m+1}$. Furthermore P is an irreducible $\langle c^{q^2} \rangle$-module.

(3) $G = PQR$, where P is elementary abelian of order p^n, Q is cyclic of order q^m and $R = \langle x \rangle$ is cyclic of order r^2 that $q \nmid p-1$. Also P is an irreducible Q-module. Furthermore $H = PQ$ and $K = PQ\langle x^r \rangle$ are nonabelian subgroups of G and H is minimal nonabelian.

(4) $G = P \rtimes Q$ is a Schmidt group, where P is a special nonabelian p-group of order p^3 and $Q = \langle c \rangle$ is cyclic of order q^2 that $q \nmid p-1$. Also $K = P\langle c^q \rangle$ is nonabelian.

(5) $G = G'Q$, where G' is a nonabelian special p-group of order p^3 and of rank 2, $Q = \langle c \rangle$ is cyclic of order q^2, $q \nmid p-1$, $q \mid p+1$ and the order of p modulo q^2 being 2. Also $G'\langle c^q \rangle$ is a Schmidt group, $G'/\Phi(G')$ is a faithful irreducible Q-module and $[\Phi(G'), Q] = 1$. Furthermore $|G''| = p$.

(6) $G = P \rtimes Q$, where $P = G'$ is elementary abelian of order p^2 and $Q \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$. Also $H = P\langle a \rangle$ and $K = P\langle a^p \rangle$ are minimal nonabelian, $|Z(G)| = 2^m$ and $Z(G) = \langle a^2 \rangle \times \langle b \rangle$.

2. Proofs

In the following Theorem we state the classification of Schmidt groups.

Theorem 2.1 [12, 16] Let G be a finite Schmidt group. Then $G = P \rtimes Q$, where P is a Sylow p-subgroup and $Q = \langle z \rangle$ is a cyclic Sylow q-subgroup of order $q^r > 1$. Furthermore $Z(G) = \Phi(G) = \Phi(P) \times \langle z^q \rangle$; $G' = P$, $P' = G'' = \Phi(P)$, and one of the following cases hold:

(1) q does not divide $p-1$ and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q. The subgroup P is elementary abelian minimal normal p-subgroup of order p^l where l is the order of p modulo q.

(2) P is a nonabelian special p-group, $|P/\Phi(P)| = p^{2m}$, $|P'| \leq p^m$, the order of p modulo q being $2m$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful irreducible Q-module, and z centralizes $\Phi(P)$.

(3) q divides $p-1$, $P = \langle a \rangle$ is cyclic of order p, and $a^z = a^i$, where i is the least primitive q-th root of unity modulo p.

Note that Schmidt groups satisfying (1) and (2) of Theorem 2.1 are nonsupersolvable and Schmidt groups satisfying (3) are supersolvable. In fact, if G is supersolvable and satisfies (1), then since P is a minimal normal subgroup G, Theorem 5.4.7 of [13] implies that $|P| = p$. It follows that $q \mid p-1$, a contradiction. Also, if G is supersolvable and satisfies (2), then $P/\Phi(P)$ is a minimal normal subgroup of supersolvable Schmidt group $G/\Phi(P)$ which implies that P is cyclic, a contradiction. Finally, if G satisfies (3), then $1 \lhd P \lhd G$ is a normal series with cyclic factors and so G is supersolvable.

We need the following concept introduced by Berkovich.

Definition 2.2 [3, Page 233] A p-group, where p is a prime, is called an $\mathscr{A}_n$-group if every subgroup of index p^n is abelian but at least one subgroup of index p^{n-1} is nonabelian.

The structure of $\mathscr{A}_2$-groups are determined in Section 71 of [3]. We need the following result about $\mathscr{A}_2$-groups.

Theorem 2.3 [3, Proposition 71.2] Let G be a metacyclic $\mathscr{A}_2$-group of order $p^k > p^4$, where p is a prime. Then $G' \cong C_{p^2}$ and we have one of the following possibilities:

(i) $G = \langle a, b \mid a^{p^m} = 1,\ b^{p^n} = a^{\varepsilon p^{m-1}},\ a^b = a^{1+p^{m-2}} \rangle$, where $\varepsilon \in \{0, 1\}$, $n \geq 1$, $m \geq 3$, $m + n \geq 5$, in case $p = 2$, $m \geq 4$.

(ii) $p = 2$, $G = \langle a, b \mid a^8 = 1,\ b^{2^n} = a^{4\varepsilon},\ a^b = a^{-1+4\eta} \rangle$, where $n \geq 2$, $\varepsilon, \eta \in \{0, 1\}$.

We begin with the following straightforward lemma.

Lemma 2.4 Let G be a group with exactly two nonabelian proper subgroups H and K. We have

(1) If $H \subset K$, then

   (a) H is a maximal subgroup of K and K is a maximal subgroup of G.

   (b) H and K are characteristic subgroups of G. In particular G/K and K/H have prime orders.

   (c) H is the unique nonabelian subgroup of K and H is minimal nonabelian.

(2) If $H \nsubseteq K$ and $K \nsubseteq H$, then

   (a) H and K are maximal subgroups of G which are normal in G.

   (b) H and K are minimal nonabelian.

Proof To prove (1), we can easily see that H is a maximal subgroup of K, as otherwise, there exists a maximal subgroup M of K containing H and clearly M is nonabelian, which contradicts the hypothesis. Similarly one can see that K is a maximal subgroup of G, which proves (a).
Now we show that H and K are characteristic subgroups of G. Let $\psi \in \mathrm{Aut}(G)$ be an automorphism of G. We have $\psi(H) \cong H$. Therefore $\psi(H)$ is nonabelian. By assumption, we have either $\psi(H) = H$ or $\psi(H) = K$. Noticing that $H \subset K$, it follows that $\psi(H) \neq K$. Thus $\psi(H) = H$ and then H is a characteristic subgroup of G. Similarly K is a characteristic subgroup of G. Now, since, by (a), H is a maximal subgroup of K and K is a maximal subgroup of G, it follows that K/H and G/K have prime orders, which proves (b).
Finally, as H and K are the only nonabelian proper subgroups of G, each subgroup of H is abelian and H is the only nonabelian proper subgroup of K, which proves (c).

To prove (2), it is not difficult to see that H and K are maximal subgroups of G which are also minimal nonabelian. We show that H and K are normal in G. If G is nilpotent, then clearly H and K are normal. So, assume that G is nonnilpotent. Suppose contrary that H is nonnormal. It follows that $H = N_G(H)$. As $|G : H| \geq 3$, we have $|G : N_G(H)| \geq 3$. It shows incidentally that H has more than three conjugates, a contradiction. Therefore H is normal in G. Similarly K is normal in G. □

Now we prove Theorem A.

Proof Theorem A First we note that groups (1) and (2) have exactly two nonabelian proper subgroups. For the group (1), $H = P$ and $K = H \times \langle b^q \rangle$ are the only nonabelian proper subgroups. If G is the group (2), then G is metacyclic. Since $d(G) = 2$, we have $|G/\Phi(G)| = 2^2$ and so G has exactly three maximal subgroups. By virtue of [6, Theorem 2.2], the maximal subgroups of G are $H = \langle a^2, b\rangle$, $K = \langle ba, a^2\rangle$ and $L = \langle b^2, a\rangle$. Clearly H and K are nonabelian and L is abelian.

Now to prove the converse of Theorem, suppose that G has exactly two nonabelian proper subgroups H and K, say. We cosider 2 cases:
Case 1. $H \subseteq K$. Since, by Lemma 2.4, H is minimal nonabelian, it follows from Theorem 1.1, that $|H| = p^n$, for some positive integer n and a prime number p. By [18, Lemma 2.5], K is not a p-group. So there exists a prime number q distinct from p such that $|K| = p^n q$. If $|G : K| = p$, then a Sylow p-subgroup of G is abelian and contains H, which is a contradiction.
If $|G : K| = r \notin \{p, q\}$, then $G = H \times \mathbb{Z}_q \times \mathbb{Z}_r$, which is impossible, because this group has 3 nonabelian proper subgroups. Therefore $|G : K| = q$ and thus $|G| = p^n q^2$. Hence $G = H \times Q$, where Q is a Sylow q-subgroup

of G. It is readily seen that Q is cyclic, as otherwise $Q \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and then G has three nonabelian proper subgroups, contradicting the hypothesis. Thus G is the group mentioned in (1).

Case 2. $H \nsubseteq K$ and $K \nsubseteq H$. First let us prove that G is a 2-group. By Lemma 2.4, H and K are minimal nonabelian and so, by Theorem 1.1, they are both p-groups. Also, according to Lemma 2.4, H and K are normal maximal subgroups. So the indices of H and K in G are prime. It follows that $|H| = |K| = p^n$ and $|G| = p^{n+1}$ (if $|G| = p^n q$, where q is a prime distinct from p, then, as H , K are Sylow subgroups of G and G is nilpotent, $H = K$, a contradiction). We claim that G has only one abelian maximal subgroup. Suppose contrary that G has more than one abelian maximal subgroup. Noticing that H is minimal nonabelian, it follows from [10] that $|H : Z(H)| = |H : \Phi(H)| = p^2$. It is readily seen that $Z(G)$ is equal to the intersection of all abelian maximal subgroups. It indicates that $\Phi(G) = Z(G) \cap H \cap K \subseteq Z(H) = \Phi(H)$ and then $|\Phi(H)| \mid |\Phi(G)|$. We deduce that $|\Phi(G)| = |\Phi(H)| = p^{n-2}$. Thus $G/\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ which yields that G has exactly $1 + p + p^2$ maximal subgroups. From [4, Lemma 3] it follows that G has $p + 1$ abelian maximal subgroups. Hence G has $p + 3$ maximal subgroups. So $p + 3 = 1 + p + p^2$, which is impossible. Therefore G has only one abelian maximal subgroup and so G has exactly 3 maximal subgroups. It follows that $G/\Phi(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and G is a 2-group.

If $|G| = 16$, then by inspection on 14 groups of order 16, we see that three groups D_{16}, SD_{16} and Q_{16} have two nonabelian proper subgroups, which are item (2) for $n = 1$. In fact, they are groups $G(1, 0, 0)$, $G(1, 0, 1)$ and $G(1, 1, 1)$.

Now suppose that $|G| > 16$. If G is nonmetacyclic, then, as G is a 2-group, by [3, Proposition 71.3], we have $G' \subseteq Z(G)$. By using the Definition 2.2, G is an $\mathscr{A}_2$-group. So [3, Proposition 71.4] indicates that $d(G) = 3$. Thus $|G/\Phi(G)| = 2^3$ and hence G has 7 maximal subgroups, which is impossible. Therefore G is metacyclic and is one of the groups in Theorem 2.3. If G is the group mentioned in Theorem 2.3(a), then according to [6, Theorem 2.2], the maximal subgroups of G are $M_1 = \langle b, a^2\rangle$, $M_2 = \langle ba, a^2\rangle$ and $M_3 = \langle b^2, a\rangle$. It is not difficult to see that these subgroups are nonabelian, a contradiction. Thus G is the group in Theorem 2.3(b), which is the item (2), when $n \geq 2$. □

In order to prove Theorem B, we prove the following proposition.

Proposition 2.5 Let G be a finite nonnilpotent supersolvable group with exactly two nonabelian proper subgroups H and K. If $H \subset K$, then G is isomorphic to one of the following groups

(1) $\langle a, b \mid a^p = b^{q^{m+2}} = 1,\ [a, b^{q^3}] = 1,\ a^b = a^i\rangle$, where $m \geq 1$, $0 \leq i \leq p-1$, $i^{q^3} \equiv 1 \pmod p$ and $q^3 \mid p-1$.

(2) $\langle a, b, c \mid a^p = b^{q^m} = c^{r^2} = 1,\ [a, c] = [b, c] = [a, b^q] = 1,\ a^b = a^i\rangle$, where $m \geq 1$, $q \mid p-1$, $0 \leq i \leq p-1$ and $i^{q^m} \equiv 1 \pmod p$.

(3) $\langle a, b, c \mid a^p = b^{p^2} = c^{q^m} = 1,\ [a, b] = [c, b] = [c^q, a] = 1,\ a^c = a^i b^j\rangle$, where $m \geq 1$, $q \mid p-1$, $0 \leq i \leq p-1$, $0 \leq j \leq p^2 - 1$, $i^{q^m} \equiv 1 \pmod p$, $1 + i + \cdots + i^{q^m-1} \equiv 0 \pmod{p^2}$ (if $j \neq 0$) and $(i, j) \neq (1, 0)$.

Proof We claim that K is not nilpotent. Suppose, for a contradiction, that K is nilpotent. By Lemma 2.4, K is a normal maximal subgroup of G and so $K = F(G)$. From Lemma 2.4, H is minimal nonabelian and it is a unique nonabelian proper subgroup of K. So by Theorem 1.1, $|H| = p^n$, where p is a prime number and n is a positive integer. Now Lemma 2.4 indicates that $|K : H| = q$, where q is a prime. As K has a unique nonabelian proper subgroup, from [18, Lemma 2.5] we know that K is not a p-group, and so $q \neq p$. Hence $|K| = p^n q$. If $|G : K| = p$, then $|G| = p^{n+1}q$ and H is properly contained in a Sylow p-subgroup P of G. Noticing that $P \neq K$, it follows that P and hence H is abelian, a contradiction. Therefore $|G : K| = q$ and $|G| = p^n q^2$. Thus $G = H \rtimes Q$, where Q is a Sylow q-subgroup of G. Clearly G is a Schmidt group. So by [13, Theorem 5.4.7] and Theorem 2.1 (3), $|H| = p$, which is impossible.

Thus our claim is true, that is K is not nilpotent. If H is nilpotent, then H is a p-group (by Theorem 1.1) and K is a Schmidt group. By [13, Theorem 5.4.7] and Theorem 2.1(3), we see that H is of prime order, which is impossible. Thus H is nonnilpotent of order pq^m, where m is a positive integer and $q \mid p-1$. Also, by Theorem 2.1, $|\Phi(H)| = q^{m-1}$. Assume that P_1 is a Sylow p-subgroup of H and Q_1 is a Sylow q-subgroup of H. Also assume that P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G. Then $|P_1| = p$ and $|Q_1| = q^m$. We have $P_1 \leq P$. We consider the following two cases.

Case 1. $P_1 = P$. First suppose that $|Q_1| < |Q|$. Then $K = HQ$ and so G has two prime divisors and $|G : K| = q$. Hence $|G| = pq^{m+2}$. Now we find the presentation of G.

If $F(G) \subseteq K$ and $F(G) \nsubseteq H$, then $|F(G)| = |F(K)| = pq^m$ and $\Phi(G) = Z(G)$ (as $\Phi(G) = Z(G) \cap K$). Since G has a normal subgroup of order q^m that intersect G' trivially, we have $q^m \mid |Z(G)|$. So $|\Phi(G)| = |Z(G)| = q^m$. On the other hand, since $F(G) \subseteq K$, we have $Z(G) \subseteq Z(K)$, which is a contradiction, because according to [18, Theorem F], $|Z(K)| = q^{m-1}$. If $F(G) \nsubseteq K$, then $|F(G)| = pq^{m+1}$. Since $\Phi(G) = Z(G) \cap K$ and $|\Phi(H)| = q^{m-1} \mid |\Phi(G)|$, we have $|\Phi(G)| = q^{m-1}$. Notice that $F(G)$ has a normal subgroup N of order q^{m+1} and it is clear that N is a normal subgroup of G. As $N \cap G' = 1$, $N \subseteq Z(G)$ and then $|Z(G)| = q^{m+1}$. We should have $pq^3 = |G : \Phi(G)| \mid |G : Z(G)||G : K| = pq^2$, which is impossible. Therefore $F(G) \subseteq H$ and so $|F(G)| = |F(H)| = pq^{m-1}$ and $|\Phi(G)| = |Z(G)| = q^{m-1}$. Let Q be a Sylow q-subgroup of G. We have $G \cong \mathbb{Z}_p \rtimes Q$. We claim that Q is cyclic. Suppose contrary that Q is not cyclic. Hence it is metacyclic and so $Q = \langle a, b\rangle$. Let $a \notin K$ (so $a \notin H$). Since $P\langle ab\rangle$ and $P\langle a\rangle$ are proper subgroups of G distinct from H and K, they are abelian and so $[P, ab] = [P, a] = 1$. It follows that $[P, b] = 1$, a contradiction. Therefore Q is cyclic, proving the claim. Let $P = \langle a\rangle$ and $Q = \langle b\rangle$. As $|Z(G)| = q^{m-1}$, we can assume that $Z(G) = \langle b^{q^3}\rangle$. Put $b^{-1}ab = a^i$, where $0 \leq i \leq p-1$ and $i^{q^3} \equiv 1 \pmod{p}$. Clearly $C_Q(P) = Z(G)$. It follows that $|C_G(P)| = pq^{m-1}$. Since $G/C_G(P)$ is isomorphic to a subgroup of $\mathrm{Aut}(P)$, $q^3 \mid p-1$. Therefore G is the group mentioned in (1).

Now suppose that $|Q_1| = |Q|$. In this case $|K : H| = r$, which is a prime distinct from p and q. It is easy to see that $|G : K| = r$ and so $|G| = pq^m r^2$. Let R be a Sylow r subgroup of G. Since G is solvable and Hall subgroups PR and QR as they are distinct from H and K, they are abelian and so $R \leq Z(G)$ and $G \cong R \times H$. Therefore $|Z(G)| = r^2 q^{m-1}$. Obviously R is cyclic, as otherwise $R = \langle a, b\rangle$. Suppose $a \notin K$ and so $a \notin H$. As $H\langle a\rangle$ is a proper subgroup of G distinct from H and K, it is abelian, which follows that H is abelian, a contradiction. Suppose that $P = \langle a\rangle$, $Q = \langle b\rangle$ and $R = \langle c\rangle$. We can assume that $Z(G) = \langle c\rangle \times \langle b^q\rangle$. Thus G is the group mentioned in (2), which is also a special case of [9, Theorem 3.4].

Case 2. $P_1 < P$. Then there exists a subgroup M of P such that P_1 is maximal in M. As HM is nonabelian, $K = HM$. If $|Q_1| < |Q|$, then $HQ \neq K$ is nonabelian, a contradiction. Thus Q_1 is a Sylow q-subgroup of G. With the same argument, if R is a Sylow subgroup distinct from P and Q, then HR is nonabelian, a contradiction. Hence G has exactly two prime divisors, $|G : K| = p$ and then $|G| = p^3 q^m$. Noticing that H is a normal subgroup of G, it follows that all conjugates of Q_1 are contained in H and so $|G : N_G(Q_1)| = p$. If P is cyclic, then $P_1 \leq N_G(Q_1)$, a contradiction. Therefore P is noncyclic but abelian. If P is elementary abelian, then, by Maschke's theorem, P_1 has an invariant complement, P_2 say, and since P_2Q_1 is distinct from H and K, it is abelian. Now for any two elements x, y, the subgroups $\langle P_1, x\rangle$ and $\langle P_1, y\rangle$ are both nonabelian and containing H, a contradiction. It follows that P is of type (p, p^2). Suppose that $P = \langle a, b\rangle$, where $|a| = p$ and $|b| = p^2$. Since $|G : N_G(Q_1)| = p$ and $a \notin N_G(Q_1)$, $b \in N_G(Q_1)$ and so $[b, Q_1] = 1$. It leads to the conclusion that $G = \langle b\rangle \times H$ and we have $|Z(G)| = p^2q^{m-1}$. Suppose that P_2 is a Sylow p-subgroup of K. By virtue of [18, Theorem E], $P_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $|Z(K)| = pq^{m-1}$. It is readily seen that $\Phi(G) = Z(G) \cap K \subseteq Z(K)$. So, we deduce that $Z(G) \nsubseteq K$ and then $Z(G) \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^{m-1}}$. Assume that $Q_1 = \langle c\rangle$. We may assume $Z(G) = \langle b\rangle \times \langle c^q\rangle$. Thus G is the group mentioned in (3). □

Now we are ready to prove Theorem B.

Proof Theorem B First we prove that all groups (1)-(4) have exactly two nonabelian proper subgroups.

(1) Put $H = \langle a, b^{q^2}\rangle$ and $K = \langle a, b^q\rangle$. We show that H and K are the only nonabelian subgroups of G. To prove this, we show that K is the only nonabelian maximal subgroup of G, H is minimal nonabelian and it is the only nonabelian maximal subgroup of K. Let T be a maximal subgroup of H. As H is supersolvable, the index of T in H is prime. Thus $|T| \in \{q^m, pq^{m-1}\}$. If $|T| = q^m$, then clearly T is abelian. So suppose that $|T| = pq^{m-1}$. As a Sylow q-subgroup of G is cyclic, it has a unique subgroup of order q^{m-1}. On the other hand, according to the presentation of G, $|Z(G)| = q^{m-1}$. Hence $Z(G)$ is a Sylow q-subgroup of T. It follows that T is the direct product of its Sylow subgroups and then it is abelian. Thus H is minimal nonabelian. Similarly one can see that each maximal subgroup of K distinct from H is abelian.
Let M be a maximal subgroup of G distinct from K. We show that M is abelian. Since G is supersolvable, the index of each maximal subgroup is prime. So $|M| = q^{m+2}$ or $|M| = pq^{m+1}$. Suppose $|M| = pq^{m+1}$. Let Q_M, Q_K and Q_G be Sylow q-subgroups of M, K and G, respectively. As Q_G is cyclic, it has a unique subgroup of order q^{m+1} and so $Q_M = Q_K$. It follows that $K = M$, a contradiction. Hence $|M| = q^{m+2}$ and it is clear that M is abelian. therefore H and K are the only nonabelian proper subgroups of G.

(2) Put $H := \langle a, b\rangle$ and $K := \langle a, b, c^r\rangle$. According to [18, Theorem D], H is minimal nonabelian and it is a unique nonabelian proper subgroup of K. From the presentation of G we can see that $|G'| = p$, $|Z(G)| = q^{m-1}r$ and $|F(G)| = pq^{m-1}r^2$. Also clearly $G \cong H \times \mathbb{Z}_{r^2}$ which yields that $|\Phi(G)| = q^{m-1}r$ (by Theorem 2.1, $|\Phi(H)| = q^{m-1}$).
Now we show that K is the only nonabelian maximal subgroup of G. To prove this, let M be a maximal subgroup of G distinct from K. Noticing that G is supersolvable, it follows that the index of each maximal subgroup of G is prime and then $|M| \in \{pq^mr, pq^{m-1}r^2, q^mr^2\}$. First suppose that $|M| = q^mr^2$. Thus $G' \nsubseteq M$ and so $MG' = G$, which implies that M is abelian. Now suppose that $|M| = pq^mr$. Since $G' \subseteq M$, M is normal in G. As $G = MK$, we have $|M \cap K| = pq^m$, which contradicts the fact $q^{m-1}r = |\Phi(G)| \mid |M \cap K| = pq^m$. Finally suppose that $|M| = pq^{m-1}r^2$. If $F(G) \neq M$, then $F(G)M = G$ and so $|F(G) \cap M| = pq^{m-2}r^2$. Thus $q^{m-1}r = |\Phi(G)| \mid |F(G) \cap M| = pq^{m-2}r^2$, which is impossible. Therefore $M = F(G)$ is abelian. Hence H and K are the only nonabelian proper subgroups of G.

(3) Put $H := \langle a, c\rangle$ and $K := \langle a, b^p, c\rangle$. According to [18, Theorem E], H is minimal nonabelian and it is a unique nonabelian proper subgroup of K. From the presentation of G we can see easily that $|G'| = p$

and $G' \subset H$. Also we can see that $G \cong H \times \mathbb{Z}_{p^2}$. According to Theorem 2.1, $|\Phi(H)| = q^{m-1}$ and so $|\Phi(G)| = pq^{m-1}$. Of course, $|F(H)| = pq^{m-1}$.

We show that each maximal subgroup M of G distinct from K is abelian. As G is supersolvable, the index of each maximal subgroup of G is prime and then $|M| \in \{p^3q^{m-1}, p^2q^m\}$. Suppose that $|M| = p^3q^{m-1}$. Since $\Phi(H) \subseteq M$, $\Phi(H)$ is a Sylow q-subgroup of M. So $M = P \times \Phi(H)$, where P is a Sylow p-subgroup of G. It follows that M is abelian. Now suppose that $|M| = p^2q^m$. We claim that M is not normal. Suppose contrary that M is normal. Then, since $MK = G$, $|M \cap K| = pq^m$. If $H \subseteq M$, then $M \cap K = H$. Hence $G/H \cong G/M \times G/K$ and $\Phi(G/H) = \Phi(G/M) \times \Phi(G/K) = 1$, which yields that $\Phi(G) \subseteq H$, whence $\Phi(G) = F(H)$. Now, since $G' \subseteq H$, $G' \subseteq F(H) = \Phi(G)$, which follows that G is nilpotent, a contradiction. Thus $H \not\subseteq M$ and so $HM = G$ and $|H \cap M| = q^m$, whence a Sylow q-subgroup of H is normal in H, a contradiction. Therefore M is nonnormal, which follows that $G' \not\subseteq M$ and so $MG' = G$. Since $M \cap G' = 1$, it follows that M is abelian. We deduce that H and K are the only nonabelian proper subgroups of G.

(4) Put $H := \langle a^2, b\rangle$ and $K := \langle a, b^p\rangle$. We show that H and K are minimal nonabelian. For this aim, we show that each maximal subgroup of H and each maximal subgroup of K is abelian. We can see from the presentation of G that $|Z(H)| = 2^{m-1}$ and $Z(G) \cong Z(H) \times \mathbb{Z}_2$. Let N be a maximal subgroup of H. As H is supersolvable, the index of N in H is prime and so $|N| \in \{p2^{m-1}, 2^m\}$. If $|N| = 2^m$, then N is abelian. So let $|N| = p2^{m-1}$. Since a Sylow 2-subgroup of H is cyclic, it has a unique subgroup of order 2^{m-1}. Thus $Z(H)$ is a Sylow 2-subgroup of N. Let P be a Sylow p-subgroup of N, which is also a Sylow p-subgroup of H. From the presentation of G, we can see that P is normal in G and so in N. Therefore $N = P \times Z(H)$ and so it is abelian. Hence H is minimal nonabelian. Similarly one can see that K is minimal nonabelian.
Now we show that H and K are the only nonabelian proper subgroups of G. For this aim, let M be a maximal subgroup of G distinct from H and K. We show that M is abelian. Clearly $|M| \in \{2^{m+1}, p2^m\}$. If $|M| = 2^{m+1}$, then, according to the presentation of G, M is abelian. So suppose that $|M| = p2^m$. Let Q_M be a Sylow 2-subgroup of M and P be a Sylow p-subgroup of M, which is also a Sylow p-subgroup of G. If Q_M is cyclic, then $M = H$ or $M = K$, a contradiction. If Q_M is noncyclic, then $Q_M \cong Z(H) \times \mathbb{Z}_2 \cong Z(G)$ (by using Theorem 2.1, $\Phi(H) = Z(H) \cong \mathbb{Z}_{2^{m-1}}$), which follows that $M \cong P \times Z(G)$ and then it is abelian. It indicates that H and K are the only nonabelian proper subgroups of G.

Now we prove the converse of Theorem. Let H and K are the only nonabelian proper subgroups of G. First suppose that $H \subseteq K$. By virtue of Proposition 2.5, G is one of the groups (1), (2) or (3).
Now suppose that $H \not\subseteq K$ and $K \not\subseteq H$. We show that G is the group mentioned in (4). By Lemma 2.4, H and K are normal in G. By a simple calculation we obtain that $|H| = |K|$. Since H and K are supersolvable Schmidt groups, using [13, Theorem 5.4.7] and Theorem 2.1 (3), $|H| = |K| = pq^m$, $p > q$ and $q \mid p-1$. Evidently G has two prime divisors and $|G : H| = |G : K| \in \{p, q\}$. If $|G : H| = |G : K| = p$, then $|G| = p^2q^m$. Since $HK = G$, we have $|H \cap K| = q^m$ and it is impossible. Thus $|G : H| = |G : K| = q$ and $|G| = pq^{m+1}$. Suppose P_1 a Sylow p-subgroup of H, Q_1 a Sylow q-subgroup of H, P_2 a Sylow p subgroup of K, Q_2 a Sylow q-subgroup of K and also P a Sylow p-subgroup G and Q a Sylow q-subgroup G. Clearly $Q_1 = Q \cap H$ and $Q_2 = Q \cap K$ are maximal subgroups of Q. If Q has two maximal subgroups distinct from Q_1 and Q_2 as M_1 and M_2, since P_1M_1 and P_1M_2 are abelian, then we arrive to the contradiction $[P_1, Q] = [P_1, M_1M_2] = 1$. It follows that Q has only three maximal subgroups and so $q = 2$. If Q is cyclic, then it has a unique subgroup of order q^m which follows that $H = K$, a contradiction. Hence $Q \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_2$ and $G \cong \mathbb{Z}_2 \times H$, whence

$Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$ (by using Theorem 2.1, $Z(H) \cong \mathbb{Z}_{2^{m-1}}$). Let $P = \langle a \rangle$ and $Q = \langle b \rangle \times \langle c \rangle$. We may assume that $Z(G) = \langle b^2 \rangle \times \langle c \rangle$. Therefore G is the group mentioned in (4). □

In order to prove Theorem C, we prove the following proposition.

Proposition 2.6 Let G be a finite nonsupersolvable group with exactly two nonabelian proper subgroups H and K. If $H \subset K$, then G is isomorphic to one of the following groups

(1) $G = P \rtimes Q$, where P is abelian of type $(p, \ldots, p, p^2)$ of order p^{n+2} and Q is cyclic of order q^m that $q \nmid p-1$. Also $H = P_1 \rtimes Q$ and $K = P_2 \rtimes Q$ are nonabelian subgroups of G that $H \subset K$, H is minimal nonabelian, $|P_1| = p^n$ and $|P_2| = p^{n+1}$. Furthermore P_1 is an irreducible Q-module and P_2 is elementary abelian.

(2) $G = P \rtimes Q$, where P is elementary abelian of order p^n and $Q = \langle c \rangle$ is cyclic of order q^{m+1} that $q \nmid p-1$. Also $H = P \rtimes Q_1$ and $K = P \rtimes Q_2$ are nonabelian subgroups of G that $H \subset K$, H is minimal nonabelian, $|Q_1| = q^m$ and $|Q_2| = q^{m+1}$. Furthermore P is an irreducible $\langle c^{q^2} \rangle$-module.

(3) $G = PQR$, where P is elementary abelian of order p^n, Q is cyclic of order q^m and $R = \langle x \rangle$ is cyclic of order r^2 that $q \nmid p-1$. Also P is an irreducible Q-module. Furthermore $H = PQ$ and $K = PQ\langle x^r \rangle$ are nonabelian subgroups of G and H is minimal nonabelian.

(4) $G = P \rtimes Q$ is a Schmidt group, where P is a special nonabelian p-group of order p^3 and $Q = \langle c \rangle$ is cyclic of order q^2 that $q \nmid p-1$. Also $K = P\langle c^q \rangle$ is nonabelian.

(5) $G = G'Q$, where G' is a nonabelian special p-group of order p^3 and of rank 2, $Q = \langle c \rangle$ is cyclic of order q^2, $q \nmid p-1$, $q \mid p+1$ and the order of p modulo q^2 being 2. Also $G'\langle c^q \rangle$ is a Schmidt group, $G'/\Phi(G')$ is a faithful irreducible Q-module and $[\Phi(G'), Q] = 1$. Furthermore $|G''| = p$.

Proof We consider two cases. In Case 1, we assume that H is nonnilpotent and in Case 2 we assume that H is nilpotent.

Case 1. H is nonnilpotent. So K is nonnilpotent. According to Lemma 2.4, H is minimal nonabelian and so it is a Schmidt group. Let $H = P_1 \rtimes Q_1$ and $|H| = p^n q^m$. If H is supersolvable, then by Theorem 2.1 and [13, Theorem 5.4.7], $|P_1| = p$. Thus we have a cyclic normal series $1 < P_1 < H < K < G$, which implies that G is supersolvable, a contradiction. Thus H is nonsupersolvable and it is the group (1) of Theorem 2.1. Suppose P a Sylow p-subgroup G and Q a Sylow q-subgroup G.

First suppose that $P_1 < P$. With a same argument to the proof of Proposition 2.5, Case 2, we infer that $Q_1 = Q$, $|G|$ has exactly two prime divisors, $|G : K| = p$ and then $|G| = p^{n+1}q^m$. Also P is noncyclic but abelian and it is of type $(p, \ldots, p, p^2)$. Thus G is the group mentioned in (1).

Now suppose that $P_1 = P$. If $|Q_1| < |Q|$, then $K = HQ$ and so G has two prime divisors and $|G : K| = q$. Thus $|G| = p^n q^{m+2}$. Similar to the proof of Proposition 2.5, Case 1, we can see that Q is cyclic. Thus G is the group mentioned in (2).
If $|Q| = |Q_1|$, then $|K : H| = r$, which is a prime distinct from p and q. It is easy to see that $|G : K| = r$ and so $|G| = p^n q^m r^2$. Hence G is the group mentioned in (3), which is also a special case of [9, Theorem 3.4].

Case 2. H is nilpotent. Suppose that K is nilpotent. Then G is a Schmidt group. By Lemma 2.4, H is a normal maximal subgroup of K. So the index of H in K is a prime number. Similar to the proof of Proposition 2.5, we have $|H| = p^n$, $|K : H| = q$ and $|G : K| = q$. Thus $|K| = p^n q$ and $|G| = p^n q^2$. Noticing that H is a nonabelian p-group of G, it follows from Theorem 2.1(2) that H is a nonabelian special p-group, that is $Z(H) = H' = \Phi(H)$. On the other hand, since H is minimal nonabelian, by [10], $|Z(H)| = |\Phi(H)| = p^{n-2}$ and $|H'| = p$. It follows that $|H| = p^3$. So G is the group mentioned in (4).

Now suppose that K is nonnilpotent. Then K is a Schmidt group. Similar as above we have $|H| = p^3$, $|K| = p^3 q$ and $|G| = p^3 q^2$. It is easy to see that G is a 2-$\mathscr{N}$-critical group. So it is group (V) of [19, Theorem B] and clearly $l = 1$. Noticing that the order of p modulo q^2 is $2l = 2$ and $q \nmid p-1$, it follows that $q \mid p+1$. Also, as $|G''| \leq p^l = p$, we have $|G''| = p$. Therefore G is the group mentioned in (5). □

Now we prove Theorem C.

Proof Theorem C First we prove that all groups (1)-(6) have exactly two nonabelian proper subgroups.

(1) We show that H and K are the only nonabelian proper subgroups of G. To prove this, we show that K is the only nonabelian maximal subgroup of G and H is the only nonabelian maximal subgroup of K. Let M be a maximal subgroup of G distinct from K. We have $|M| \in \{p^i q^m, p^{n+2} q^j\}$, where $0 \leq i \leq n+1$ and $0 \leq j \leq m-1$. First assume that $|M| = p^i q^m$. If $H \subset M$, then, as $H \neq M$, $|M| = |K| = p^{n+1} q^m$ (that is $i = n+1$). The maximality of K ensures that $G = MK$ and then $|M \cap K| = p^n q^m$. Since $M \cap K$ is normal in M, a Sylow q-subgroup of M, which is also a Sylow q-subgroup of H, is normal in M and then is normal in H, a contradiction. If $H \nsubseteq M$, then $|H \cap M| = p^{i-2} q^m$ and similarly we obtain a contradiction.
Now assume that $|M| = p^{n+2} q^j$. We have $MH = G$ and then $|H \cap M| = p^n q^j$. As $H \cap M$ is normal in M, a Sylow q-subgroup of M is normal in M. Thus M is the direct product of its Sylow subgroups and then it is abelian. Similarly one can see that each maximal subgroup of K distinct from H is abelian. Therefore H and K are the only nonabelian proper subgroups of G.

(2), (3) With a similar argument to (1), we can see that H and K are the only nonabelian proper subgroups of G.

(4) Put $H := P$. It is clear that H is minimal nonabelian. Since G is a Schmidt group, K is nilpotent and then $K = P \times \langle c^q \rangle$. Immediately $H = P$ is the only nonabelian proper subgroup of K. We show that $K = P \times \langle c^q \rangle$ is the only nonabelian maximal subgroup of G. To prove this, suppose that M is a maximal subgroup of G distinct from K. Then $|M| \in \{pq^2, p^2q^2, p^3q\}$. Assume that $|M| = p^3 q$. As Q is cyclic, it has a unique subgroup of order q. So a Sylow q-subgroup of K equals to a Sylow q-subgroup of M. It follows that $K = M$, a contradiction. Hence $|M| \in \{pq^2, p^2q^2, \}$. Since G is a Schmidt group, each proper subgroup of G is the direct product of its Sylow subgroups. Hence M is abelian. We conclude that H and K are the only nonabelian proper subgroups of G.

(5) According to [19, Theorem B], G is 2-$\mathscr{N}$-critical and $K = G'\langle c^q \rangle$ is a unique nonnilpotent proper subgroup of G. Put $H := G'$. Now similar to (4), we can see that H and K are the only nonabelian proper subgroups of G.

(6) Evidently, H and K are maximal subgroups of G. Since $G' = P$ is contained in H and K, they are normal in G. We show that each maximal subgroup M of G distinct from H and K is abelian. We have $|M| \in \{p^i 2^{m+1}, p^n 2^j\}$, where $0 \leq i \leq n-1$ and $j \in \{m-1, m\}$ (as H is a nonnilpotent minimal nonabelian

group, it is a Schmidt group and, by Theorem 2.1, $|\Phi(H)| = 2^{m-1}$ and so $2^{m-1} \mid |\Phi(G)|$. Thus $2^{m-1} \mid |M|$). First assume that $|M| = p^n 2^j$. If $j = m-1$, then $M = P \times \Phi(H)$, which follows that M is abelian. So let $|M| = p^n 2^m$. If a Sylow q-subgroup of M, Q_M say, is cyclic, then $Q_M = \langle a \rangle$ or $Q_M = \langle a^p \rangle$ and hence $M = H$ or $M = K$, a contradiction. Hence Q_M is noncyclic. Since $\Phi(H) \subseteq Q_M$, we have $Q_M = \Phi(H) \times \langle b \rangle$. Therefore $Q_M = Z(G)$ and so $M = P \times Z(G)$, which follows that M is abelian. Now assume that $|M| = p^i 2^{m+1}$. Let P_M be a Sylow p-subgroup of M. The maximality of M ensures that $MG' = G$. So $|G' \cap M| = p^i$. As $G' \cap M$ is a normal subgroup of G contained in H and P is an irreducible $\mathbb{Z}_{2^m}$-module, so $i = 0$ and then $|M| = 2^{m+1}$. Thus M is abelian, which follows that H and K are the only nonabelian proper subgroups of G.

Now we prove the converse of Theorem. Let H and K be the only nonabelian proper subgroups of G. First suppose that $H \subseteq K$. According to Proposition 2.6, G is one of the groups (1)-(5).
Now suppose that $H \nsubseteq K$ and $K \nsubseteq H$. By Lemma 2.4, H and K are normal in G and they are minimal nonabelian. As H and K are nonnilpotent, they are Schmidt groups and since they are nonsupersolvable, by Theorem 2.1(1), $|H| = |K| = p^n q^m$ where $n > 1$ and $q \nmid p-1$. With a same argument to the proof of Theorem B, we conclude that $|G:H| = |G:K| = q = 2$ and $|G| = p^n 2^{m+1}$. Also a Sylow q-subgroup of G, Q say, is isomorphic to $\mathbb{Z}_{2^m} \times \mathbb{Z}_2$ and $G \cong \mathbb{Z}_2 \times H$. It follows that $Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$. Suppose that $Q \cong \langle a \rangle \times \langle b \rangle$, $|a| = 2^m$ and $|b| = 2$. If $n > 2$, then $P\langle a^{p^i} \rangle$, where $1 \leq i \leq n-1$, are nonabelian subgroups of G, a contradiction. Therefore $n = 2$ and G is the group mentioned in (6). $\square$

References

[1] Baer R. Supersoluble immersion. Canadian Journal of Mathematics 1959; 11: 353-369. doi:10.4153/CJM-1959-036-2

[2] Ballester-Bolinches A and Esteban-Romero R. On minimal non-supersoluble group. Revista matemática Iberoamericana 2007; 23 (1): 127-142.

[3] Berkovich Y. Groups of prime power order. Vol. 2 Walter de Gruyter de Gruyter, 2008.

[4] Berkovich Y. On abelian subgroups of p-groups. Jornal of Algebra 1998; 199: 262-280.

[5] Doerk K, Hawkes T. Finite soluble groups. Berlin: de Gruyter Expositions in Mathematics, 4 (Walter de Gruyter & Co., Berlin), 1992.

[6] Fouladi S. Pairwise non-commuting elements in finite metacyclic 2-groups and some finite p-groups. Bulletin of Iranian Mathematical Society 2014; 40 (6): 1573-1585. doi:10.1007/s41980-020-00473-8

[7] Fouladi S, Orfi R. The automorphism group of a finite minimal non-abelian p-group. Mathematical Reports 2014; 16 (1): 133-139. doi:10.1142/S0219498813501508

[8] Itô N. Note on (LM)-groups of finite order. Technical report, Kodai Mathematical Seminar Reports 1951.

[9] Meng W. Finite Groups with Few Non-Abelian Subgroups. Communications in Algebra 2015; 43: 909-915. doi:10.1080/00927872.2013.851208

[10] Miller GA, Moreno HC. Non-abelian groups in which every subgroup is abelian. Transactions of the American Mathematical Society 1903; 4 (4): 398-404.

[11] Rédei L, Das “Schiefe Produkt“ in der Gruppentheorie. Commentarii Mathematici Helvetici 1947; 20: 225–264.

[12] Rédei L. Die endlichen einstufig nichtnilpotenten Gruppen. Publicationes Mathematicae Debrecen 1956; 4: 303-324.

[13] Robinson DJS. A course in the theory of groups, 2nd ed., New York: Springer-Verlag, 1996.

[14] Rose JS. The influence on a finite group of its proper abnormal structure. Journal of the London Mathematical Society 1965; 40: 348-361. doi:10.1112/JLMS/S1-40.1.348

[15] Russo FG. Minimal non-nilpotent groups which are supersolvable. JP Journal of Algebra, Number Theory and Applications 2010; 18: 79-88.

[16] Schmidt OYu. Groups whose all subgroups are special. Matematicheskii Sbornik 1924; 31: 366-372.

[17] Shi JT, Zhang C. Some sufficient conditions on the number of non-abelian subgroups of a finite group to be solvable. Acta Mathematica Sinica (English Series) 2011; 27 (5): 891-896. doi:10.1007/s10114-011-9476-1

[18] Taeri B, Tayanloo-Beyg F. Finite groups with a unique non-abelian proper subgroup. Journal of Algebra and its Applications 2019; 18 (11): 1950210. doi: 10.1142/S0219498819502104

[19] Taeri B, Tayanloo-Beyg F. Characterization of finite groups with a unique non-nilpotent proper subgroup. International Journal of Group Theory 2021; 10 (1): 47-53. doi: 10.22108/ijgt.2019.116209.1543

[20] Zarrin M. A generalization of Schmidt's Theorem on groups with all subgroups nilpotent. Archiv der Mathematik 2012; 99: 201-206. doi:10.1007/s00013-012-0411-1

[21] Zarrin M. On solvability of groups with a few non-cyclic subgroups. Algebra Colloquium 2016; 23 (1): 105-110. doi:10.1142/S1005386716000134