Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2022) 46: 157 – 166
© TÜBİTAK
doi:10.3906/mat-2108-107

Examination of eigenvalues and spectral singularities of a discrete Dirac operator with an interaction point

Şerifenur CEBESOY*
Department of Mathematics, Faculty of Science, Çankırı Karatekin University, Çankırı Turkey

Received: 24.08.2021 • **Accepted/Published Online:** 21.10.2021 • **Final Version:** 19.01.2022

Abstract: In this paper, the main content is the consideration of the concepts of eigenvalues and spectral singularities of an operator generated by a discrete Dirac system in $\ell_2(\mathbb{Z}, \mathbb{C}^2)$ with an interior interaction point. Defining a transfer matrix M enables us to present a relationship between the M_{22} component of this matrix and Jost functions of mentioned Dirac operator so that its eigenvalues and spectral properties can be studied. Finally, some special cases are examined where the impulsive condition possesses certain symmetries.

Key words: Impulsive conditions, discrete Dirac systems, eigenvalues, spectral singularities, symmetries

1. Introduction and preliminaries

Mathematical simulations of evolution of a real process usually depends on the abrupt changes or perturbations in its state. In general, while these sudden changes occur, the duration of the time passing is imperceptible in comparison with the time passing during the whole process. It is supposed that these instantaneous changes are momentary and the state of the process changes through jumps. An existence of a jump discontinuity causes more complicated and challenging theory which is richer than the theory of ordinary differential equations. Therefore, it becomes necessary to study equations or dynamical systems with such short-term rapid changes (that is, jumps), or with impulse effects, for the sake of brevity, as they called, impulsive differential equations, or sometimes, differential equations with impulse effects. Theory of impulsive differential equations dates back to the end of last century [4, 10, 17] and the solutions of some problems for impulsive equations led to a significant improvement over the existing literature. In the recent years, most of the researchers were interested in such problems under the influence of impulsive actions for its wide applications in physics, engineering, quantum mechanics and spectral theory. For Sturm–Liouville operator and its discrete analogue, there are a couple of equivalent descriptions such as point interactions, impulsive conditions, transmission conditions, and interface conditions used by authors that disturb the continuity [5, 8, 9, 13, 14, 19, 20].

Before presenting the main results on impulsive discrete Dirac equations, let us recall the concepts related with our study.

Eigenvalues and spectral singularities are two parts of continuous spectrum of a nonselfadjoint Sturm–Liouville operator which have been investigated with the pioneering study of Naimark [15]. In quantum mechanics, an eigenvalue is said to be a bound state that is a state in Hilbert space corresponding to two

*Correspondence: scebesoy@karatekin.edu.tr
2010 *AMS Mathematics Subject Classification:* 34B37, 34L05, 35Q41, 81Q05

or more particles whose interaction energy is less than the total energy of each separate particle, and therefore these particles cannot be separated unless energy is spent. On the other hand, spectral singularities correspond to the resonance states having a real energy. Spectral singularities were discovered by Naimark and subsequently studied by others [16, 18]. Since then, it was understood that spectral singularities cause obstructions for the completeness of the eigenfunctions and they have unignorable physical meanings.

Marchenko investigated the Sturm–Liouville boundary value problem [12]

$$-y^{''} + q(x)y = \lambda^2 y, \quad -\infty < x < \infty, \tag{1.1}$$

where λ is a spectral parameter, q is a complex valued function. (1.1) admits a pair of solutions fulfilling the conditions

$$\lim_{x\to\pm\infty} e(x,\lambda)e^{\mp i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}. \tag{1.2}$$

These bounded solutions are called Jost solutions of (1.1), which play an important role in the solutions of direct and inverse problems of spectral theory [2, 3, 12].

In spectral theory, definitions of an eigenvalue and a spectral singularity are given in terms of these particular solutions of the equation. For example, a spectral singularity is a point of the continuous spectrum of H satisfying the eigenvalue equation

$$Hy = \lambda^2 y \tag{1.3}$$

such that the Jost solutions are linearly dependent at these points, in other words, they have a vanishing Wronskian. Note that, the Wronskian of any two y, z solutions of (1.1) is defined as

$$W[y,z] := yz' - y'z.$$

In the present paper, we consider the following discrete Dirac system

$$\begin{cases} y_{n+1}^{(2)} - y_n^{(2)} = \lambda y_n^{(1)} \\ y_n^{(1)} + y_{n-1}^{(1)} = \lambda y_n^{(2)} \end{cases}, \quad n \in \mathbb{Z}\backslash\{-1,0,1\} \tag{1.4}$$

and the impulsive condition

$$y_1(z) = \mathbf{P}y_{-1}(z), \quad \mathbf{P} = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}, \quad p_i \in \mathbb{C}, \quad i = 1, \dots 4, \tag{1.5}$$

where

$$\lambda := 2\sin\left(\frac{z}{2}\right)$$

is a complex spectral parameter. (1.5) is an impulsive condition, namely, point interaction for (1.4) at the point $n = 0$ and the matrix $\mathbf{P}$ works out to continue the solution of (1.4) from negative integer numbers to the positive integer numbers.

Many mathematical studies have been conducted and some useful results have been obtained for the system (1.4) without impulsive effects [3, 6, 7, 11]. This paper aims to investigate the eigenvalues and spectral singularities of the discrete Dirac operator with a point interaction corresponding to the boundary value problem (1.4)–(1.5) subject to the choice of the constants p_i, $i = 1, \dots 4$ and to examine the presences of $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$-symmetries that the point interaction has.

2. Corresponding operator

In this section, we first introduce the Hilbert space $\ell_2(\mathbb{Z},\mathbb{C}^2)$ consisting of all vector sequences $y=\{y_n\}$, $y_n=\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $y_n^{(i)}\in\mathbb{C}$, $i=1,2$, $n\in\mathbb{N}$ with the inner product

$$\langle y,z\rangle_{\ell_2}:=\sum_{n\in\mathbb{Z}}\left[\left(y_n^{(1)},z_n^{(1)}\right)+\left(y_n^{(2)},z_n^{(2)}\right)\right],\qquad y_n,z_n:\mathbb{Z}\to\mathbb{C}^2$$

and the norm

$$\|y\|_{\ell_2}^2:=\sum_{n\in\mathbb{Z}}\left(\left|y_n^{(1)}\right|^2+\left|y_n^{(2)}\right|^2\right)<\infty.$$

Let T be the operator in $\ell_2(\mathbb{Z},\mathbb{C}^2)$ created by the difference expression (1.4) and the point interaction (1.5) and also let $z\in\mathbb{C}\backslash\mathbb{C}_1$, where $\mathbb{C}_1:=\{z:z=(2n+1)\pi,n\in\mathbb{Z}\}$. Then it is easy to verify that

$$\varphi_n(z)=\begin{pmatrix} \varphi_n^{(1)}(z) \\ \varphi_n^{(2)}(z) \end{pmatrix}=\begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix}e^{inz}$$

and

$$\psi_n(z)=\begin{pmatrix} \psi_n^{(1)}(z) \\ \psi_n^{(2)}(z) \end{pmatrix}=\begin{pmatrix} -i \\ e^{i\frac{z}{2}} \end{pmatrix}e^{-inz}$$

are two linearly independent solutions of (1.4), since the Wronskian of these two functions is defined as

$$W[\varphi,\psi]:=\varphi_n^{(1)}(z)\psi_n^{(2)}(z)-\varphi_n^{(2)}(z)\psi_n^{(1)}(z)=e^{iz}+1.$$

Moreover, there exist $e_n^{\pm}(z)$ solutions of the Equation (1.4) with the following asymptotical behaviors:

$$e_n^{+}(z)\longrightarrow\varphi_n(z),\quad z\in\mathbb{C}\backslash\mathbb{C}_1,\quad n\to\infty \tag{2.1}$$

and

$$e_n^{-}(z)\longrightarrow\psi_n(z),\quad z\in\mathbb{C}\backslash\mathbb{C}_1,\quad n\to-\infty. \tag{2.2}$$

These are celebrated as Jost solutions of (1.4) and they are both analytic with respect to z in $\mathbb{C}_+:=\{z:z\in\mathbb{C},\operatorname{Im}\lambda>0\}$ and continuous up to the real axis.

Now, by the help of the linearly independent solutions, we can express the general solution $y_n=\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$ of (1.4) by

$$y_n(z)=A\varphi_n(z)+B\psi_n(z),\quad n\in\mathbb{Z}^-\cup\mathbb{Z}^+$$

and we clearly get

$$y_n^{(1)}(z)=\begin{cases} A_-e^{iz(\frac{1}{2}+n)}-iB_-e^{-inz}, & n\in\mathbb{Z}^- \\ A_+e^{iz(\frac{1}{2}+n)}-iB_+e^{-inz}, & n\in\mathbb{Z}^+, \end{cases} \tag{2.3}$$

$$y_n^{(2)}(z) = \begin{cases} -iA_- e^{inz} + B_- e^{iz(\frac{1}{2}-n)}, & n \in \mathbb{Z}^- \\ -iA_+ e^{inz} + B_+ e^{iz(\frac{1}{2}-n)}, & n \in \mathbb{Z}^+, \end{cases} \tag{2.4}$$

where $A, B, A_\pm$ and $B_\pm$ are constant coefficients.

According to the point interaction (1.5), we write

$$\begin{pmatrix} y_1^{(1)}(z) \\ y_1^{(2)}(z) \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} y_{-1}^{(1)}(z) \\ y_{-1}^{(2)}(z) \end{pmatrix} \tag{2.5}$$

and finally (2.5) turns into

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_- \\ B_- \end{pmatrix}, \tag{2.6}$$

where

$$\mathbf{M} := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} := N_1^{-1} \mathbf{B} N_2 \tag{2.7}$$

with

$$N_1 := \begin{pmatrix} e^{\frac{3iz}{2}} & -ie^{-iz} \\ -ie^{iz} & e^{-\frac{iz}{2}} \end{pmatrix}, \quad N_2 := \begin{pmatrix} e^{-\frac{iz}{2}} & -ie^{iz} \\ -ie^{-iz} & e^{\frac{3iz}{2}} \end{pmatrix}.$$

By (2.5) and (2.7), we get

$$\begin{aligned} M_{11}(z) &:= \frac{1}{e^{iz}+1} \left[p_1 e^{-iz} - p_2 i e^{-\frac{3iz}{2}} + p_3 i e^{-\frac{3iz}{2}} + p_4 e^{-2iz} \right], \\ M_{12}(z) &:= \frac{1}{e^{iz}+1} \left[-p_1 i e^{\frac{iz}{2}} + p_2 e^{iz} + p_3 + i p_4 e^{\frac{iz}{2}} \right], \\ M_{21}(z) &:= \frac{1}{e^{iz}+1} \left[p_1 i e^{\frac{iz}{2}} + p_2 + p_3 e^{iz} - i p_4 e^{\frac{iz}{2}} \right], \\ M_{22}(z) &:= \frac{1}{e^{iz}+1} \left[p_1 e^{2iz} + p_2 i e^{\frac{5iz}{2}} - p_3 i e^{\frac{5iz}{2}} + p_4 e^{3iz} \right]. \end{aligned}$$

Now, let us consider any two $f_n^\pm(z)$ solutions of (1.4) satisfying the impulsive condition (1.5), denoting the coefficients $A_\pm$ and $B_\pm$ by $A_\pm^\pm$ and $B_\pm^\pm$ which are expressed as

$$f_n^+(z) = \begin{cases} A_+^+ \varphi_n(z) + B_+^+ \psi_n(z), & n \in \mathbb{Z}^+, \\ A_-^+ \varphi_n(z) + B_-^+ \psi_n(z), & n \in \mathbb{Z}^-, \end{cases} \tag{2.8}$$

$$f_n^-(z) = \begin{cases} A_+^- \varphi_n(z) + B_+^- \psi_n(z), & n \in \mathbb{Z}^+, \\ A_-^- \varphi_n(z) + B_-^- \psi_n(z), & n \in \mathbb{Z}^- \end{cases} \tag{2.9}$$

for all $z \in \mathbb{C} \backslash \mathbb{C}_1$, where $A_\pm^\pm$ and $B_\pm^\pm$ are complex coefficients. If we associate $f_n^+(z)$ and $f_n^-(z)$ with the Jost solutions $e_n^+(z)$ and $e_n^+(z)$, respectively, we write following asymptotics

$$f_n^+(z) \longrightarrow \varphi_n(z), \quad z \in \mathbb{C} \backslash \mathbb{C}_1, \quad n \to \infty$$

and

$$f_n^-(z) \longrightarrow \psi_n(z), \quad z \in \mathbb{C}\backslash\mathbb{C}_1, \quad n \to -\infty.$$

In view of these asymptotics, we uniquely obtain

$$A_+^+ = B_-^- = 1, \qquad A_-^- = B_+^+ = 0. \tag{2.10}$$

Furthermore, taking into account the expressions (2.6) and (2.10), we get

$$A_-^+ = \frac{M_{22}}{\det \mathbf{M}}, \qquad B_-^+ = -\frac{M_{21}}{\det \mathbf{M}} \tag{2.11}$$

for the solution f_n^+ by solving

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_-^+ \\ B_-^+ \end{pmatrix}.$$

Similarly, for the solution f_n^-, we get

$$A_+^- = M_{12}, \qquad B_+^- = M_{22} \tag{2.12}$$

by solving

$$\begin{pmatrix} A_+^- \\ B_+^- \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly, inserting $A_+^+, A_-^+, B_+^+, B_-^+$ in (2.8), we obtain the solution f_n^+ providing following asymptotic

$$f_n^+(z) = \begin{cases} \varphi_n(z), & n \to +\infty \\ \dfrac{M_{22}}{\det \mathbf{M}}\varphi_n(z) - \dfrac{M_{21}}{\det \mathbf{M}}\psi_n(z), & n \to -\infty. \end{cases} \tag{2.13}$$

Similarly, inserting $A_-^-, A_+^-, B_-^-, B_+^-$ in (2.9), we obtain the solution f_n^- providing following asymptotic

$$f_n^-(z) = \begin{cases} M_{12}\varphi_n(z) + M_{22}\psi_n(z), & n \to +\infty \\ \psi_n(z), & n \to -\infty. \end{cases} \tag{2.14}$$

The solutions $f_n^\pm(z)$ are called the Jost solutions of the impulsive boundary value problem (1.4)-(1.5). By using the definition of the Wronskian of the Jost solutions of the operator T acting in $\ell_2(\mathbb{Z}, \mathbb{C}^2)$, we can give the following theorem.

Theorem 2.1 *The following asymptotics hold.*

$$W[f_n^+, f_n^-](z) = M_{22}\left(e^{iz} + 1\right), \quad n \to +\infty \tag{2.15}$$

$$W[f_n^+, f_n^-](z) = \frac{M_{22}}{\det \boldsymbol{M}}\left(e^{iz} + 1\right), \quad n \to -\infty \tag{2.16}$$

Proof (2.13) and (2.14) can be used to calculate the Wronskian of the Jost solutions for $n \to +\infty$ and for $n \to -\infty$. □

We remark that the transfer matrix $\mathbf{M}$ has unit determinant. Therefore, the point interaction which does not satisfy this property is called *anomalous point interaction.* Recall that a necessary and sufficient condition to investigate the eigenvalues and spectral singularities of a discrete Dirac operator is to investigate the zeros of the Wronskian of the Jost functions [1]. For the operator T (or the impulsive boundary value problem (1.4)-(1.5)), the eigenvalues and spectral singularities correspond to λ values for which $W[f_n^+, f_n^-](z) = 0$. Therefore, we can give the next lemma without proof.

Lemma 2.2 *In order to examine the eigenvalues and spectral singularities of the operator T, we need to examine the zeros of the function M_{22}, i.e.*

$$p_1 + i(p_2 - p_3)e^{i\frac{z}{2}} + p_4 e^{iz} = 0. \tag{2.17}$$

For simplicity, (2.17) can be rewritten as

$$M_{22}(\tau) := p_4\tau^2 + i(p_2 - p_3)\tau + p_1 = 0, \tag{2.18}$$

where $\tau := e^{i\frac{z}{2}}$.

Next, we proceed by giving the definitions of eigenvalues and spectral singularities of the operator T denoted by $\sigma_d(T)$ and $\sigma_{ss}(T)$, respectively. Hence, in accordance with the traditional definition, we can write [1, 15]

$$\sigma_d(T) = \left\{ \lambda \in \mathbb{C}\backslash(-2,2) : \lambda = i\left(\frac{1}{\tau} - \tau\right), M_{22}(\tau) = 0 \right\} \tag{2.19}$$

and

$$\sigma_{ss}(T) = \left\{ \lambda \in (-2,2) : \lambda = i\left(\frac{1}{\tau} - \tau\right), M_{22}(\tau) = 0 \right\}. \tag{2.20}$$

In order to examine the zeros of (2.18), we consider some cases obtained in the following two lemmas so that we see the conditions for the existence of eigenvalues and spectral singularities.

Lemma 2.3 *Assume $p_4 \neq 0$. In this case, (2.18) gives*

$$\tau_{1,2} = -i\left(\mu \pm \sqrt{\mu^2 + \nu}\right),$$

where

$$\mu := \frac{p_2 - p_3}{2p_4} \quad , \quad \nu := \frac{p_1}{p_4}. \tag{2.21}$$

This implies that

$$\lambda = -\frac{1}{\mu \pm \sqrt{\mu^2 + \nu}} + \left(\mu \pm \sqrt{\mu^2 + \nu}\right),$$

and thus, two complex spectral singularities appear if

$$\left|\mu \pm \sqrt{\mu^2 + \nu}\right| = 1 \quad \textit{and} \quad 0 < \left|\mathrm{Re}\left(\mu \pm \sqrt{\mu^2 + \nu}\right)\right| < 1,$$

i.e.

$$|\tau| = 1 \quad \textit{and} \quad 0 < |\mathrm{Re}\,\tau| < 1.$$

Otherwise, there exist two eigenvalues.

Lemma 2.4 *Assume $p_4 = 0$. In this case, (2.18) gives*

$$i\,(p_2 - p_3)\,\tau + p_1 = 0,$$

and thus there appear two special cases:

(i) If $p_2 \neq p_3$, then

$$\lambda = \frac{p_2 - p_3}{p_1} + \frac{p_1}{p_2 - p_3}$$

is a spectral singularity if and only if

$$\left|\frac{p_2 - p_3}{p_1}\right| = 1 \quad \textit{and} \quad 0 < \left|\mathrm{Re}\left(\frac{p_2 - p_3}{p_1}\right)\right| < 1.$$

Otherwise, there exists an eigenvalue.

(ii) If $p_2 = p_3$, then the condition of the existence of a spectral singularity or an eigenvalue, namely $M_{22} = 0$, implies that $p_1 = 0$. In this case,

$$\mathbf{P} = \mathbf{M} = \begin{pmatrix} 0 & p_2 \\ p_2 & 0 \end{pmatrix},$$

and $\det \mathbf{M} = -p_2^2$. In particular, $\mathbf{M}$ is independent of z, M_{22} vanishes identically, and the interaction is anomalous for $p_2 \neq \pm i$.

3. $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$-symmetries

This section is devoted to deduce some consequences as a result of symmetries imposed on the impulsive condition (1.5) leading to the discrete Dirac operator T.

3.1. $\mathcal{P}$-symmetry

Definition 3.1 *The operator $\mathcal{P}$ acting on complex vector sequences $y = (y_n) = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $y_n : \mathbb{Z} \to \mathbb{C}^2$ defined by*

$$(\mathcal{P} y_n) := y_{-n}, \quad n \in \mathbb{Z}$$

is called the parity (reflection) operator.

Definition 3.2 *The point interaction (1.5) is called $\mathcal{P}$-invariant (or has $\mathcal{P}$-symmetry) if and only if*

$$\mathcal{P} y_1 = \boldsymbol{P} \mathcal{P} y_{-1}. \tag{3.1}$$

Theorem 3.3 *The point interaction* (1.5) *has* $\mathcal{P}$*-symmetry if and only if*

$$\mathbf{P} = \mathbf{P}^{-1}. \tag{3.2}$$

Proof Assume that the point interaction (1.5) has $\mathcal{P}$-symmetry. Then, (3.1) is satisfied. Using Definition 3.1 and (1.5), we can easily get (3.2). On the contrary, the proof can be completed in the same way. □

Theorem 3.4 *If the point interaction* (1.5) *has* $\mathcal{P}$*-symmetry, then the operator* T *does not have any spectral singularity, just has eigenvalues.*

Proof Suppose that, (1.5) has $\mathcal{P}$-symmetry. Since $\det \mathbf{P} = 1$, we obtain

$$p_1 = p_4 = \pm 1, \; p_2 = p_3 = 0$$

by (3.2). Therefore, $\tau = \pm i$ are the zeros of $M_{22}(\tau)$ which yields $\lambda = \pm 2$ and this means that there is no spectral singularity and they are eigenvalues. This completes the proof. □

3.2. $\mathcal{T}$-symmetry

Definition 3.5 *The operator* $\mathcal{T}$ *acting on complex vector sequences* $y = (y_n) = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $y_n : \mathbb{Z} \to \mathbb{C}^2$ *defined by*

$$\mathcal{T} y_n = y_n^*, \quad n \in \mathbb{Z}$$

is called the time-reversal operator, where "$*$" denotes the complex conjugate of y*.*

Definition 3.6 *The point interaction* (1.5) *is called* $\mathcal{T}$*-invariant (or has* $\mathcal{T}$*-symmetry) if and only if*

$$\mathcal{T} y_1 = \boldsymbol{P} \mathcal{T} y_{-1}. \tag{3.3}$$

Theorem 3.7 *The point interaction* (1.5) *has* $\mathcal{T}$*-symmetry if and only if*

$$\mathbf{P}^* = \mathbf{P}$$

holds; that is, $\mathbf{P}$ *is real.*

Proof Taking into account Definition 3.6, it is easy to see that this relation is equivalent to the requirement that $\mathbf{P}$ is a real matrix, i.e. p_1, p_2, p_3, p_4 must be real. □

Theorem 3.8 *Assume that the point interaction* (1.5) *has* $\mathcal{T}$*-symmetry. In the case of*

$$p_4 \neq 0 \quad \text{and} \quad \left(\frac{p_2 - p_3}{2p_4} \right)^2 + \frac{p_1}{p_4} < 0,$$

there exist two spectral singularities for $p_1, p_2, p_3, p_4 \in \mathbb{C}$ *such that*

$$p_1 + p_4 = 0 \quad \text{and} \quad |p_2 - p_3| < 2\,|p_4|\,.$$

3.3. $\mathcal{PT}$-symmetry

Definition 3.9 *The point interaction (1.5) is called $\mathcal{PT}$-invariant (or has $\mathcal{PT}$-symmetry) if and only if*

$$\mathcal{PT}y_1 = \boldsymbol{P}\mathcal{PT}y_{-1}. \tag{3.4}$$

Theorem 3.10 *The point interaction (1.5) has $\mathcal{PT}$-symmetry if and only if*

$$\mathbf{P}^* = \mathbf{P}^{-1} \tag{3.5}$$

holds.

Proof Taking into account Definition 3.9, it is easy to see that this relation is equivalent to the requirement that $\mathbf{P}^* = \mathbf{P}^{-1}$. □

Theorem 3.11 *Assume that the point interaction (1.5) has $\mathcal{PT}$-symmetry. In case of*

$$p_4 \neq 0 \quad \text{and} \quad |\text{Im}(p_2 + p_3)| \leq 2,$$

there exist two spectral singularities for $p_i \in \mathbb{C}$, $i = 1, \ldots 4$ such that

$$2\,\text{Im}(p_1) \neq |\text{Im}(p_3 - p_2)|\,.$$

Proof Let the point interaction (1.5) has $\mathcal{PT}$-symmetry, and let p_4 be different from 0. From Theorem 3.10, we can write p_1, p_2, p_3, p_4 as $p_1 = x + iy$, $p_2 = i\zeta$, $p_3 = i\delta$, $p_4 = x - iy$, respectively such that $x, y, \zeta, \delta \in \mathbb{R}$. Using this relation, $\det \mathbf{P} = 1$ and (2.21), we get

$$\sqrt{\mu^2 + \nu} = \frac{x + iy}{2(1 - \zeta\delta)}\sqrt{4 - (\delta + \zeta)^2}.$$

In this case we need to examine two different situations:

$$(i) \quad |\delta + \zeta| \leq 2, \qquad (ii) \quad |\delta + \zeta| > 2.$$

Examining these situations in detail, the proof of this theorem is obtained clearly.

□

4. Conclusion

This paper is devoted to discuss the eigenvalues and spectral singularities of a discrete canonical Dirac system in $\ell_2(\mathbb{Z}, \mathbb{C}^2)$ having a point interaction. We first present the Jost solutions of this impulsive boundary value problem by the help of a transfer matrix which puts forth an alternative approach to the theory of impulsive discrete Dirac equations. The transfer matrix enables us to form the sets of eigenvalues and spectral singularities in terms of the zeros of a second–order polynomial function. Then, we examine some special cases depending on the choice of the constants given in impulsive condition. Therefore, one can specialise the impulsive condition and see the effects of a point interaction. The remainder of the paper deals with fundamental symmetries such as $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$-symmetries, which are mostly related to the mathematical physics. We finally analyzed some consequences about the existence of eigenvalues and spectral singularities in the event of presence of these symmetries. In further research, the results may be generalized into impulsive discrete Dirac equations with more than one impulsive point which has not been treated elsewhere yet.

References

[1] Adıvar M, Bairamov E. Spectral properties of non-selfadjoint difference operators. Journal of Mathematical Analysis and Applications 2001; 261 (2): 461-478.

[2] Aygar Y, Bairamov E. Jost solution and the spectral properties of the matrix valued difference operators. Applied Mathematics and Computation 2012; 218: 9676-9681.

[3] Aygar Y, Olgun M. Investigation of the spectrum and the Jost solutions of discrete Dirac system on the whole axis. Journal of Inequalities and Applications 2014; 73: 9 pp.

[4] Bainov D, Simeonov P. Oscillation theory of impulsive differential equations. International Publications, Orlando, FL, 1998.

[5] Bairamov E, Cebesoy S, Erdal I. Difference equations with a point interaction. Mathematical Methods in the Applied Sciences 2019; 42: 5498–5508.

[6] Bairamov E, Çelebi AO. Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators. The Quarterly Journal of Mathematics. Oxford Second Series 1999; 50 (200): 371-384.

[7] Bairamov E, Koprubasi T. Eigenparameter dependent discrete Dirac equations with spectral singularities. Applied Mathematics and Computation 2010; 215 (12): 4216-4220.

[8] Bairamov E, Solmaz S. Scattering theory of Dirac operator with the impulsive condition on whole axis. Mathematical Methods in the Applied Sciences 2021; 44 (9): 7732-7746.

[9] Erdal I, Yardımcı S. Eigenvalues and scattering properties of difference operators with impulsive condition. Communications Faculty of Science University of Ankara Series A1 Mathematics and Statistics 2019; 68 (1): 663-671.

[10] Lakshmikantham V, Bainov DD, Simeonov PS. Theory of impulsive differential equations, volume 6 of Series in Modern Applied Mathematics. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[11] Levitan BM, Sargsjan IS. Sturm–Liouville and Dirac Operators. Springer, 1991.

[12] Marchenko VA. Sturm–Liouville operators and applications, volume 22 of Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel. Translated from the Russian by A. Iacob, 1986.

[13] Mostafazadeh A. Spectral singularities of a general point interaction. Journal of Physics. A. Mathematical and Theoretical 2011; 44 (37): 375302, 9.

[14] Mukhtarov O, Kadakal M, Mukhtarov FS. On discontinuous Sturm–Liouville problems with transmission conditions. Journal of Mathematics of Kyoto University 2004; 44 (4): 779-798.

[15] Naimark MA. Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis. American Mathematical Society Translations: Series 2 1960; 16: 103-193.

[16] Pavlov BS. On the spectral theory of non-selfadjoint differential operators. Doklady Akademii. Nauk SSSR 1962; 146: 1267-1270.

[17] Samoilenko AM, Perestyuk NA. Impulsive differential equations, volume 14 of World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[18] Schwartz J. Some non-selfadjoint operators. Communications on Pure and Applied Mathematics 1960; 13: 609-639.

[19] Ugurlu E. Investigation of the spectral properties of a fourth order differential operator with finite transmission conditions. Mathematical Methods in the Applied Sciences 2016; 36 (6): 1298-1310.

[20] Zhang M, Wang Y. Dependence of eigenvalues of Sturm–Liouville problems with interface conditions. Applied Mathematics and Computation 2015; 265: 31-39.