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On the paper "Regular equivalence relations on ordered $*$-semihypergroups"

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Abstract: If $(S, \circ, \leq)$ is an ordered hypersemigroup, an equivalence relation ρ on S is called congruence if $(a,b) \in \rho$ implies $(a \circ x, b \circ x) \in \rho$ and $(x \circ a, x \circ b) \in \rho$ for every $x \in S$; in the sense that for every $u \in a \circ x$ there exists $v \in b \circ x$ such that $(u,v) \in \rho$ and for every $u \in x \circ a$ there exists $v \in x \circ b$ such that $(u,v) \in \rho$. It has been proved in Turk J Math 2021(5) [On the paper "A study on (strong) order-congruences in ordered semihypergroups"] that if S is an ordered hypersemigroup, then there exists a congruence ρ on S such that S/ρ is an ordered hypersemigroup. This result, is the main result for an involution ordered hypersemigroup by Xinyang Feng, Jian Tang and Yanfeng Luo in U.P.B. Sci. Bull, Series A, 2018, but its proof is wrong; the correct proof is given in the present paper. Examples illustrate the results.

Key words: Ordered hypersemigroup, involution, quasi pseudoorder, congruence, pseudoorder, strong congruence

1. Introduction

The aim is to correct the main Theorem 3.6 in [2].

From an involution ordered hypersemigroup in which the involution is the identity mapping, only results on *commutative* ordered hypersemigroups can be obtained, that is a very restricted case.

The concept of regular equivalent relation in [2] (and in the existed bibliography, in general) should be corrected. It contains conditions that should be omitted (see also [7]). In [2], for example (and the same in any related paper), the regular equivalent relation is defined as follows: An equivalence relation ρ on an hypersemigroup S is called regular equivalent relation if $(a,b) \in \rho$ implies $(a \circ x, b \circ x) \in \rho$ and $(x \circ a, x \circ b) \in \rho$ for every $x \in S$ in the sense that

(1) for every $u \in a \circ x$ there exists $v \in b \circ x$ such that $(u,v) \in \rho$ and for every $u \in b \circ x$ there exists $v \in a \circ x$ such that $(v,u) \in \rho$; and

(2) for every $u \in x \circ a$ there exists $v \in x \circ b$ such that $(u,v) \in \rho$ and for every $u \in x \circ b$ there exists $v \in x \circ a$ such that $(v,u) \in \rho$.

In other words, an equivalence relation ρ is called regular if, for any $x \in S$, the $(a,b) \in \rho$ $(a, b \in S)$ implies

(1) for every $u \in a \circ x$ there exists $v \in b \circ x$ such that $(u,v) \in \rho$

(2) for every $u \in b \circ x$ there exists $v \in a \circ x$ such that $(v,u) \in \rho$

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(3) for every $u \in x \circ a$ there exists $v \in x \circ b$ such that $(u, v) \in \rho$

(4) for every $u \in x \circ b$ there exists $v \in x \circ a$ such that $(v, u) \in \rho$.

But as ρ is symmetric, $(a, b) \in \rho$ implies $(b, a) \in \rho$. Then, by (1), for every $u \in b \circ x$ there exists $v \in a \circ x$ such that $(u, v) \in \rho$. Again since ρ is symmetric, $(u, v) \in \rho$ implies $(v, u) \in \rho$; so from $(a, b) \in \rho$ and (1), condition (2) is satisfied. Similarly, $(a, b) \in \rho$ and (3) imply (4).

We will use the term "congruence relation" instead of "regular equivalence relation" given in [2] as this is the standard notion for semigroups or ordered semigroups we try to extend. In the proof of the main Theorem 3.6 in [2], the authors apply the Theorem 2.1 of the same paper in a wrong way as the sets $\{(z)_\rho \mid z \in x \circ y\}$ and $\bigcup\limits_{z \in x \circ y} (z)_\rho$ are not equal; details have been given in [7]. The section 4 of their paper, based on the wrong proof of Theorem 3.6, is also wrong. Besides, the concept of homomorphism has been used in it (see [2, p. 138]) that is without sense; but this is out of the scope of the present paper.

In addition, as it is very difficult to check the examples on hypersemigroups given by a table of multiplication by hand and, especially, since they are obtained from ordered semigroups, the authors should explain how the Example 2.5 of their paper in [2] has been constructed. The following might be mentioned: The examples on ordered hypersemigroups given by a table of multiplication and an order can be constructed, in a very easy way, from corresponding examples on ordered semigroups; so reference to ordered semigroups should be given. Using the Light's associativity test together with its extended form for ordered semigroups [1, 3], it is very easy to check the examples of ordered semigroups by hand. Then, from the examples on ordered semigroups given by a table of multiplication and an order, we can immediately pass to corresponding examples on ordered hypersemigroups and we are sure that our examples are correct.

The first two conditions of ordered $*$-hypersemigroup given in [2, Definition 2.4] should define the hypersemigroups (without order). In ordered hypersemigroups, the order plays the essential role, so the term "order preserving involution" should be deleted from the Definition 2.4 in [2] and should be the third condition defining the involution.

An ordered semigroup $(S, \cdot, \leq)$ is called *involution ordered semigroup* (or *ordered $*$-semigroup*) if there exists a unary operation $\rho : S \to S \mid a \to a^*$ on S such that (1) $(a^*)^* := a^{**} = a$ for every $a \in S$; (2) $(ab)^* = b^* a^*$ for every $a, b \in S$ and (3) if $a \leq b$, then $a^* \leq b^*$ [6].

If the ordered semigroup S is commutative and the involution is the identity mapping on S (that is, $x^* = x$ for every $x \in S$), then S is an involution ordered semigroup. So, *every commutative ordered semigroup is an involution ordered semigroup.*

It has been proved in [7, Theorem 3.2] that if S is an ordered hypersemigroup, then there exists a congruence ρ on S such that S/ρ is an ordered hypersemigroup. One can get the same result as a corollary to [7, Theorem 3.4], as every strong congruence is a congruence on S. As a continuation of the paper in [7], in Theorem 2.5 of present paper, we prove that if S is an involution ordered hypersemigroup, then there exists a congruence ρ on S such that the set S/ρ is an involution ordered hypersemigroup. The same result can be obtained as a corollary to Theorem 2.10, as every strong congruence is a congruence on S (see Corollary 2.11). This is the open problem in [2] (after correcting the concept of "regular equivalence relation" in it); and the theorems of the present paper correct its proof. For more detailed information we refer to [7].

2. Main results

The concept of involution ordered semigroup can be extended, in a natural way, to an ordered hypersemigroup as follows:

Definition 2.1 *An ordered hypersemigroup* $(S, \circ, \leq)$ *is called an involution ordered hypersemigroup (or ordered* $*$*-hypersemigroup), if there exists a unary operation* $* : S \to S \mid a \to a^*$ *on* S *such that*

(1) $(a^*)^* = a$ *and* $(a \circ b)^* = b^* \circ a^*$ *for every* $a, b \in S$ *and*

(2) if $a \leq b$*, then* $a^* \leq b^*$.

The mapping $*$ is called an *involution* on S. The element $(a^*)^*$ is also denoted by a^{**}. For a nonempty subset A of S, we define $A^* := \{a^* \mid a \in A\}$.

Remark 2.2 *If* S *is an ordered hypersemigroup and* $*$ *is an involution on* S *defined by* $a^* = a$ *for every* $a \in S$ *then, for every* $a, b \in S$*, we have* $(a \circ b)^* = a \circ b$. Indeed, if $z \in (a \circ b)^*$, then $z = u^*$ for some $u \in a \circ b$. Since $u \in S$, we have $u^* = u$. Then $z = u \in a \circ b$ and so $(a \circ b)^* \subseteq a \circ b$. If $z \in a \circ b$, then $z = z^* \in (a \circ b)^*$ and so $a \circ b \subseteq (a \circ b)^*$. If S is commutative (that is, $a \circ b = b \circ a \ \forall\, a, b \in S$), then $(a \circ b)^* = a \circ b = b \circ a = b^* \circ a^*$.

As a consequence, *every commutative ordered hypersemigroup with the involution* $a^* = a \,\forall\, a \in S$ *(: the identity mapping) is an involution ordered hypersemigroup.*

Definition 2.3 *Let* $(S, \circ, \leq, *)$ *be an ordered* $*$*-hypersemigroup. A relation* σ *on* S *is called quasi pseudoorder on S if the following conditions are satisfied:*

(1) $\leq \subseteq \sigma$

(2) if $(a, b) \in \sigma$ *and* $(b, c) \in \sigma$*, then* $(a, c) \in \sigma$

(3) if $(a, b) \in \sigma$ *and* $c \in S$*, then* $(a \circ c, b \circ c) \in \sigma$ *and* $(c \circ a, c \circ b) \in \sigma$ *in the sense that for every* $u \in a \circ c$ *there exists* $v \in b \circ c$ *such that* $(u, v) \in \sigma$ *and* $(v, u) \in \sigma$ *and for every* $u \in c \circ a$ *there exists* $v \in c \circ b$ *such that* $(u, v) \in \sigma$ *and* $(v, u) \in \sigma$

(4) if $(a, b) \in \sigma$*, then* $(a^*, b^*) \in \sigma$.

That is, σ is a quasi pseudoorder of the ordered hypersemigroup $(S, \circ, \leq)$ [7] and, in addition,

$$(a, b) \in \sigma \text{ implies } (a^*, b^*) \in \sigma.$$

Definition 2.4 *Let* S *be an ordered* $*$*-hypersemigroup. An equivalence relation* ρ *on* S *is called congruence if* $(a, b) \in \rho$ *implies* $(a \circ x, b \circ x) \in \rho$ *and* $(x \circ a, x \circ b) \in \rho$ *for every* $x \in S$*; in the sense that for every* $u \in a \circ x$ *there exists* $v \in b \circ x$ *such that* $(u, v) \in \rho$ *and for every* $u \in x \circ a$ *there exists* $v \in x \circ b$ *such that* $(u, v) \in \rho$.

Theorem 2.5 *Let* $(S, \circ, \leq, *)$ *be an ordered* $*$*-hypersemigroup and* σ *a quasi pseudoorder on S. Then there exists a congruence* ρ *on S such that the set* S/ρ *is an ordered* $*$*-hypersemigroup.*

Proof Since $(S,\circ,\leq)$ is an ordered hypersemigroup and σ a quasi pseudoorder on $(S,\circ,\leq)$, the relation ρ on $(S,\circ,\leq)$ defined by

$$\rho := \big\{(a,b) \mid (a,b) \in \sigma \text{ and } (b,a) \in \sigma\big\}$$

is a congruence on $(S,\circ,\leq)$ and the set S/ρ with the hyperoperation $\odot$ and the order $\preceq$ on S/ρ defined by:

$$(a)_\rho \odot (b)_\rho := \big\{(c)_\rho \mid c \in a \circ b\big\} \tag{2.1}$$

and

$$(a)_\rho \preceq (b)_\rho \text{ if and only if } (a,b) \in \sigma$$

is an ordered hypersemigroup (see the proof of Theorem 3.2 in [7]). The mapping:

$$* : S/\rho \to S/\rho \mid (a)_\rho \to (a)_\rho^* := (a^*)_\rho$$

is an involution on $(S/\rho,\odot,\leq,*)$. In fact: First of all, it is well defined: If $(a)_\rho = (b)_\rho$, then $(a,b) \in \rho$, then $(a,b) \in \sigma$ and $(b,a) \in \sigma$. Since σ is a quasi pseudoorder on S, we have $(a^*,b^*) \in \sigma$ and $(b^*,a^*) \in \sigma$, then $(a^*,b^*) \in \rho$ and so $(a^*)_\rho = (b^*)_\rho$; that is, $(a)_\rho^* = (b)_\rho^*$. The following properties hold:

(1) $\big((a)_\rho^*\big)^* = (a)_\rho$. Indeed:

If $x \in \big((a)_\rho^*\big)^*$, then $x = c^*$ for some $c \in (a)_\rho^* = (a^*)_\rho$, then $(c,a^*) \in \rho$. Thus we have $(c,a^*) \in \sigma$ and $(a^*,c) \in \sigma$; since $*$ is an involution on S, we have $(c^*,a) \in \sigma$ and $(a,c^*) \in \sigma$, then $(c^*,a) \in \rho$, that is $(x,a) \in \rho$ and so $x \in (a)_\rho$. If $x \in (a)_\rho$, then $(x,a) \in \rho$, $(x,a) \in \sigma$ and $(a,x) \in \sigma$. Since $*$ is an involution on S, we have $(x^*,a^*) \in \sigma$ and $(a^*,x^*) \in \sigma$; thus $(x^*,a^*) \in \rho$, then $x^* \in (x^*)_\rho = (a^*)_\rho$ and so $x = x^{**} \in \big((a^*)_\rho\big)^* = \big((a)_\rho^*\big)^*$.

(2) $\Big((a)_\rho \odot (b)_\rho\Big)^* = (b)_\rho^* \odot (a)_\rho^*$. Indeed:

Let $x \in \Big((a)_\rho \odot (b)_\rho\Big)^*$. Then $x = d^*$ for some $d \in (a)_\rho \odot (b)_\rho$. Since $d \in (a)_\rho \odot (b)_\rho$, by (2.1), we have $d = (c)_\rho$ for some $c \in a \circ b$. Since $c \in a \circ b$ and $*$ is an involution on S, we have $c^* \in (a \circ b)^* = b^* \circ a^*$. By (2.1), we have $x = d^* = (c)_\rho^* = (c^*)_\rho \in (b^*)_\rho \odot (a^*)_\rho = (b)_\rho^* \odot (a)_\rho^*$. Let now $x \in (b)_\rho^* \odot (a)_\rho^*$. Then $x \in (b^*)_\rho \odot (a^*)_\rho$ and, by (2.1), we have $x = (c)_\rho$ for some $c \in b^* \circ a^*$. Then $c^* \in (b^* \circ a^*)^* = a^{**} \circ b^{**} = a \circ b$. By (2.1), $x^* = (c)_\rho^* = (c^*)_\rho \in (a)_\rho \odot (b)_\rho$ and so $x = x^{**} \in \Big((a)_\rho \odot (b)_\rho\Big)^*$.

(3) If $(a)_\rho \preceq (b)_\rho$, then $(a)_\rho^* \preceq (b)_\rho^*$. In fact:
$(a)_\rho \preceq (b)_\rho$ implies $(a,b) \in \sigma$. Since $*$ is an involution on S, we have $(a^*,b^*) \in \sigma$ and so $(a^*)_\rho \preceq (b^*)_\rho$; then $(a)_\rho^* \preceq (b)_\rho^*$. $\square$

We apply Theorem 2.5 to the following examples.

Example 2.6 Let $S = \{a,b,c,d\}$ be the ordered semigroup defined by Table 1 and Figure 1.

For an easy way to check that this is an ordered semigroup see [3]. From this ordered semigroup, in the way indicated in [5], the ordered hypersemigroup $(S,\circ,\leq)$ given by Table 2 and the same Figure 1 can be obtained. We define

$$a^* = a,\ b^* = c,\ c^* = b \text{ and } d^* = d.$$

Then $(S,\circ,\leq,*)$ is an ordered $*$-hypersemigroup.

Table 1. The multiplication of the ordered semigroup of Example 2.6.

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	d	c	d
d	a	d	a	a

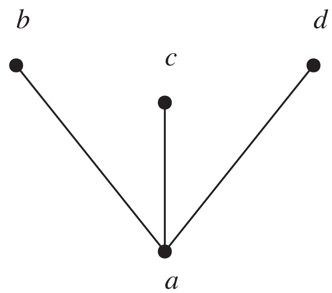

Figure 1. Figure that shows the order of Example 2.6.

Table 2. The hyperoperation of the ordered hypersemigroup of Example 2.6.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a,d\}$	$\{a,c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a\}$	$\{a\}$

One can check that the relation σ defined by

$$\sigma := \big\{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(d,d),(d,a),(d,b),(d,c)\big\}$$

is a quasi pseudoorder on $(S,\circ,\leq,*)$. Then, by Theorem 2.5, the relation

$$\begin{aligned}\rho &= \big\{(x,y) \mid (x,y)\in\sigma \text{ and } (y,x)\in\sigma\big\}\\ &= \big\{(a,a),(a,d),(b,b),(c,c),(d,d),(d,a)\big\}\end{aligned}$$

is a congruence on $(S,\circ,\leq,*)$. We have

$$(a)_\rho=(d)_\rho=\{a,d\},\ (b)_\rho=\{b\},\ (c)_\rho=\{c\}.$$

And the set $S/\rho=\big\{(a)_\rho,(b)_\rho,(c)_\rho\big\}$ with the hyperoperation given by Table 3, the order $\preceq$ below and the mapping

$$*: S/\rho\to S/\rho \mid (a)_\rho\to(a^*)_\rho=(a)_\rho,\ (b)_\rho\to(b^*)_\rho=(c)_\rho,\ (c)_\rho\to(c^*)_\rho=(b)_\rho$$

is an ordered $*$-hypersemigroup.

$$\preceq=\Big\{\big((a)_\rho,(a)_\rho\big),\big((a)_\rho,(b)_\rho\big),\big((a)_\rho,(c)_\rho\big),\big((b)_\rho,(b)_\rho\big),\big((c)_\rho,(c)_\rho\big)\Big\}.$$

Table 3. The hyperoperation on S/ρ in Example 2.6.

$\odot$	$(a)_\rho$	$(b)_\rho$	$(c)_\rho$
$(a)_\rho$	$\{(a)_\rho\}$	$\{(a)_\rho\}$	$\{(a)_\rho\}$
$(b)_\rho$	$(a)_\rho\}$	$\{(a)_\rho, (b)_\rho\}$	$\{(a)_\rho\}$
$(c)_\rho$	$\{(a)_\rho\}$	$\{(a)_\rho, (d)_\rho\}$	$\{(a)_\rho, (c)_\rho\}$

If we write, for short, $(a)_\rho := x,\ (b)_\rho := y,\ (c)_\rho := z$ then, we can independently check, that the set $S = \{x, y, z\}$ with the hyperoperation defined by Table 4,

Table 4. The hyperoperation of Table 3 given in a short way.

$\circ$	x	y	z
x	$\{x\}$	$\{x\}$	$\{x\}$
y	$\{x\}$	$\{x, y\}$	$\{x\}$
z	$\{x\}$	$\{x\}$	$\{x, z\}$

the order

$$\leq = \big\{(x, x), (x, y), (x, z), (y, y), (z, z)\big\}$$

and the mapping

$$* : S \to S \mid x \to x^* = x,\ y \to y^* = z,\ z \to z^* = y$$

is an ordered $*$-hypersemigroup.

Another example that is more difficult to check by hand, is the following.

Example 2.7 (see also [4]). Let $S = \{a, b, c, d, e\}$ be the ordered semigroup defined by Table 5 and Figure 2.

Table 5. The multiplication of the ordered semigroup of Example 2.7.

$\cdot$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

From this ordered semigroup, the ordered hypersemigroup given by Table 6 and the same Figure 2 can be obtained. We define

$$a^* = e,\ b^* = c,\ c^* = b,\ d^* = d \text{ and } e^* = a.$$

Then $(S, \circ, \leq, *)$ is an ordered $*$-hypersemigroup. The relation σ on $(S, \circ, \leq, *)$ defined by

$$\sigma = \big\{(a, a), (a, b), (b, b), (c, c), (d, b), (d, c), (d, d), (e, c), (e, e), (b, a), (c, e), (d, a), (d, e)\big\}$$

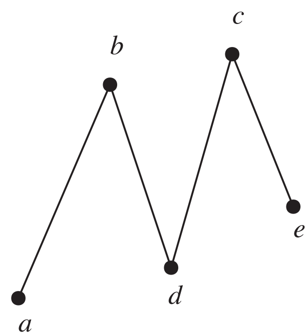

Figure 2. Figure that shows the order of Example 2.7.

Table 6. The hyperoperation of the ordered hypersemigroup of Example 2.7.

$\circ$	a	b	c	d	e
a	$\{a,b,d\}$	$\{a,b,d\}$	$\{d\}$	$\{d\}$	$\{d\}$
b	$\{a,b,d\}$	$\{a,b,d\}$	$\{d\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{c,d,e\}$	$\{d\}$	$\{c,d,e\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{d\}$	$\{d\}$	$\{c,d,e\}$	$\{d\}$	$\{c,d,e\}$

is a quasi pseudoorder on $(S,\circ,\leq,*)$. By Theorem 2.5, the relation ρ on S defined by

$$\begin{aligned}\rho &= \big\{(x,y) \mid (x,y)\in\sigma \text{ and } (y,x)\in\sigma\big\}\\ &= \big\{(a,a),(a,b),(b,b),(c,c),(d,d),(e,c),(e,e),(b,a),(c,e)\big\}\end{aligned}$$

is a congruence on $(S,\circ,\leq,*)$. We have $(a)_\rho=(b)_\rho=\{a,b\}$, $(c)_\rho=(e)_\rho=\{c,e\}$, $(d)_\rho=\{d\}$, and the set $S/\rho=\big\{(a)_\rho,(c)_\rho,(d)_\rho\big\}$ with the hyperoperation $\odot$ given by Table 7, the order

$$\preceq=\Big\{\big((a)_\rho,(a)_\rho\big),\big((c)_\rho,(c)_\rho\big),\big((d)_\rho,(a)_\rho\big),\big((d)_\rho,(c)_\rho\big),\big((d)_\rho,(d)_\rho\big)\Big\}$$

and the involution

$$*: S/\rho \mid S/\rho \mid (a)_\rho\to(a^*)_\rho=(c)_\rho,\ (c)_\rho\to(c^*)_\rho=(b)_\rho=(a)_\rho,\ (d)_\rho\to(d^*)_\rho=(d)_\rho$$

is an involution $*$-hypersemigroup. If we write, for short, $(a)_\rho:=x$, $(c)_\rho:=y$, $(d)_\rho:=z$, then we have that

Table 7. The hyperoperation on S/ρ in Example 2.7.

$\odot$	$(a)_\rho$	$(c)_\rho$	$(d)_\rho$
$(a)_\rho$	$\{(a)_\rho,(d)_\rho\}$	$\{(d)_\rho\}$	$\{(d)_\rho\}$
$(c)_\rho$	$\{(d)_\rho\}$	$\{(c)_\rho,(d)_\rho\}$	$\{(d)_\rho\}$
$(d)_\rho$	$\{(d)_\rho\}$	$\{(d)_\rho\}$	$\{(d)_\rho\}$

the set $S=\{x,y,z\}$ with the hyperoperation $\circ$ defined by Table 8, the order given by

$$\leq=\big\{(x,x),(y,y),(z,x),(z,y),(z,z)\big\}$$

and the involution $x^*=y$, $y^*=x$, $z^*=z$ is an ordered $*$-hypersemigroup.

Table 8. The hyperoperation on S/ρ in Example 2.7 written in a short way.

$\circ$	x	y	z
x	$\{x,z\}$	$\{z\}$	$\{z\}$
y	$\{z\}$	$\{y,z\}$	$\{z\}$
z	$\{z\}$	$\{z\}$	$\{z\}$

Theorem 2.5 provides us with a condition under which S/ρ is an involution ordered hypersemigroup; this has been done using the concept of quasi pseudoorder. However, we get the same result using the concept of pseudoorder as well.

As we refer to [7], to avoid any misunderstanding, we will still keep the concept "pseudoorder".

Definition 2.8 *Let $(S,\circ,\le,*)$ be an ordered $*$-hypersemigroup and σ a relation on S. We will say that σ is a pseudoorder on S if the following conditions are satisfied:*

(1) $\le\subseteq\sigma$

(2) if $(a,b)\in\sigma$ and $(b,c)\in\sigma$, then $(a,c)\in\sigma$

(3) if $(a,b)\in\sigma$ and $c\in S$, then $(a\circ c,b\circ c)\in\sigma$ and $(c\circ a,c\circ b)\in\sigma$ in the sense that for every $u\in a\circ c$ and every $v\in b\circ c$ we have $(u,v)\in\sigma$ and for every $u\in c\circ a$ and every $v\in c\circ b$ we have $(u,v)\in\sigma$

(4) if $(a,b)\in\sigma$, then $(a^*,b^*)\in\sigma$.

That is, σ is a pseudoorder of the ordered hypersemigroup $(S,\circ,\le)$ [7] and, in addition,

$$(a,b)\in\sigma \text{ implies } (a^*,b^*)\in\sigma.$$

Definition 2.9 *Let S be an ordered $*$-hypersemigroup. An equivalence relation ρ on S is called strong congruence if $(a,b)\in\rho$ implies $(a\circ x,b\circ x)\in\rho$ and $(x\circ a,x\circ b)\in\rho$ for every $x\in S$; in the sense that for every $u\in a\circ x$ and every $v\in b\circ x$ we have $(u,v)\in\rho$ and for every $u\in x\circ a$ and every $v\in x\circ b$ we have $(u,v)\in\rho$.*

The following theorem holds.

Theorem 2.10 *Let $(S,\circ,\le,*)$ be an ordered $*$-hypersemigroup and σ a pseudoorder on S. Then there exists a strong congruence ρ on S such that S/ρ is an ordered $*$-hypersemigroup.*

Proof The relation $\rho:=\big\{(a,b)\mid(a,b)\in\sigma \text{ and } (b,a)\in\sigma\big\}$ is a strong congruence on S and so the set S/ρ endowed with the hyperoperation $(a)_\rho\odot(b)_\rho=\big\{(c)_\rho\big\}$, where c is an arbitrary element of $a\circ b$, and the order "$(a)_\rho\preceq(b)_\rho$ if and only if $(a,b)\in\sigma$" is an ordered hypersemigroup (see the proof of Theorem 3.4 in [7]). The mapping

$$*:S/\rho\to S/\rho\mid(a)_\rho\to(a)_\rho^*=(a^*)_\rho$$

is an involution on S/ρ (see the proof of Theorem 2.5) and the set $(S/\rho,\odot,\preceq,*)$ is an ordered $*$-hypersemigroup. $\square$

Regarding the proof of Theorem 2.10, the following might be reminded: As ρ is a strong congruence on S, the singleton $\{(c)_\rho\}$, where c is an arbitrary element of $a \circ b$ is equal to the set $\{(d)_\rho \mid d \in a \circ b\}$ (see [7, Proposition 2.2].

Since every strong congruence is a congruence on S [7, Remark 1.5], from Theorem 2.10, we have the following corollary.

Corollary 2.11 *Let $(S, \circ, \leq, *)$ be an ordered $*$-hypersemigroup and σ a pseudorder on S. Then there exists a congruence ρ on S such that S/ρ is an ordered $*$-hypersemigroup.*

We apply Theorem 2.10 to the following two examples.

Example 2.12 We consider the Example 2.6 and the quasi pseudoorder

$$\sigma := \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(d,d),(d,a),(d,b),(d,c)\}$$

given in it. This is not a pseudoorder on S as, for example, $(a,b) \in \sigma$, but $(a \circ b, b \circ b) \notin \sigma$ ($a \circ b = \{a\}$, $b \circ b = \{a,b\}$, while $(b,a) \notin \sigma$). If we try to obtain a pseudoorder $\overline{\sigma}$ from σ, then we have to add in σ the elements $(b,a),(c,a),(b,c),(b,d),(c,b),(c,d)$. But then $\overline{\sigma} = S \times S$, $\rho = S \times S$, $(a)_\rho = (b)_\rho = (c)_\rho = (d)_\rho = S$; and the set $S/\rho = \{(a)_\rho\}$ with the hyperoperation $(a)_\rho \odot (a)_\rho = \{(a)_\rho\}$, the order $\preceq = \Big\{\big((a)_\rho, (a)_\rho\big)\Big\}$, and the mapping $* : S/\rho \to S/\rho \mid (a)_\rho \to (a)_\rho$ is an involution ordered hypersemigroup (independently one can immediately see that this is true).

The quasi pseudoorder

$$\begin{aligned}\sigma \;=\; &\{(a,a),(a,b),(b,b),(c,c),(d,b),(d,c),(d,d),(e,c),(e,e),(b,a),\\ &(c,e),(d,a),(d,e)\}\end{aligned}$$

defined in Example 2.7 is not a pseudoorder on S. Indeed, for example, $(a,a) \in \sigma$ but $(a \circ a, a \circ a) \notin \sigma$ as $a \circ a = \{a,b,d\}$ while $(a,d) \notin \sigma$. If we want to get a pseudoorder $\overline{\sigma}$ from σ, then we have to add in σ the elements $(a,d),(b,d),(c,d),(e,d),(a,c),(a,e),(b,c),(b,e),(c,b),(e,b),(e,a),(c,a)$. But then, again $\overline{\sigma} = S \times S$.

An example for which the pseudoorder σ is proper (i.e. $\sigma \neq S \times S$) is the following.

Example 2.13 Consider the ordered semigroup $S = \{a,b,c\}$ given by Table 9 and the order $\leq$.

Table 9. The multiplication of the ordered semigroup of the Example 2.13.

$\cdot$	a	b	c
a	b	b	c
b	b	b	c
c	c	c	c

$$\leq = \{(a,a),(a,b),(b,b),(c,c)\}.$$

From this ordered semigroup, the ordered hypersemigroup given by Table 10 and the same order $\leq$ can be obtained.

Table 10. The hyperoperation of the ordered hypersemigroup of the Example 2.13.

$\circ$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	$\{c\}$
b	$\{a,b\}$	$\{a,b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

We define the involution on $(S,\circ,\le)$ as follows:

$$a^*=a,\ b^*=b,\ c^*=c.$$

Then $(S,\circ,\le,*)$ is an ordered $*$-hypersemigroup (see also Remark 2.2).
The relation

$$\sigma=\big\{(a,a),(a,b),(b,b),(c,c),(b,a),(c,a),(c,b)\big\}$$

is a pseudoorder on $(S,\circ,\le,*)$. The relation

$$\begin{aligned}\rho &= \big\{(x,y)\mid (x,y)\in\sigma \text{ and } (y,x)\in\sigma\big\}\\ &= \big\{(a,a),(a,b),(b,b),(c,c),(b,a)\big\}\end{aligned}$$

is a strong congruence (and so a congruence) on S as well. We have $(a)_\rho=(b)_\rho=\{a,b\}$ and $(c)_\rho=\{c\}$. The set $S/\rho=\big\{(a)_\rho,(c)_\rho\big\}$ with the hyperoperation given by Table 11, the order

$$\preceq=\Big\{\Big((a)_\rho,(a)_\rho\Big),\Big((c)_\rho,(a)_\rho\Big),\Big((a)_\rho,(a)_\rho\Big),\Big((c)_\rho,(c)_\rho\Big)\Big\}$$

and the involution

$$*:S/\rho\to S/\rho\mid (a)_\rho\to(a^*)_\rho=(a)_\rho,\ (b)_\rho\to(b^*)_\rho=(b)_\rho,\ (c)_\rho\to(c^*)_\rho=(c)_\rho$$

is an ordered $*$-hypersemigroup. If we write, for short, $(a)_\rho:=x$, $(c)_\rho:=y$, then we can immediately see

Table 11. The ordered hypersemigroup S/ρ of the Example 2.13.

$\odot$	$(a)_\rho$	$(c)_\rho$
$(a)_\rho$	$\{(a)_\rho\}$	$\{(c)_\rho\}$
$(c)_\rho$	$\{(c)_\rho\}$	$\{(c)_\rho\}$

that the ordered hypersemigroup defined by Table 12, the order $\le=\big\{(x,x),(y,x),(y,y)\big\}$ and the involution $x^*=x,\ y^*=y$ is an ordered $*$-hypersemigroup.

Table 12. The ordered hypersemigroup S/ρ of the Example 2.13 written in a short way.

$\circ$	x	y
x	$\{x\}$	$\{y\}$
y	$\{y\}$	$\{y\}$

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