Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 167 – 176
© TÜBİTAK
doi:10.3906/mat-2108-4

# Various operators in relation to fractional order calculus and some of their applications to normalized analytic functions in the open unit disk

**Hüseyin IRMAK***
Department of Mathematics, Faculty of Science, Çankırı Karatekin University, Uluyazı Campus, Çankırı, TURKEY

**Received:** 02.08.2021 • **Accepted/Published Online:** 17.12.2021 • **Final Version:** 19.01.2022

**Abstract:** The main object of this scientific work is firstly to introduce various operators of fractional calculus (that is that fractional integral and fractional derivative(s)) in certain domains of the complex plane, then to determine certain results correlating with normalized analytic functions, which are analytic in certain domains in the complex plane, as a few applications of those operators, and also to present a number of extensive implications of them as special results.

**Key words:** Fractional calculus, domains, complex exponentiation, normalized analytic functions, Taylor-Maclaurin series expansions, equations-inequalities in certain complex domains

## 1. Submission of preliminary information and concerned applications

As far as the literature presents us, the fundamental theory of fractional calculus (that is that fractional derivative(s) and integral) has very important roles in terms of literature. We also encounter a wide variety of scientific researches from mathematics to almost all engineering sciences. Some of them are in several forms of extensive applications and others are in various forms of theoretical studies as topics associating with special functions, operators, equations, inequalities, and so on...

For the theory of fractional calculus and also (some of) its mentioned implications, one may refer to numerous basic works given in [6], [17], [24], [25], [27], [28] and also to various examples [1], [2], [5], [9], [10], [16], [18], [19], [21], [22], [26] and [30]-[34] in the references of this investigation.

As stated in the abstract of this paper, the main purpose of this present research is to remind certain basic information about fractional calculus and then to introduce numerous operators of fractional order in the complex plane. After this extensive information, it is aimed to constitute several extensive applications of those operators to certain normalized analytic functions and also a number of associated implications of those applications.

We now begin by introducing related information *which* will be required in our investigation pertaining to various applications and characteristic properties of those fractional-order type operators.

Firstly, by the familiar notations:

$$\mathbb{N} \quad , \quad \mathbb{R} \quad , \quad \mathbb{C} \quad \text{and} \quad \mathbb{U},$$

*Correspondence: hirmak@karatekin.edu.tr; hisimya@yahoo.com

2010 *AMS Mathematics Subject Classification:* 26A33, 30A10, 26E05, 30C45, 30C55.

respectively, we denote the sets of positive integers, real numbers, complex numbers, and also the open unit disk, *namely,* the well-known open set:

$$\mathbb{U} := \big\{ z \, : \, z \in \mathbb{C} \text{ and } |z| < 1 \big\}.$$

And also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad , \quad \mathbb{R}^* := \mathbb{R} - \{0\} \quad \text{and} \quad \mathbb{C}^* := \mathbb{C} - \{0\}.$$

In addition, let $\varphi(z)$ be any (complex) function that is analytic in the set $\mathbb{U}$ and is also normalized by the Taylor-Maclaurin series expansions given by

$$\varphi := \varphi(z) = z + w_{1+n} z^{1+n} + w_{2+n} z^{2+n} + \cdots \quad , \tag{1.1}$$

*where* all coefficients $w_{1+n} \in \mathbb{C}$ for all $n \in \mathbb{N}$.

Secondly, for any (complex) function $\varphi := \varphi(z)$, *which* is analytic in a simply-connected region of the complex plane containing the origin, the definition of fractional integral of the function $\varphi(z)$ is usually denoted by the following-equivalent notations:

$$\mathbf{D}_z^{-\lambda}\{\varphi\} \equiv \mathbf{D}_z^{-\lambda}\{\varphi(z)\} \equiv \mathbf{D}_z^{-\lambda}\{\varphi\}(z) \quad \big(\lambda > 0\big)$$

and it is also defined by

$$\mathbf{D}_z^{-\lambda}\{\varphi\} = \frac{1}{\Gamma(\lambda)} \int_0^z (z-s)^{\lambda-1}\, \varphi(s)\, ds \quad \big(\lambda > 0\big), \tag{1.2}$$

*where* the multiplicity of the expression $(z-s)^{\lambda-1}$ just above is removed by requiring $log(z-s)$ to be real *when* $z - s > 0$.

Moreover, for $\varphi := \varphi(z)$, the definition of fractional derivative(s) of the function $\varphi$ is generally denoted by the following-equivalent notations:

$$\mathbf{D}_z^{\lambda}\{\varphi\} \equiv \mathbf{D}_z^{\lambda}\{\varphi(z)\} \equiv \mathbf{D}_z^{\lambda}\{\varphi\}(z) \quad \big(0 \le \lambda < 1\big)$$

and it is also defined by

$$\mathbf{D}_z^{\lambda}\{\varphi\} = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z (z-s)^{-\lambda}\, \varphi(s)\, ds \quad \big(0 \le \lambda < 1\big), \tag{1.3}$$

*where* the multiplicity of the expression $(z-s)^{-\lambda}$ just above is removed by requiring $log(z-s)$ to be real *when* $z - s > 0$.

Of course, we also note here that, in (1.2) and (1.3) and throughout this paper, the notation $\Gamma$ denotes the well-known Gamma function occasionally encountered in the mathematical literature.

Under the conditions of the definition given in (1.3), for the function $\varphi$, the well-known derivative of the fractional order $\vartheta + \lambda$ is also introduced by

$$\mathbf{D}_z^{\vartheta+\lambda}\{\varphi\} = \frac{d^{\vartheta}}{dz^{\vartheta}} \Big( \mathbf{D}_z^{\lambda}\{\varphi\} \Big) \tag{1.4}$$

*for* some $\lambda$ $(0 \leq \lambda < 1)$ and for all $\vartheta \in \mathbb{N}_0$.

Most especially, for that function $\varphi$, we note down here that

$$\mathbf{D}_z^{0+\lambda}\{\varphi\} = \mathbf{D}_z^{\lambda}\{\varphi\} \quad \text{and} \quad \mathbf{D}_z^{1+\lambda}\{\varphi\} = \frac{d}{dz}\Big(\mathrm{D}_z^{\lambda}\{\varphi\}\Big), \tag{1.5}$$

*where* $0 \leq \lambda < 1$.

Lastly, through the instrumentality of the operator of fractional order given by (1.3) and for the function $\varphi := \varphi(z)$, we also describe the Tremblay operator by the equivalent notations given by

$$\mathbf{T}_z^{\kappa,\lambda}\{\varphi\} \equiv \mathbf{T}_z^{\kappa,\lambda}\{\varphi(z)\}$$

and also denote by the definition given by

$$\mathbf{T}_z^{\kappa,\lambda}\{\varphi\} = \frac{\Gamma(\lambda)}{\Gamma(\kappa)} z^{1-\lambda}\mathbf{D}_z^{\kappa-\lambda}\big\{z^{\kappa-1}\varphi(z)\big\}, \tag{1.6}$$

*where* $0 < \kappa \leq 1,\ 0 < \lambda \leq 1$ and $0 \leq \kappa - \lambda < 1$.

We specially point out here that the notation:

$$\mathbf{D}_z^{\kappa-\lambda}\{\cdot\} \quad \big(0 \leq \kappa - \lambda < 1\big)$$

represents the well-known-Srivastava-Owa operator of fractional derivative of order $\kappa-\lambda$, *which* is also presented by the definition of the operator defined as in (1.3). For its details, see [8], [14], [15], and [27]. In addition, for more information, the details of the mentioned definitions above and also numerous related examples, we think that it will be useful that the related researchers may focus on the mentioned works (*or* certain earlier results) given by the references in [3], [4], [8] and [12]-[15].

Moreover, as a result of simple analysis, numerous relationships between the operators related to fractional calculations, *which* are given between (1.3)-(1.6), can be easily seen. In the next section, some of extensive results including those specific relationships will be presented as related implications of our fundamental results *which* will be composed. However, for both some of those indicated-specific relationships and some of their applications, as example, one can especially concentrate on the earlier papers signified in the references given in [8], [14] and [15].

## 2. Formation of main results and related implications

As it is well known, for setting and then proving of theories, techniques (which will be used there) play important roles in all scientific investigations. So, for proofs of our main results, we also need two important auxiliary theorems given just below. For their details, one can center on the associated references given in [14], [20], [23] and [29].

**Lemma 2.1** *Let* $z \in \mathbb{C}^*$ *and also let* $w \in \mathbb{C}$. *Then,*

$$z^w = |z|^w e^{iwArg(z)}, \tag{2.1}$$

*or, equivalently,*

$$z^w = |z|^{\Re e(w)} e^{-\Im m(w) Arg(z)} \Big(\cos\Theta + i\sin\Theta\Big), \tag{2.2}$$

*where*

$$\Theta := \Re e(w) Arg(z) + \Im m(w) \log|z| \,. \tag{2.3}$$

**Lemma 2.2** *Let a function $\wp(z)$ be of the forms given by*

$$\wp(z) = 1 + \ell_1 z + \ell_2 z^2 + \ell_3 z^3 + \cdots + \ell_n z^n + \cdots \quad , \tag{2.4}$$

*where $\ell_n \in \mathbb{C}$, $n \in \mathbb{N}$, $z \in \mathbb{U}$ and $\wp(z) \not\equiv 1$. At that time, if there exists a point $z_0 \in \mathbb{U}$ such that*

$$\Re e\Big(\wp(z)\Big) > 0 \quad for \quad |z| < |z_0| < 1 \quad \big(z \in \mathbb{U}\big) \tag{2.5}$$

*and*

$$\Re e\Big(\wp(z_0)\Big) = 0 \quad and \quad \wp(z_0) \neq 0 \,, \tag{2.6}$$

*then*

$$\wp(z_0) = i\eta \quad and \quad z\wp'(z)\Big|_{z=z_0} = i\mu\Big(\eta + \frac{1}{\eta}\Big)\wp(z)\Big|_{z=z_0} \,, \tag{2.7}$$

*where $\mu \geq 1/2$ and $\eta \in \mathbb{R}^*$.*

**Theorem 2.3** *Let $w \in \mathbb{C}^*$ and also let the function $\varphi := \varphi(z)$ be of the form in (1.1). Then, under the conditions designated by*

$$0 < \kappa \leq 1 \quad , \quad 0 < \lambda \leq 1 \quad , \quad 0 \leq \kappa - \lambda < 1 \quad and \quad 0 \leq \alpha < \frac{\kappa}{\lambda} \,, \tag{2.8}$$

*and, for all $z \in \mathbb{U}$, the following proposition holds:*

$$\left| \left[ z \frac{d^2}{dz^2} \Big( \boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\} \Big) \right]^w \right|$$

$$\begin{cases} < e^{-\pi \Im m(w)} \left[ \frac{1}{2} \Big( \frac{\kappa}{\lambda} - \alpha \Big) \right]^{\Re e(w)} & when \ \Re e(w) \geq 0 \\ > e^{-\pi \Im m(w)} \left[ \frac{1}{2} \Big( \frac{\kappa}{\lambda} - \alpha \Big) \right]^{\Re e(w)} & when \ \Re e(w) \leq 0 \end{cases} \tag{2.9}$$

$$\Longrightarrow \quad \Re e \left[ \frac{d}{dz} \Big( \boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\} \Big) \right] > \alpha \,, \tag{2.10}$$

*where, here and throughout this paper, the value of the complex power above is taken into account as its principal value.*

**Proof** Under the conditions indicated in (2.8) and by means of the definitions of the mentioned operator given by (1.3) together with (1.4), as an application of the operator defined by (1.6), for a function $\varphi := \varphi(z)$ being of the form given in (1.1), the following result:

$$\mathbf{T}_z^{\kappa,\lambda}\{\varphi\} = \frac{\Gamma(\lambda)}{\Gamma(\kappa)} z^{1-\lambda} \mathbf{D}_z^{\kappa-\lambda}\big\{z^{\kappa-1}\varphi(z)\big\}$$

$$= \frac{\kappa}{\lambda} z + \sum_{t=n+1}^{\infty} \frac{\Gamma(t+\kappa)\Gamma(\lambda)}{\Gamma(t+\lambda)\Gamma(\kappa)} w_t z^t \tag{2.11}$$

can be easily determined (cf., e.g., [8] and [15]). By the help of the first derivative of the result obtained in (2.11), we consider an implicit function $q(z)$ as the form given by

$$\alpha + \left(\frac{\kappa}{\lambda} - \alpha\right) q(z) = \frac{d}{dz}\Big(\mathbf{T}_z^{\kappa,\lambda}\{\varphi\}\Big). \tag{2.12}$$

It is clear that the function $q(z)$ in the implicit form just above both has the form in the series expansion given in (2.4) and is analytic in the disk $\mathbb{U}$. In short, under the conditions in (2.8), the function $q(z)$ satisfies the condition $q(0) = 1$ of Lemma 2.2. It follows from (2.12) that

$$\left(\frac{\kappa}{\lambda} - \alpha\right) z q'(z) = z \frac{d^2}{dz^2}\Big(\mathbf{T}_z^{\kappa,\lambda}\{\varphi\}\Big). \tag{2.13}$$

We assume now that there is a point $z_0 \in \mathbb{U}$ such that

$$\Re e\Big(q(z_0)\Big) = 0\,. \tag{2.14}$$

Then, from (2.9), it is clear that $q(z_0) \neq 0$, *which* is the condition (of Lemma 2.2) given by (2.6). In this case, by making use of the assertions (of Lemma 2.2) given in (2.7) together with Lemma 2.1 to the equation given by (2.13), we firstly get that

$$\begin{aligned}
\Xi(z_0) &:= \left[z \frac{d^2}{dz^2}\Big(\mathbf{T}_z^{\kappa,\lambda}\{\varphi(z)\}\Big)\right]^w \Bigg|_{z:=z_0} \\
&= \left[\left(\frac{\kappa}{\lambda} - \alpha\right) z q'(z)\right]^w \Bigg|_{z:=z_0} \\
&= \left[\left(\frac{\kappa}{\lambda} - \alpha\right) i\mu\Big(\eta + \frac{1}{\eta}\Big) q(z_0)\right]^w \\
&= \left[-\mu\big(1+\eta^2\big)\left(\frac{\kappa}{\lambda} - \alpha\right)\right]^w \\
&= e^{-\pi \Im m(w) + i\Psi}\left[\mu\big(1+\eta^2\big)\Big(\frac{\kappa}{\lambda} - \alpha\Big)\right]^{\Re e(w)},
\end{aligned} \tag{2.15}$$

*where*

$$\Psi := \pi \Re e(w) + \Im m(w) \ln\left[\mu\big(1+\eta^2\big)\Big(\frac{\kappa}{\lambda} - \alpha\Big)\right].$$

Indeed, in consideration of (2.1) (*or* (2.2)) along with (2.3) (of Lemma 2.1) and under the conditions in (2.7) and (2.8), by taking the modulus of each one of the mentioned results in (2.15) and also considering the condition $\mu \geq 1/2,$ we also arrive at the following relations:

$$\begin{aligned} \left|\Xi(z_0)\right| &= \cdots \\ &= \left| e^{-\pi \Im m(w)+i\Psi} \left[\mu\left(1+\eta^2\right)\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} \right| \\ &= e^{-\pi \Im m(w)} \left[\mu\left(1+\eta^2\right)\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)}. \end{aligned} \tag{2.16}$$

Now we have to specify some special cases regarding some of the parameters contained in (2.16). For both those and researchers, we think that it will be important to underline the following inequalities.

Firstly, let $\Re e(w) \geq 0.$ Then, we get that

$$\mu \geq \frac{1}{2} \;\Rightarrow\; \mu^{\Re e(w)} \geq \left(\frac{1}{2}\right)^{\Re e(w)}$$

and

$$\eta \in \mathbb{R}^* \;\Rightarrow\; 1+\eta^2 > 1 \Rightarrow \; \left(1+\eta^2\right)^{\Re e(w)} \geq 1\,.$$

Secondly, let $\Re e(w) \leq 0.$ Then, we also get that

$$\mu \geq \frac{1}{2} \;\Rightarrow\; 0 < \mu^{\Re e(w)} \leq \left(\frac{1}{2}\right)^{\Re e(w)}$$

and

$$\eta \in \mathbb{R}^* \;\Rightarrow\; 1+\eta^2 > 1 \;\Rightarrow\; 0 < \left(1+\eta^2\right)^{\Re e(w)} \leq 1\,.$$

Thus, under favour of the extra information relating to the detailed inequalities identified just above, the result (2.15) immediately gives us the following inequalities.

$$\left|\Xi(z_0)\right| \begin{cases} \geq e^{-\pi \Im m(w)} \left[\frac{1}{2}\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} & \textit{when} \quad \Re e(w) \geq 0 \\ \leq e^{-\pi \Im m(w)} \left[\frac{1}{2}\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} & \textit{when} \quad \Re e(w) \leq 0 \end{cases},$$

*which* are contradictions with the inequalities consisting of the cases (of Theorem 2.3) given by (2.9), respectively. This shows that there is not a point like $z_0 \in \mathbb{U}$ satisfying the condition of Lemma 2.2 indicated by (2.14) (*or* (2.6)). This means that the inequality:

$$\Re e\Big(q(z)\Big) > 0$$

has to be true for all $z \in \mathbb{U}$. By this way, the definition constituted in (2.12) easily leads us to the inequality given by

$$\Re e\left(\frac{\frac{d}{dz}\left(\mathbf{T}_z^{\kappa,\lambda}\{\varphi\}\right)-\alpha}{\frac{\kappa}{\lambda}-\alpha}\right) > 0\,,$$

*which* also requires the inequality given in (2.10). Thereby, this also finishes the desired proof. □

As various implications of Theorem 2.3, when focusing on the main result created as above, new possible theorems can be easily recreated, and, of course, a great number of specific results of those theorems can be revealed again. For related researchers, we want to constitute (*or* reveal) only some of them.

Firstly, through the instrumentality of all steps in the proof of Theorem 2.3, the following propositions can be easily proved. In consideration of the proof of Theorem 2.3, the interested researchers can easily construct the related proofs in detail. For those, it will be also enough to consider Theorem 2.3 together with the well-known simple properties in complex analysis, *which* are created by

$$\Re e\big\{\Omega(z)\big\} \leq \big|\Re e\big\{\Omega(z)\big\}\big| \leq \big|\Omega(z)\big|$$

and

$$\Im m\big\{\Omega(z)\big\} \leq \big|\Im m\big\{\Omega(z)\big\}\big| \leq \big|\Omega(z)\big|$$

for all suitable function with complex variable like $\Omega(z)$. Therefore, we think that it is not necessary to prove some of those propositions given by the following theorem (below).

**Theorem 2.4** *Let $w \in \mathbb{C}^*$ and also let $\varphi := \varphi(z)$ be of the form in (1.1). Then, under the conditions given by (2.8) and for all $z \in \mathbb{U}$, if any one of the following propositions:*

$$\Re e\left\{\left[z\frac{d^2}{dz^2}\Big(\boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\}\Big)\right]^w\right\} \leq \left|\Re e\left\{\left[z\frac{d^2}{dz^2}\Big(\boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\}\Big)\right]^w\right\}\right| \begin{cases} < e^{-\pi\Im m(w)}\left[\frac{1}{2}\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} & when\ \Re e(w) \geq 0 \\ > e^{-\pi\Im m(w)}\left[\frac{1}{2}\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} & when\ \Re e(w) \leq 0 \end{cases}$$

*and*

$$\Im m\left\{\left[z\frac{d^2}{dz^2}\Big(\boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\}\Big)\right]^w\right\} \leq \left|\Im m\left\{\left[z\frac{d^2}{dz^2}\Big(\boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\}\Big)\right]^w\right\}\right| \begin{cases} < e^{-\pi\Im m(w)}\left[\frac{1}{2}\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} & when\ \Re e(w) \geq 0 \\ > e^{-\pi\Im m(w)}\left[\frac{1}{2}\left(\frac{\kappa}{\lambda}-\alpha\right)\right]^{\Re e(w)} & when\ \Re e(w) \leq 0 \end{cases}$$

*is true, then*

$$\Re e\left[\frac{d}{dz}\Big(\boldsymbol{T}_z^{\kappa,\lambda}\{\varphi\}\Big)\right] > \alpha$$

*is also true, where the value of each one of the complex powers above is considered as its principle value.*

Secondly, if the mentioned parameters in both the definitions in the first section and the theorems in this section above are appropriately chosen, of course, a great number of special results of the main results can be also determined. For certain examples, we would like to present only two of them as implications of our main results.

As the first-special implication, it can be generated by choosing the special values of the parameters $\kappa$ and $\lambda$ in Theorem 2.1 as $\kappa := 1 =: \lambda$. In this present case, we both can see the relation between the Tremblay operator and the function $\varphi(z)$, given by

$$\mathbf{T}_z^{1,1}\{\varphi(z)\} \equiv \varphi(z)$$

and then present it as one of the special results, *which* is Proposition 2.5 (just below).

**Proposition 2.5** *Let $z \in \mathbb{U}$ and $w \in \mathbb{C}^*$. For a function $\varphi(z)$ being the form given in (1.1), if any one of the conditions of the assertion:*

$$\left|\left[z\varphi''(z)\right]^w\right| \begin{cases} < e^{-\pi \Im m(w)}\left[\frac{1}{2}\left(1-\alpha\right)\right]^{\Re e(w)} & if \ \ \Re e(w) \geq 0 \\ > e^{-\pi \Im m(w)}\left[\frac{1}{2}\left(1-\alpha\right)\right]^{\Re e(w)} & if \ \ \Re e(w) \leq 0 \end{cases}$$

*is satisfied, then*

$$\Re e\Big(\varphi'(z)\Big) > \alpha \quad \big(0 \leq \alpha < 1\big)$$

*is satisfied, where the value of the complex power just above is taken into account as its principle value.*

As the second-special implication, by setting $w := 1$ in Proposition 2.5, the following special result, *which* is associated with geometric properties of analytic (and also univalent) functions (see the works in [7], [11] and [27] for their details and also the paper in [15] for instance), can be also established as Proposition 2.6, which is given by the following assertion.

**Proposition 2.6** *Let $z \in \mathbb{U}$. Then, for a function $\varphi(z)$ having the series form in (1.1), the following assertion holds true.*

$$2\big|z\varphi''(z)\big| < (1-\alpha)e^{-\pi} \quad \Longrightarrow \quad \Re e\Big(\varphi'(z)\Big) > \alpha \quad \big(0 \leq \alpha < 1\big)\,.$$

## References

[1] Anastassiou GA. Basics of right nabla fractional calculus on time scales. Journal of Nonlinear Evolution Equations and Applications 2011; 111-118.

[2] Aouf MK, Bulboaca T, El-Ashwash RM. Subordination properties of multivalent functions involving an extended fractional differintegral operator. Acta Mathematica Scientia Series B 2014; 34: 367-379.

[3] Chen MP, Irmak H, Srivastava, HM. Some families of multivalently analytic functions with negative coefficients. Journal of Mathematical Analysis and Applications 1997; 214: 674-690.

[4] Chen MP, Irmak H, Srivastava HM. A certain subclass of analytic functions involving operators of fractional calculus. Computers & Mathematics with Applications 1998; 35: 83-91.

[5] Cho NE, Aouf MK. Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients. Turkish Journal of Mathematics 1996; 20: 553-562.

[6] Debnath L. A brief historical introduction to fractional calculus. International Journal of Mathematical Education in Science and Technology 2004; 35: 487-501.

[7] Duren PL. Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heilderberg and Tokyo, 1983.

[8] Esa Z, Srivastava HM, A. Kilicman A, Ibrahim RW. A novel subclass of analytic functions specified by a family of fractional derivatives in the complex domain. Filomat 2017; 9: 2837-2849.

[9] Frasin BA, Murugusundaramoorthy G. Fractional calculus to certain family of analytic functions defined by convolution. Indian Journal of Mathematics 2009; 51: 537-548.

[10] Grozdev S. On the appearance of the fractional calculus. Journal of Theoretical and Applied Mechanics 1997; 27: 11-20.

[11] Goodman AW. Univalent Functions. Vol. I, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[12] Ibrahim RW, Jahangiri JM. Boundary fractional differential equation in a complex domain. Boundary Value Problems 2014; 2014: Article ID 66, 1-11.

[13] Irmak H. Some families of $p$-valently analytic functions established by fractional derivatives and their certain consequences. Applied Mathematics and Computation 2011; 218: 822-826.

[14] Irmak H. Characterizations of some fractional-order operators in complex domains and their extensive implications to certain analytic functions. Annals of the University of Craiova - Mathematics and Computer Science Series 2021; 48: 349-357.

[15] Irmak H. Geometric properties of some applications of the Tremblay operator. General Mathematics 2020; 28: 87-96.

[16] Khodabakhshi N, Vaezpour SM. The existence of extremal solutions to nonlinear fractional integro-differential equations with advanced arguments. Hacettepe Journal of Mathematics and Statistics 2016; 45: 1411-1419.

[17] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematical Studies, Vol. 204, Elsevier, Amsterdam, London and New York, 2006.

[18] Luo MJ, Raina RK. The decompositional structure of certain fractional integral operators. Hokkaido Mathematical Journal 2019; 48: 611-650.

[19] Nguyen VT. Fractional calculus in probability. Probability and Mathematical Statistics 1984; 3: 173-189.

[20] Nunokawa, M. On properties of non-Caratheodory functions. Proceedings of the Japan Academy, Ser. A, Mathematical Sciences 1992; 68: 152-153.

[21] Nishimoto K. Some equalities and N-fractional calculus for Riemann's zeta function. Advances in Applied Clifford Algebras 2001; 11: 247-256.

[22] Nishimoto K, Kalla S. Application of fractional calculus to a third order linear ordinary differential equation. Serdica 1988; 14: 385-391.

[23] Olver FWJ, Lozier DW, Boiswert RF, Clark CW. NIST Handbook of Mathematical Functions. Cambridge University Press, New York, USA, 2010.

[24] Ross B. Origins of fractional calculus and some applications. International Journal of Mathematics and Mathematical Sciences 1992; 1: 21-34.

[25] Ross B. The development of fractional calculus 1695-1900. Historia Mathematica 1977; 4: 75-89.

[26] Saxena RK, Ram, J, Suthar DL. Fractional calculus of generalized Mittag-Leffler functions. Journal of Indian Academy of Mathematics 2009; 31: 165-172.

[27] Srivastava HM, Owa S. Univalent functions, fractional calculus and their applications. Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1989.

[28] Srivastava HM. Fractional-order derivatives and integrals: introductory overview and recent developments. Kyungpook Mathematical Journal 2020; 60: 73-116.

[29] Stein EM, Shakarchi, R. Princeton Lecture in Analysis II: Complex Analysis. Princeton University Press, Princeton, USA, 2003.

[30] Taberski R. Contributions to fractional calculus and exponential approximation. Functiones et Approximatio, Commentarii Mathematici 1986; 15: 81-106.

[31] Tarasova VV, Tarasov VE. Elasticity for economic processes with memory: Fractional differential calculus approach. Fractional Differential Calculus 2016; 6: 219-232.

[32] Vasantha KWB. On multidifferentiation of decimal and fractional calculus. Mathematics Education 1994; 28: 220-224.

[33] Vinagre J, Blas M, Calderon G, Antonio J, Suarez M et al. Control theory and fractional calculus. Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 2005; 99: 241-258.

[34] Zhuravkov M, Romanova N. Modification of Hertz contact problem solution on the basis of fractional calculus for biostructures mechanical properties determination. Nonlinear Phenomena in Complex Systems 2015; 18: 31-37.