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Global existence and blow-up of solutions for parabolic equations involving the Laplacian under nonlinear boundary conditions

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Abstract: This paper is concerned with the existence and blow-up of solutions to the following linear parabolic equation: $u_t - \Delta u + u = 0$ in $\Omega \times (0,T)$, under nonlinear boundary condition in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary. We obtain a threshold result for the global existence of solutions, next we shall prove the existence time T of solution is finite when the initial energy satisfies certain condition.

Key words: Parabolic problem, heat equation, nonlinear boundary condition, blow-up

1. Introduction and main results

In the present paper, we study the following parabolic problem with nonlinear boundary conditions:

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{p-1} u & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, u_t denote the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x, $\lambda > 0$, and p satisfies

$$(H) \quad \begin{cases} 1 \leq p \leq \frac{n}{n-2} & \text{if } n > 2, \\ 1 \leq p < \infty & \text{if } n \leq 2. \end{cases}$$

Define the functions

$$E(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{\lambda}{p+1}\|u\|_{p+1,\partial\Omega}^{p+1}, \tag{1.2}$$

and

$$F(u) = \|u\|_{H^1}^2 - \lambda \|u\|_{p+1,\partial\Omega}^{p+1}. \tag{1.3}$$

The parabolic problems are fundamental to the modeling of space and time-dependent problems such as problems from physics or biology. To be specific, evolutionary equations and systems are likely to be used to model physical

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processes like heat conduction or diffusion processes. Let us take as an illustration the Navier-Stokes equation, the basic equation in fluid mechanics. Moreover, we would like to refer to [5], where fluids in motion are studied. Applications involve climate modeling and climatology as well (see [3, 4]).

The global existence, nonexistence and blow-up of solutions for semilinear parabolic equations with Dirichelet boundary conditions were studied by several authors, for example in [6], Fujita considered the following semilinear problem:

$$(P_1)\quad \begin{cases} u_t - \Delta u = |u|^p & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x;0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

and obtained the following results

(a) When $1 < p < 1 + 2/n$ and $u_0 \geqslant 0$, problem (P_1) possesses no global positive solution.

(b) When $p > 1 + 2/n$ and u_0 is smaller than a small Gaussian, then (P_1) has global positive solutions.

The above problem was investigated by Zhang in [14], the author has proved under certain conditions the following conclusions

(a) If $p < 1 + 2/n, \int_{\mathbb{R}^n} u_0(x)\mathrm{d}x > 0$, then (P_1) has no global solutions.

(b) If $p = 1 + 2/n$, and $\int_{\mathbb{R}^n} u_0(x)\mathrm{d}x > 0$, then (P_1) has no global solutions provided that u_0^- is compactly supported.

(c) If $p > 1 + 2/n$, then (P_1) has global solutions for some u_0.

(d) For any $p > 1$, there exists u_0 satisfying $\int_{\mathbb{R}^n} u_0(x)\mathrm{d}x < 0$ such that (P_1) has global solutions.

The semilinear parabolic problem with Dirichelet boundary conditions of the following form

$$(P_2)\quad \begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

was studied by Filippo Gazzola and Tobias Weth [7], they analyzed the behavior of the solutions when the initial data varies in the phase space $H_0^1(\Omega)$, the authors obtained both global solutions and finite time blow-up solutions. Their main tools are the comparison principle and variational methods (see also [2, 11, 13]).

For more general Laplace equation, the authors of [12] established the existence and uniqueness results for the following problem:

$$(P_3)\quad \begin{cases} u_t - \Delta u = g(u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a C^1 -function satisfies $g(0) = 0$ and $g'(0) < \lambda_{\min}$. Here, $\lambda_{\min}$ denotes the smallest eigenvalue of $-\Delta$.

In the present work, we will study the existence and blow-up of the solutions to problem (1.1). We will, in this paper, adopt a method of approximation to obtain the existence of the solution. Moreover, we show that this solution explodes in finite time in the following sense

Definition 1.1 *Let u be a weak solution of problem (1.1). We call u a blowup in finite time if the maximal existence time T is finite and*

$$\lim_{t\to T}\int_0^T \|u\|^2 \mathrm{d}\tau = +\infty.$$

More precisely, we have the following results.

Theorem 1.2 *Let p satisfy* (H)*, $u_0(x) \in H^1(\Omega)$. Assume that $E(u_0) < k, F(u_0) > 0$. Then, there exists a global weak solution $u(t) \in L^\infty\left(0,\infty; H^1(\Omega)\right) \cap C\left([0,T]; L^2(\Omega) \times L^2(\partial\Omega,\rho)\right)$ of (1.1) with $u_t(t) \in L^2\left(0,\infty; L^2(\Omega)\right)$ and $u(t) \in X$ for $0 \le t < \infty$.*

Theorem 1.3 *Let p satisfy (H), $u_0(x) \in H^1(\Omega)$. Then the weak solution $u(x,t)$ of problem (1.1) must blows up in finite time provided that:*

$$0 < E(u_0) < \frac{p-1}{2A(p+1)}\|u_0\|^2, \tag{1.4}$$

where

$$A = \underset{u\in H^1(\Omega)}{Sup}\frac{\|u\|^2}{\|u\|^2_{H^1}} \ . \tag{1.5}$$

This article is divided into three sections. In Section 2, we present the basic preliminary results and give some definitions. The proofs of Theorems (1.2) and (1.3) are presented in Section 3.

2. Preliminaries

We shall denote by $d\rho$ the restriction to $\partial\Omega$. The Lebesgue norm of $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$, and the Lebesgue norm of $L^q(\partial\Omega,\rho)$ by $\|\cdot\|_{q,\partial\Omega}$, for $q \in [1,\infty]$, and $\|u\|_{L^2(\Omega)} = \|u\|$. The scalar product of $L^2(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$ and the scalar product of $L^2(\partial\Omega,\rho)$ will be denoted by $\langle\cdot,\cdot\rangle_0$:

$$\langle u,v\rangle = \int_\Omega uv\ dx, \quad \langle u,v\rangle_0 = \oint_{\partial\Omega} uv\ d\rho.$$

We denote the usual Sobolev space on Ω

$$H^1(\Omega) = \left\{u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega)\right\},$$

equipped by the norm

$$\|u\|^2_{H^1} = \|u\|^2 + \|\nabla u\|^2.$$

Throughout this paper, we denote

$$p^\partial = \begin{cases} \frac{p(n-1)}{n-p} & \text{if } 1 < p < n, \\ \infty & \text{if } \ p \ge n. \end{cases}$$

Proposition 2.1 *(See [1])*
The trace operator $W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ is continuous if and only if $1 \leq q \leq p^\partial$ if $p \neq n$ and for $1 \leq q < \infty$ if $p = n$. Especially for $q = 2$, the trace operator is well-defined and continuous under the following condition:

$$W^{1,p}(\Omega) \to L^2(\partial\Omega, \rho) \Longleftrightarrow p \geq p_1 := \frac{2n}{n+1}.$$

Let us introduce some functionals and sets as follows:

$$Y = \left\{u \in H^1(\Omega) \mid F(u) = 0, \|u\|_{H^1} \neq 0\right\},$$

$$X = \left\{u \in H^1(\Omega) \mid F(u) > 0,\ E(u) < k\right\} \cup \{0\},$$

where

$$k = \inf_{u \in Y} E(u).$$

For $\sigma > 0$, we further define

$$F_\sigma(u) = \sigma\|u\|_{H^1}^2 - \lambda\|u\|_{p+1,\partial\Omega}^{p+1},$$

$$k(\sigma) = \inf_{u \in Y_\sigma} E(u),$$

where

$$Y_\sigma = \left\{u \in H^1(\Omega) \mid F_\sigma(u) = 0, \|u\|_{H^1} \neq 0\right\}.$$

$$X_\sigma = \left\{u \in H^1(\Omega) \mid F_\sigma(u) > 0,\ E(u) < k(\sigma)\right\} \cup \{0\}.$$

we end this section with the definition of the weak solution of problem (1.1).

Definition 2.2 *A function $u \in L^\infty\left(0, T; H^1(\Omega)\right) \cap C\left([0, T]; L^2(\Omega) \times L^2(\partial\Omega, \rho)\right)$ with $u_t \in L^2\left(0, T; L^2(\Omega)\right)$ is said to be a weak solution of problem (1.1) on $\Omega \times [0, T)$ if the following conditions are satisfied:*

$$(i) \quad \langle u_t, v\rangle + \langle \nabla u, \nabla v\rangle + \langle u, v\rangle = \lambda\langle |u_{|\partial\Omega}|^{p-1}\ u_{|\partial\Omega}, v\rangle_0, \quad \forall v \in H^1(\Omega), t \in [0, T), \tag{2.1}$$

$$(ii) \quad u(x, 0) = u_0(x) \quad in \quad H^1(\Omega),$$

$$(iii) \quad \int_0^t \|u_\tau\|^2\ \mathrm{d}\tau + E(u) \leq E(u_0), \quad \forall t \in [0, T), \tag{2.2}$$

where $u_{|\partial\Omega}$ denotes the trace of u on $\partial\Omega$.

3. Proof of main results

3.1. Proof of Theorem 1.1

To derive the Theorem (1.2) we need the following results.

Lemma 3.1 *Assume that $0 < E(u) < k$ for some $u \in H^1(\Omega)$, and $\sigma_1 < \sigma_2$ are the two roots of equation $k(\sigma) = E(u)$. Then the sign of $F_\sigma(u)$ does not change for $\sigma_1 < \sigma < \sigma_2$.*

Proof Arguing by contradiction, we assume that the sign of $F_\sigma(u)$ is changeable for $\sigma_1 < \sigma < \sigma_2$, then there exist a $\sigma_0 \in (\sigma_1, \sigma_2)$ such that $F_{\sigma_0}(u) = 0$. Since $E(u) > 0$ implies $\|u\|_{H^1} \neq 0$, hence $u \in Y_{\sigma_0}$, thus we have $E(u) \geq k(\sigma_0)$, which contradicts

$$E(u) = k(\sigma_1) = k(\sigma_2) < k(\sigma_0).$$

□

Lemma 3.2 *Let p satisfy* (H), $u_0(x) \in H^1(\Omega), 0 < c < k,\ \sigma_1 < \sigma_2$ *be the two roots of equation* $k(\sigma) = c$. *Then all weak solutions u of problem* (1.1) *with* $E(u_0) = c$ *belong to* X_σ *for* $\sigma_1 < \sigma < \sigma_2$, $0 \leq t < T$, *provided* $F(u_0) > 0$.

Proof First from $E(u_0) = c, F(u_0) > 0$ and Lemma (3.1) it follows that $F_\sigma(u_0) > 0$ and $E(u_0) < k(\sigma)$ i.e. $u_0(x) \in X_\sigma$ for $\sigma_1 < \sigma < \sigma_2$. Next, let $u(t)$ be any weak solution of problem (1.1) with $E(u_0) = c$ and $F(u_0) > 0$, and T be the maximal existence time of $u(t)$. Arguing by contradiction, we assume that there exist a $\sigma_0 \in (\sigma_1, \sigma_2)$ and $t_0 \in (0, T)$ such that $F_{\sigma_0}(u(t_0)) = 0, \|u(t_0)\|_{H^1} \neq 0$ or $E(u(t_0)) = k(\sigma_0)$. By (2.2), we can deduce

$$\int_0^t \|u_\tau\|^2 \, \mathrm{d}\tau + E(u) \leq E(u_0) < k(\sigma), \quad \sigma_1 < \sigma < \sigma_2,\ 0 \leq t < T, \tag{3.1}$$

therefore $E(u(t_0)) \neq k(\sigma_0)$. If $F_{\sigma_0}(u(t_0)) = 0, \|u(t_0)\|_{H^1} \neq 0$, thus the definition of $k(\sigma)$ implies that $E(u(t_0)) \geq k(\sigma_0)$, which contradicts (3.1). □

Proof of Theorem (1.2). We start by constructing a sequence such that its limit equal to the solution in (1.1). Let $\{\xi_j(x)\}$ be a system of base functions in $H^1(\Omega)$, define the approximate solution to (1.1) as follows:

$$u_m(x,t) = \sum_{j=1}^{m} c_{jm}(t)\xi_j(x), \quad m = 1, 2, \dots$$

satisfying

$$\langle u_{mt}, \xi_s\rangle + \langle \nabla u_m, \nabla \xi_s\rangle + \langle u_m, \xi_s\rangle = \lambda \langle |u_{m_{|\partial\Omega}}|^{p-1} u_{m_{|\partial\Omega}}, \xi_s\rangle_0\ , \quad s = 1, 2, \dots, m \tag{3.2}$$

$$u_m(x,0) = \sum_{i=1}^{m} a_{jm}\xi_j(x) \to u_0(x) \quad \text{in } H^1(\Omega). \tag{3.3}$$

Multiplying (3.2) by u_{mt} and integrating on Ω implies

$$\|u_{mt}\|_2^2 = -\frac{1}{2}\frac{d}{dt}\|u_m\|_{H^1}^2 + \frac{\lambda}{p+1}\frac{d}{dt}\|u_m\|_{p+1,\partial\Omega}^{p+1},$$

which together with (1.2) gives

$$\frac{d}{dt}E(u_m) = -\|u_{mt}\|_2^2 \quad \text{for } t \in [0, r), \tag{3.4}$$

hence

$$\int_0^t \|u_{m\tau}\|^2 \, \mathrm{d}\tau + E(u_m) = E(u_m(0)), \quad 0 \leq t < \infty.$$

From (3.3), we have $E(u_m(0)) \to E(u_0)$, thus

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau + E(u_m) < k, \quad 0 \le t < \infty, \tag{3.5}$$

from (3.5) and an argument similar to that in the proof of Lemma (3.2) we can show that $u_m(t) \in X$ for $0 \le t < \infty$ and sufficiently large m.
Furethermore, by (3.5) and

$$E(u_m) = \frac{p-1}{2(p+1)} \|u_m\|_{H^1}^2 + \frac{1}{p+1} F(u_m),$$

we get

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau + \frac{p-1}{2(p+1)} \|u_m\|_{H^1}^2 < k, \quad 0 \le t < \infty, \tag{3.6}$$

for sufficiently large m, thus

$$\|u_m\|_{H^1}^2 < \frac{2(p+1)}{p-1} k, \quad 0 \le t < \infty\ . \tag{3.7}$$

Moreover, the trace operator $H^1(\Omega) \to L^{p+1}(\partial\Omega, \rho)$ is continuous, then there exist a constant $C^* > 0$ such that

$$\|u\|_{p+1,\partial\Omega}^{p+1} \le C_*^{p+1} \|u\|_{H^1}^{p+1} \le C_*^{p+1} \left(\frac{2(p+1)}{p-1} k\right)^{\frac{p+1}{2}}, \quad 0 \le t < \infty, \tag{3.8}$$

$$\left\| |u_m|^{p-1} u_m \right\|_{r,\partial\Omega}^r = \|u_m\|_{p+1,\partial\Omega}^{p+1} \le C_*^{p+1} \left(\frac{2(p+1)}{p-1} k\right)^{\frac{p+1}{2}}, \quad r = \frac{p+1}{p}, 0 \le t < \infty, \tag{3.9}$$

$$\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau < k, \quad 0 \le t < \infty, \tag{3.10}$$

then there exists a u and a subsequence $\{u_v\}$ of $\{u_m\}$ such that

$u_v \to u$ in $L^\infty(0,\infty; H^1(\Omega)) \cap C([0,T]; L^2(\Omega) \times L^2(\partial\Omega,\rho))$ weak star and a.e. in $\Omega \times [0,\infty)$,
$|u_v|^{p-1} u_v \to |u|^{p-1}u$ in $L^\infty(0,\infty; L^r(\Omega)) \cap C([0,T]; L^2(\Omega) \times L^2(\partial\Omega,\rho))$ weak star,
$u_{vt} \to u_t$ in $L^2(0,\infty; L^2(\Omega))$ weakly.

In (3.2) for fixed s, we may now pass to the limit as $m = v \to \infty$ in every term of (3.2), thus

$$\langle u_t, \xi_s \rangle + \langle \nabla u, \nabla \xi_s \rangle + \langle u, \xi_s \rangle = \lambda \langle |u_{|\partial\Omega}|^{p-1} u_{|\partial\Omega}, \xi_s \rangle_0, \quad \forall v \in H^1(\Omega), \quad \forall s$$

and (2.1). On the other hand, (3.3) gives $u(x,0) = u_0(x)$ in $H^1(\Omega)$. Finally, from Lemma (3.2), we obtain $u(t) \in X$ for $0 \le t < \infty$.

3.2. Proof of Theorem 1.2

In order to prove our main result, we will give the following lemma.

Lemma 3.3 *[8] Suppose that a positive, twice-differentiable function $\theta(t)$ satisfies the inequality*

$$\theta^{''}(t)\theta(t)-(1+\beta)(\theta^{'}(t))^2\geq 0, \quad t>0,$$

where $\beta>0$ is some constant. If $\theta(0)>0$ and $\theta^{'}(0)>0$, then exists $0<t_1\leq \frac{\theta(0)}{\beta\theta^{'}(0)}$ such that $\lim\limits_{t\to t_1}\theta(t)=+\infty$.

Proof of Theorem (1.3). In order to prove that the solution blows up in finite time under condition (1.4), we assume the solution $u(x,t)$ is global to get a contradiction.
Multiplying the first equation of (1.1) by u_t and integrating on Ω,

$$\int_\Omega |u_t|^2dx-\int_\Omega \Delta u.u_tdx+\int_\Omega u.u_tdx=0,$$

$$\int_\Omega |u_t|^2dx=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}.u_td\rho-\int_\Omega \nabla u.\nabla u_tdx-\int_\Omega u.u_tdx,$$

then

$$\|u_t\|^2=-\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^1}+\frac{\lambda}{p+1}\frac{d}{dt}\|u\|^{p+1}_{p+1,\partial\Omega}. \tag{3.11}$$

From (1.2), we get

$$\frac{d}{dt}E(u)=-\|u_t\|^2\leq 0 \text{ for } t\in[0,T), \tag{3.12}$$

by virtue of (1.2) and (1.3) we can deduce

$$E(u_0)=\frac{p-1}{2(p+1)}\|u_0\|^2_{H^1}+\frac{1}{p+1}F(u_0), \tag{3.13}$$

by (1.5)

$$E(u_0)\geq\frac{p-1}{2A(p+1)}\|u_0\|^2+\frac{1}{p+1}F(u_0),$$

which combines (1.4) to give $F(u_0)<0$.
The following task is to claim that $F(u)<0$ for all $t\in[0,T)$. If it is false, by the continuity of $F(t)$ in t there exists a first time $t_0\in(0,T)$ such that

$$F(u(t))<0 \text{ for } t\in[0,t_0), \tag{3.14}$$

and

$$F(u(t_0))=0, \tag{3.15}$$

this inequality along with (1.2) and (1.5) given

$$\begin{aligned} E(u(t_0)) &= \frac{p-1}{2(p+1)}\|u(t_0)\|_{H^1}^2 + \frac{1}{p+1}F(u(t_0)) \\ &= \frac{p-1}{2(p+1)}\|u(t_0)\|_{H^1}^2 \\ &\geq \frac{p-1}{2A(p+1)}\|u(t_0)\|^2. \end{aligned}$$

Multiplying the first equation in (1.1) by u and integrating over $\Omega \times [0,t]$ gives

$$\frac{1}{2}\|u\|^2 - \frac{1}{2}\|u_0\|^2 + \int_0^t \left(\|u\|_{H^1}^2 - \lambda\|u\|_{p+1,\partial\Omega}^{p+1}\right) d\tau = 0,$$

combining (1.3) and (3.14) we obtain

$$\frac{d}{dt}\|u(t)\|^2 = -2F(u) > 0 \text{ for } t \in [0, t_0), \tag{3.16}$$

then

$$\|u_0\|^2 < \|u(t_0)\|^2, \tag{3.17}$$

from (3.12) we can deduce

$$E(u(t_0)) \leq E(u_0). \tag{3.18}$$

Hence by (3.18) and (1.4),

$$\frac{p-1}{2A(p+1)}\|u(t_0)\|^2 \leq E(u(t_0)) \leq E(u_0) < \frac{p-1}{2A(p+1)}\|u_0\|^2,$$

which contradicts (3.17). Then $F(u) < 0$ for all $t \in [0,T)$.
Next, we discuss the following two cases.

Case I: $E(u(t_0)) < 0$ for some $t_0 > 0$
Put $v(x,t) := u(x, t+t_0)$, we get $E(v_0) = E(u(t_0)) < 0$.
By (3.12) we deduce

$$E(v) \leq E(v_0) < 0. \tag{3.19}$$

Now, we define

$$\Phi(t) := \int_0^t \|v\|^2 d\tau,$$

then

$$\Phi'(t) = \|v\|^2,$$

by (3.16) we have

$$\Phi''(t) = \frac{d}{dt}\|v\|^2 = -2F(v) \geq 0.$$

Next, integrating (3.12) results in

$$E(v)+\int_0^t \|v_\tau\|^2 \, d\tau = E(v_0), \tag{3.20}$$

since

$$E(v)=\frac{p-1}{2(p+1)}\|v\|_{H^1}^2+\frac{1}{p+1}F(v),$$

we can deduce

$$\Phi''(t) \geq 2(p+1)\int_0^t \|v_\tau\|^2 \, \mathrm{d}\tau + (p-1)\ \Phi'(t) - 2(p+1)E\left(v_0\right),$$

and further

$$\begin{aligned}\Phi(t)\Phi''(t)-\frac{p+1}{2}(\Phi'(t))^2 \geq & 2(p+1)\left(\int_0^t \|v\|^2 \, \mathrm{d}\tau \int_0^t \|v_\tau\|^2 \, \mathrm{d}\tau - \left(\int_0^t (v_\tau, v)\, \mathrm{d}\tau\right)^2\right) \\ & + (p-1)\Phi(t)\Phi'(t) - (p+1)\left\|v_0\right\|^2 \Phi'(t) - 2(p+1)E\left(v_0\right)\Phi(t).\end{aligned}$$

Making use of the Schwartz inequality, we get

$$\Phi(t)\Phi''(t)-\frac{p+1}{2}(\Phi'(t))^2 \geq (p-1)\Phi(t)\Phi'(t) - (p+1)\left\|v_0\right\|^2 \Phi'(t) - 2(p+1)E\left(v_0\right)\Phi(t),$$

hence, by $E\left(v_0\right) \leq 0$ it follows that

$$\Phi\Phi''-\frac{p+1}{2}(\Phi')^2 \geq (p-1)\Phi\Phi' - (p+1)\left\|v_0\right\|^2 \Phi' \ .$$

Since $\Phi''(t)>0$ for $t \geq 0$, then Φ'' is a convex function, therefore there exists a $t_0 \geq 0$ such that $\Phi'\left(t_0\right)>0$ and $\Phi(t) \geq \Phi'\left(t_0\right)\left(t-t_0\right)$ for $t \geq t_0$.
Thus, for sufficiently large t, we have

$$(p-1)\ \Phi > (p+1)\left\|v_0\right\|^2,$$

and

$$\Phi\Phi''-\frac{p+1}{2}(\Phi')^2 > 0.$$

Therefore, we can apply Lemma (3.3) to conclude that $\Phi(t) \to +\infty$, as $t \to t^* \leq \frac{2\Phi(0)}{(p-1)\Phi'(0)}$.

Case II: $E(u) \geq 0$ for all $t>0$

From (3.16), we can write

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= -2F(u) \\ &= -2\left(\|u\|_{H^1}^2 - \lambda\|u\|_{p+1,\partial\Omega}^{p+1}\right) \\ &= -4\left(\frac{1}{2}\|u\|_{H^1}^2 - \frac{\lambda}{p+1}\|u\|_{p+1,\partial\Omega}^{p+1}\right) + \left(2\lambda - \frac{4\lambda}{p+1}\right)\|u\|_{p+1,\partial\Omega}^{p+1}\end{aligned}$$

$$= -4E(u) + \frac{2\lambda(p-1)}{p+1}\|u\|_{p+1,\partial\Omega}^{p+1}. \tag{3.21}$$

By $F(u) < 0$ and (1.5), we know that

$$\lambda\|u\|_{p+1,\partial\Omega}^{p+1} > \|u\|_{H^1}^2 \geq \frac{1}{A}\|u\|^2. \tag{3.22}$$

Put

$$1 < \alpha < \frac{(p-1)\|u_0\|^2}{2A(p+1)E(u_0)}. \tag{3.23}$$

Combining (3.21) and (3.20) for $E(u) \geq 0$ yields

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= 4(\alpha-1)E(u) - 4\alpha E(u) + \frac{2\lambda(p-1)}{p+1}\|u\|_{p+1,\partial\Omega}^{p+1} \\ &\geq -4\alpha E(u_0) + 4\alpha\int_0^t \|u_\tau\|^2\, d\tau + \frac{2\lambda(p-1)}{p+1}\|u\|_{p+1,\partial\Omega}^{p+1}\ ,\end{aligned} \tag{3.24}$$

Substituting (3.22) into (3.24), we get

$$\frac{d}{dt}\|u\|^2 > -4\alpha E(u_0) + 4\alpha\int_0^t \|u_\tau\|^2\, d\tau + \frac{2(p-1)}{A(p+1)}\|u\|^2, \tag{3.25}$$

$$\frac{d}{dt}\|u\|^2 - \frac{2(p-1)}{A(p+1)}\|u\|^2 > -4\alpha E(u_0)\,,$$

hence

$$\|u\|^2 > \|u_0\|^2 e^{\frac{2(p-1)}{A(p+1)}t} + \frac{2\alpha A(p+1)}{p-1}E(u_0)\left(1 - e^{\frac{2(p-1)}{A(p+1)}t}\right). \tag{3.26}$$

Now, let

$$\varphi(t) := \int_0^t \|u\|^2 d\tau \text{ for } t \in [0, +\infty),$$

then

$$\varphi'(t) = \|u\|^2, \tag{3.27}$$

and

$$\varphi''(t) = \frac{d}{dt}\|u\|^2.$$

Let us introduce the auxiliary function

$$\Gamma(t) = (\varphi(t))^2 + \varepsilon^{-1}\|u_0\|^2\varphi(t) + a, \tag{3.28}$$

where

$$0 < \varepsilon < \frac{1}{2\alpha\|u_0\|^2}\left(\frac{2(p-1)}{A(p+1)}\|u_0\|^2 - 4\alpha E(u_0)\right), \tag{3.29}$$

and

$$a > \frac{1}{4}\varepsilon^{-2}\left\|u_0\right\|^4. \tag{3.30}$$

Substituting (3.26) into (3.25), we get

$$\varphi''(t) > 4\alpha\int_0^t \left\|u_\tau\right\|^2 d\tau + \left(\frac{2(p-1)}{A(p+1)}\left\|u_0\right\|^2 - 4\alpha E\left(u_0\right)\right)e^{\frac{2(p-1)}{A(p+1)}t},$$

by (3.30), we deduce

$$\begin{aligned}\varphi''(t) &> 4\alpha\int_0^t \left\|u_\tau\right\|^2 d\tau + 2\alpha\varepsilon\left\|u_0\right\|^2 \\ &:= \mathbb{M}(t)\end{aligned} \tag{3.31}$$

By (3.27) , the Young's and Hölder's inequalities imply

$$\begin{aligned}\left(\varphi'(t)\right)^2 =& \|u(t)\|^4 \\ =& \left(\left\|u_0\right\|^2 + 2\int_0^t\int_\Omega u(\tau)u_\tau(\tau)dxd\tau\right)^2 \\ \leq& \left(\left\|u_0\right\|^2 + 2\left(\int_0^t \|u(\tau)\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \left\|u_\tau(\tau)\right\|^2 d\tau\right)^{\frac{1}{2}}\right)^2 \\ =& \left\|u_0\right\|^4 + 4\left\|u_0\right\|^2\left(\int_0^t \|u(\tau)\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \left\|u_\tau(\tau)\right\|^2 d\tau\right)^{\frac{1}{2}} \\ &+ 4\varphi(t)\int_0^t \left\|u_\tau(\tau)\right\|^2 d\tau \\ \leq& \left\|u_0\right\|^4 + 4\varphi(t)\int_0^t \left\|u_\tau(\tau)\right\|^2 d\tau + 2\varepsilon\left\|u_0\right\|^2\varphi(t) \\ &+ 2\varepsilon^{-1}\left\|u_0\right\|^2\int_0^t \left\|u_\tau(\tau)\right\|^2 d\tau \\ :=& \mathbb{N}(t)\end{aligned} \tag{3.32}$$

By (3.28), we obtain

$$\Gamma'(t) = \left(2\varphi(t) + \varepsilon^{-1}\left\|u_0\right\|^2\right)\varphi'(t), \tag{3.33}$$

and

$$\Gamma''(t) = \left(2\varphi(t) + \varepsilon^{-1}\left\|u_0\right\|^2\right)\varphi''(t) + 2\left(\varphi'(t)\right)^2. \tag{3.34}$$

From (3.30) , we can set $h = 4a - \varepsilon^{-2} \|u_0\|^4 > 0$.
By (3.33), we obtain

$$\begin{aligned}
(\Gamma'(t))^2 &= \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right)^2 (\varphi'(t))^2 \\
&= \left(4\varphi^2(t) + 4\varepsilon^{-1} \|u_0\|^2 \varphi(t) + \varepsilon^{-2} \|u_0\|^4\right) (\varphi'(t))^2 \\
&= \left(4\varphi^2(t) + 4\varepsilon^{-1} \|u_0\|^2 \varphi(t) + 4a - h\right) (\varphi'(t))^2 \\
&= (4\Gamma(t) - h) (\varphi'(t))^2 .
\end{aligned} \tag{3.35}$$

The above equality yields

$$4\Gamma(t) (\varphi'(t))^2 = (\Gamma'(t))^2 + h (\varphi'(t))^2 , \tag{3.36}$$

from (3.34) we get

$$\begin{aligned}
2\Gamma(t)\Gamma''(t) =& 2 \left(\left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \varphi''(t) + 2 (\varphi'(t))^2\right) \Gamma(t) \\
=& 2 \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \varphi''(t)\Gamma(t) + 4 (\varphi'(t))^2 \Gamma(t) \\
=& 2 \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \varphi''(t)\Gamma(t) + (\Gamma'(t))^2 + h (\varphi'(t))^2 .
\end{aligned} \tag{3.37}$$

By (3.37) ,(3.35) ,(3.31) and (3.32)

$$\begin{aligned}
2\Gamma(t)\Gamma''(t) - (1+\alpha) (\Gamma'(t))^2 =& 2 \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \varphi''(t)\Gamma(t) - \alpha(4\Gamma(t) - h) (\varphi'(t))^2 + h (\varphi'(t))^2 \\
=& 2 \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \varphi''(t)\Gamma(t) - 4\alpha\Gamma(t) (\varphi'(t))^2 + h(1+\alpha) (\varphi'(t))^2 \\
>& 2\Gamma(t) \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \mathbb{M}(t) - 4\alpha\Gamma(t)\mathbb{N}(t) \\
>& 4\alpha\Gamma(t) \left(2\varphi(t) + \varepsilon^{-1} \|u_0\|^2\right) \left(2 \int_0^t \|u_\tau\|^2 d\tau + \varepsilon \|u_0\|^2\right) - 4\alpha\Gamma(t)\mathbb{N}(t) \\
=& 4\alpha\Gamma(t)\mathbb{N}(t) - 4\alpha\Gamma(t)\mathbb{N}(t) \\
=& 0.
\end{aligned}$$

Thus

$$\Gamma(t)\Gamma''(t) - \frac{1+\alpha}{2} (\Gamma'(t))^2 > 0.$$

Since $\Gamma(0) > 0$ and $\Gamma'(0) > 0$, according to Lemma (3.3) to deduce that $\Gamma(t) \to +\infty$ as

$$t \to t^* \leq \frac{2\Gamma(0}{(\alpha - 1)\Gamma'(0)}.$$

Next, due to the continuity of Γ with respect to φ, we can conclude that $\varphi(t)$ tends to infinity as some finite time, which is a contradiction.

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