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Numerical solution of a singularly perturbed Fredholm integro differential equation with Robin boundary condition

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Abstract: In this paper, we deal with singularly perturbed Fredholm integro differential equation (SPFIDE) with mixed boundary conditions. By using interpolating quadrature rules and exponential basis function, fitted second order difference scheme has been constructed on a Shishkin mesh. The stability and convergence of the difference scheme have been analyzed in the discrete maximum norm. Some numerical examples have been solved and numerical outcomes are obtained.

Key words: Fredholm integro differential equation, singular perturbation, finite difference methods, Shishkin mesh, uniform convergence

1. Introduction

This paper deals with the following (SPFIDE) with mixed boundary conditions:

$$L_N u := -\varepsilon u'' + a(x)u + \lambda \int_0^l K(x,s)u(s)ds = f(x), \quad x \in (0,l)\,, \tag{1.1}$$

$$L_0 u := -\sqrt{\varepsilon}u'(0) + \beta u(0) = A, \tag{1.2}$$

$$u(l) = B, \tag{1.3}$$

where $\varepsilon \in (0,1]$ is a perturbation parameter, $\beta > 0$, λ, A, B are given constant. It was presumed that $a(x) \geq \alpha > 0$, $f(x)$, $(x \in [0,l])$, $K(x,s)\big((x,s) \in [0,l]^2\big)$ are the sufficiently smooth functions, to be specified. Underneath these conditions, the solution $u(x)$ of the problem (1.1)-(1.3) has in general boundary layers at $x = 0$ and $x = l$.

Singularly perturbed problems (SPPs) are mostly characterized by a small parameter ε that multiplies some or all of the higher order terms in the equation, because boundary layers are generally found in their solutions. To cite a few, one can find the exact solutions of SPPs and their applications in [16, 19]. SPPs possess a vast number of applications in the fields of fluid dynamics, population dynamics, heat transport problems, nanofluid, neurobiology, mathematical biology, viscoelasticity, and simultaneous control systems etc. It is notable that, whenever a small ε parameter is multiplied with the derivative, the vast majority of the

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classic numerical methods on uniform meshes are not proper for handling problems until severe diminutions are being made on the step-size of discretization. So, as the ε perturbation parameter gets small, the truncation error happens boundless. In order to solve SPPs, mostly approaches are made called the fitted finite difference method and are used extensively [9, 13, 18, 20–22].

Most applications of engineering and various branches of science have been governed by Fredholm integro differential equations (FIDEs). Population dynamics, fluid mechanics, financial mathematics, oceanography, plasma physics, artificial neural networks, electromagnetic theory and biological processes are among these (see, e.g., [5, 7, 14]). Thus, many papers have been written about FIDEs. Solving such kinds of problems is so difficult. Therefore, we need uniform and robust numerical techniques [6, 8, 10, 11, 15, 23, 25, 28](see, also references therein).

These mentioned studies related to FIDEs were only related to the regular cases (i.e. when the boundary layers are absent). Until now, numerical investigation of SPFIDEs has not been common yet. Amiraliyev et al. recently constructed an exponential-difference scheme with an accuracy of $O(N^{-1})$ for a first-order linear SPFIDE on a uniform grid in [1], and a finite-difference scheme with an accuracy of $O\left(N^{-2}\ln N\right)$ on a Shishkin grid for the same problem in [2]. The major contribution of this study is to offer an elegant and effective numerical technique with $O\left(N^{-2}\ln N\right)$ accuracy for second order linear SPFIDE with Robin boundary condition on Shishkin mesh.

This study is organized in a subsequent manner. Properties of the solution, description of discretization are given in Section 2. Mesh and error estimates are presented in Section 3. Computational results are given to support the predicted theory in Section 4. Finally, inferences and discussions are given in the conclusion.

2. The continuous problem and difference scheme

Here, we have stated some analytical bounds that will be used later during our error analysis.

Lemma 2.1 *Let* $a, f \in C^2[0,l]$, $\frac{\partial^m K}{\partial x^m} \in C[0,l]^2$, $(m = 0,1,2)$ *and*

$$|\lambda| < \frac{\alpha}{\max\limits_{0\le x\le l} \int\limits_0^l |K(x,s)|\,ds},$$

then the solution $u(x)$ *of the problem (1.1)-(1.3) satisfies the estimates:*

$$\|u\|_\infty \le C, \tag{2.1}$$

$$\left|u^{(k)}(x)\right| \le C\left(1+\varepsilon^{-\frac{k}{2}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)\right), \quad (k=1,2), \quad x\in[0,l]. \tag{2.2}$$

Proof The proof is by like approach as in [3, 26]. □

Now, we now turn to the establishment of the difference scheme. Let ω_N be any nonuniform mesh on $[0,l]$:

$$\omega_N = \{0 < x_1 < ... < x_{N-1} < l,\ \ h_i = x_i - x_{i-1}\}$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = l\}.$$

To any mesh function $v(x)$ defined on $\bar{\omega}_N$, we use

$$v_i = v(x_i), \quad \hbar_i = \frac{h_i + h_{i+1}}{2}, \quad \|v\|_\infty \equiv \|v\|_{\infty,\bar{\omega}_N} := \max_{0\le i\le N} |v_i|, \quad v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}.$$

We construct the numerical method using the identity

$$\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} (-\varepsilon u'' + a(x)u)\,\varphi_i(x)dx \;+\; \chi_i \hbar_i^{-1} \lambda \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x) \left(\int\limits_0^l K(x,s)\,u(s)\,ds \right) dx$$
$$= \chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx, \quad 1 \le i \le N-1, \tag{2.3}$$

with the basis functions

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \frac{sinh\gamma_i(x - x_{i-1})}{sinh\gamma_i h_i}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) \equiv \frac{sinh\gamma_i(x_{i+1} - x)}{sinh\gamma_i h_{i+1}}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

where $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are the solutions of the following problems, respectively:

$$-\varepsilon\varphi''(x) + a_i\varphi(x) = 0, \qquad x_{i-1} < x < x_i,$$
$$\varphi(x_{i-1}) = 0, \quad \varphi(x_i) = 1.$$

$$-\varepsilon\varphi''(x) + a_i\varphi(x) = 0, \qquad x_i < x < x_{i+1},$$
$$\varphi(x_i) = 1, \quad \varphi(x_{i+1}) = 0.$$

and

$$\gamma_i = \sqrt{\frac{a(x_i)}{\varepsilon}},$$

$$\chi_i = \left(\hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx \right)^{-1} = \frac{\gamma_i \hbar_i}{tanh\left(\frac{\gamma_i h_i}{2}\right) + tanh\left(\frac{\gamma_i h_{i+1}}{2}\right)}.$$

Using the appropriate quadratures formulas with the remainder term in integral form [4, 12, 17, 27] (see also [24], pp. 207-214), to the first term at the right side of (2.3) we get

$$\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} (-\varepsilon u'' + a(x)u)\,\varphi_i(x)dx = -\varepsilon\delta^2 u_i + a_i u_i + \chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]\,u(x)\,\varphi_i(x)dx, \tag{2.4}$$

where

$$\delta^2 u_i = \frac{\theta_i^{(2)} u_{x,i} - \theta_i^{(1)} u_{\overline{x},i}}{\hbar_i}, \tag{2.5}$$

with

$$\theta_i^{(1)} = \frac{a_i h_i \hbar_i}{\varepsilon sinh\left(\gamma_i h_i\right)\left[tanh\left(\frac{\gamma_i h_i}{2}\right) + tanh\left(\frac{\gamma_i h_{i+1}}{2}\right)\right]},$$

$$\theta_i^{(2)} = \frac{a_i h_{i+1} \hbar_i}{\varepsilon sinh\left(\gamma_i h_{i+1}\right)\left[tanh\left(\frac{\gamma_i h_i}{2}\right) + tanh\left(\frac{\gamma_i h_{i+1}}{2}\right)\right]}.$$

By Newton interpolation formula in respect to mesh points x_i, x_{i+1} we have

$$a(x) = a(x_i) + (x - x_i)a_{x,i} + \frac{1}{2}(x - x_i)(x - x_{i+1})a''\big(\xi_i(x)\big).$$

Therefore we get

$$\begin{aligned} \chi_i \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \left[a(x) - a(x_i)\right] u(x)\varphi_i(x)dx &= \chi_i \hbar_i^{-1} a_{x,i} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)u(x)\varphi_i(x)dx \\ &+ \frac{1}{2}\chi_i \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)(x - x_{i+1})a''\big(\xi_i(x)\big)u(x)\varphi_i(x)dx. \end{aligned} \tag{2.6}$$

Also using

$$u(x) = u(x_i) + \int_{x_i}^{x} u'(t)dt,$$

in the first term at the right side of (2.6), we have

$$\chi_i h_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \left[a(x) - a(x_i)\right] u(x)\varphi_i(x)dx = a_{x,i}\chi_i \mu_i u_i + R_i^{(1)},$$

where

$$\mu_i = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)\varphi_i(x)dx,$$

$$R_i^{(1)} = \frac{1}{2}\chi_i \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)(x - x_{i+1})a''\big(\xi_i(x)\big)u(x)\varphi_i(x)dx + \chi_i \hbar_i^{-1} a_{x,i} \int_{x_{i-1}}^{x_{i+1}} (x - x_i)\varphi_i(x)\left(\int_{x_i}^{x} u'(t)dt\right)dx. \tag{2.7}$$

Easy computation gives

$$\mu_i = \hbar_i^{-1}\gamma_i^{-1}\left(\frac{h_i}{sinh\left(\gamma_i h_i\right)} - \frac{h_{i+1}}{sinh\left(\gamma_i h_{i+1}\right)}\right). \tag{2.8}$$

To fixed step size we possess $\mu_i = 0$.
Thereby the identity (2.4) degrades to

$$\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} L_1 u \varphi_i(x) dx = -\varepsilon \delta^2 u_i + \bar{a}_i u_i + R_i^{(1)}, \tag{2.9}$$

where

$$\bar{a}_i = a_i + a_{x,i} \chi_i \mu_i. \tag{2.10}$$

Similarly, we derive

$$\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} f(x) \varphi_i(x) dx = f_i + f_{x,i} \chi_i \mu_i + R_i^{(2)} = \bar{f}_i + R_i^{(2)}, \tag{2.11}$$

where

$$\bar{f}_i = f_i + f_{x,i} \chi_i \mu_i, \tag{2.12}$$

$$R_i^{(2)} = \frac{1}{2} \chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)(x - x_{i+1}) f''(\xi_i(x)) \varphi_i(x) dx. \tag{2.13}$$

For $h_i = h = const.$, will be $\bar{f}_i = f_i$.
Subsequently, for the integral part in (2.3), using the Taylor expansion

$$K(x,s) = K(x_i,s) + (x - x_i)\frac{\partial}{\partial x} K(x_i,s) + \frac{(x - x_i)^2}{2} \frac{\partial^2}{\partial x^2} K\big(\xi_i(x), s\big), \tag{2.14}$$

we get

$$\begin{aligned} \chi_i \hbar_i^{-1} \lambda \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x) \Bigg(\int\limits_0^l K(x,s) u(s) ds \Bigg) dx &= \lambda \int\limits_0^l K(x_i,s) u(s) ds + \mu_i \lambda \int\limits_0^l \frac{\partial}{\partial x} K(x_i,s) u(s) ds + R_i^{(3)} \\ &\equiv \lambda \int\limits_0^l \mathcal{K}(x_i,s) u(s) ds + R_i^{(3)}, \end{aligned} \tag{2.15}$$

where

$$\mathcal{K}(x_i,s) = K(x_i,s) + \mu_i \frac{\partial}{\partial x} K(x_i,s), \tag{2.16}$$

$$R_i^{(3)} = \frac{1}{2} \chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2 \varphi_i(x) \Bigg(\int\limits_0^l \frac{\partial^2}{\partial x^2} K\big(\xi_i(x), s\big) u(s) ds \Bigg) dx \tag{2.17}$$

and μ_i is given by (2.8).
Finally, if the first term at the right side of (2.15) is operated by applying the composite trapezoidal integration

[3], we get

$$\lambda \int_0^l \mathcal{K}(x_i,s)u(s)ds = \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} u_j + R_i^{(4)}, \tag{2.18}$$

where

$$R_i^{(4)} = \frac{1}{2}\lambda \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j-\xi)(x_{j-1}-\xi)\frac{d^2}{d\xi^2}\big(\mathcal{K}(x_i,\xi)u(\xi)\big)d\xi. \tag{2.19}$$

Combining (2.9),(2.11),(2.15) and (2.18) in (2.3) we obtain following difference scheme

$$L_N u_i := -\varepsilon\delta^2 u_i + \bar{a}_i u_i + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} u_j = \bar{f}_i - R_i \tag{2.20}$$

with remainder term

$$R_i = \sum_{k=1}^{4} R_i^{(k)}, \tag{2.21}$$

where $R_i^{(k)}, (k=1,2,3,4)$ are defined by (2.7),(2.13),(2.17), and (2.19), respectively.
To define an approximation for the boundary condition (1.2), we start with the identity

$$\chi_0^{-1}\int_0^{x_1} L_1 u\varphi_0(x)dx + \chi_0^{-1}\lambda \int_0^{x_1} \varphi_0(x)\left(\int_0^l K(x,s)u(s)ds\right)dx = \chi_0^{-1}\int_0^{x_1} f(x)\varphi_0(x)dx, \tag{2.22}$$

where

$$\chi_0 = \sqrt{\varepsilon} + \frac{a_0}{\beta}\int_0^{x_1}\varphi_0(x)dx = \sqrt{\varepsilon} + a_0\beta^{-1}\gamma_0^{-1}\tanh\left(\frac{\gamma_0 h_1}{2}\right), \quad \gamma_0 = \sqrt{\frac{a_0}{\varepsilon}},$$

$$\varphi_0(x) = \begin{cases} \frac{\sinh\gamma_0(x_1-x)}{\sinh\gamma_0 h_1}, & x_0<x<x_1, \\ 0, & x\notin(x_0,x_1) \end{cases}$$

and the basis function $\varphi_0(x)$ is the solution of the problem

$$-\varepsilon\varphi''(x) + a_0\varphi(x) = 0, \qquad x_0<x<x_1,$$
$$\varphi(x_0)=1, \quad \varphi(x_1)=0.$$

Using the appropriate quadratures formulas with the remainder term in integral form [4, 12, 17, 27] (see also [24], pp. 207-214), for the differential part from (2.22) we have

$$\chi_0^{-1}\int_0^{x_1} L_1 u\varphi_0(x)dx = -\sqrt{\varepsilon}\theta_0 u_{x,0} + \beta u_0 - \bar{A} + \chi_0^{-1}\int_0^{x_1}[a(x)-a(0)]\,u(x)\,\varphi_0(x)dx, \tag{2.23}$$

where

$$\theta_0 = \chi_0^{-1}\left(\sqrt{\varepsilon} - \frac{a_0}{\sqrt{\varepsilon}}\int\limits_0^{x_1} x\varphi_0(x)dx\right), \qquad \bar{A} = \chi_0^{-1}\sqrt{\varepsilon}A.$$

Simple calculation gives

$$\theta_0 = \frac{\gamma_0 h_1}{\sinh(\gamma_0 h_1) + 2\sqrt{a_0}\beta^{-1}\sinh^2\left(\frac{\gamma_0 h_1}{2}\right)}, \tag{2.24}$$

$$\bar{A} = \frac{A\sqrt{\varepsilon}}{\sqrt{\varepsilon} + a_0\beta^{-1}\gamma_0^{-1}\tanh\left(\frac{\gamma_0 h_1}{2}\right)}. \tag{2.25}$$

By Newton interpolation formula, we have

$$a(x) - a(0) = xa_{x,0} + \frac{1}{2}x(x - x_1)a''(\xi_0(x)).$$

Therefore we get

$$\begin{aligned}\chi_0^{-1}\int\limits_0^{x_1}[a(x) - a(0)]u(x)\varphi_0(x)dx &= \chi_0^{-1}a_{x,0}\int\limits_0^{x_1} xu(x)\varphi_0(x)dx \\ &+ \frac{1}{2}\chi_0^{-1}\int\limits_0^{x_1} x(x - x_1)a''(\xi_0(x))u(x)\varphi_0(x)dx.\end{aligned} \tag{2.26}$$

Also using

$$u(x) = u(0) + \int\limits_0^{x} u'(t)dt,$$

in the first term at the right side of (2.26), we have

$$\chi_0^{-1}\int\limits_0^{x_1}[a(x) - a(0)]u(x)\varphi_0(x)dx = \chi_0^{-1}a_{x,0}\mu_0 u_0 + r^{(1)},$$

where

$$\mu_0 = \int\limits_0^{x_1} x\varphi_0(x)dx = \frac{1}{\gamma_0^2}\sinh(\gamma_0 h_1) - \frac{h_1}{\gamma_0}, \tag{2.27}$$

$$r^{(1)} = \frac{1}{2}\chi_0^{-1}\int\limits_0^{x_1} x(x - x_1)a''(\xi_0(x))u(x)\varphi_0(x)dx + \chi_0^{-1}a_{x,0}\int\limits_0^{x_1} x\varphi_0(x)\left(\int\limits_0^{x} u'(t)dt\right)dx. \tag{2.28}$$

Thereby the identity (2.23) degrades to

$$\chi_0^{-1}\int\limits_0^{x_1} L_1 u\varphi_0(x)dx = -\sqrt{\varepsilon}\theta_0 u_{x,0} + \bar{\beta}u_0 - \bar{A} + r^{(1)}, \tag{2.29}$$

where

$$\bar{\beta} = \beta + \chi_0^{-1} a_{x,0}\mu_0 \tag{2.30}$$

and μ_0 is given by (2.27).
Analogously we derive

$$\chi_0^{-1} \int_0^{x_1} f(x)\varphi_0(x)dx = k_1 f_0 + k_2 f_{x,0} + r^{(2)}, \tag{2.31}$$

where

$$k_1 = \chi_0^{-1} \int_0^{x_1} \varphi_0(x)dx = \frac{\tanh\left(\frac{\gamma_0 h_1}{2}\right)}{\sqrt{a_0} + a_0\beta^{-1}\gamma_0^{-1}\tanh\left(\frac{\gamma_0 h_1}{2}\right)}, \tag{2.32}$$

$$k_2 = \chi_0^{-1}\mu_0 = \frac{\frac{1}{\gamma_0^2}\sinh\left(\gamma_0 h_1\right) - \frac{h_1}{\gamma_0}}{\sqrt{\varepsilon} + a_0\beta^{-1}\gamma_0^{-1}\tanh\left(\frac{\gamma_0 h_1}{2}\right)}, \tag{2.33}$$

$$r^{(2)} = \frac{1}{2}\chi_0^{-1} \int_0^{x_1} x(x-x_1)f''(\xi_0(x))\varphi_0(x)dx. \tag{2.34}$$

Using (2.14) for $i=0$ we get

$$\begin{aligned} \chi_0^{-1}\lambda \int_0^{x_1} \varphi_0(x)\left(\int_0^{l} K(x,s)u(s)ds\right)dx &= k_1\lambda \int_0^{l} K(0,s)u(s)ds + k_2\lambda \int_0^{l} \frac{\partial}{\partial x}K(0,s)u(s)ds + r^{(3)} \\ &\equiv \lambda \int_0^{l} \mathcal{M}(s)u(s)ds + r^{(3)}, \end{aligned} \tag{2.35}$$

where

$$\mathcal{M}(s) = k_1 K(0,s) + k_2 \frac{\partial}{\partial x} K(0,s), \tag{2.36}$$

$$r^{(3)} = \frac{1}{2}\chi_0^{-1}\lambda \int_0^{x_1} x^2\varphi_0(x)\left(\int_0^{l} \frac{\partial^2}{\partial x^2}K(\xi_0(x), s)u(s)ds\right)dx. \tag{2.37}$$

Further applying the composite trapezoidal rule [3] on $[0,l]$, we have

$$\lambda \int_0^{l} \mathcal{M}(s)u(s)ds = \lambda \sum_{j=0}^{N} \hbar_j \mathcal{M}_j u_j + r^{(4)}, \tag{2.38}$$

where

$$r^{(4)} = \frac{1}{2}\lambda \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j - \xi)(x_{j-1} - \xi)\frac{d^2}{d\xi^2}\left(\mathcal{M}(\xi)u(\xi)\right)d\xi. \tag{2.39}$$

After taking into consideration (2.29), (2.31), (2.35) and (2.38) in (2.22), we obtain

$$L_0 u := -\sqrt{\varepsilon}\theta_0 u_{x,0} + \bar{\beta} u_0 + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{M}_j u_j = \bar{A} + k_1 f_0 + k_2 f_{x,0} - r, \tag{2.40}$$

with remainder term

$$r = r^{(1)} - r^{(2)} + r^{(3)} + r^{(4)}, \tag{2.41}$$

where $r^{(1)}$, $r^{(2)}$, $r^{(3)}$ and $r^{(4)}$ are identified by (2.28), (2.34), (2.37) and (2.39) separately.
Based on (2.20) and (2.40), we present the following difference scheme for approximating the problem (1.1)-(1.3):

$$L_N y_i := -\varepsilon \delta^2 y_i + \bar{a}_i u_i + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} y_j = \bar{f}_i, \quad 1 \le i \le N-1, \tag{2.42}$$

$$L_0 y := -\sqrt{\varepsilon}\theta_0 y_{x,0} + \bar{\beta} y_0 + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{M}_j y_j = \bar{A} + k_1 f_0 + k_2 f_{x,0}, \tag{2.43}$$

$$y_N = B, \tag{2.44}$$

where δ^2, $\bar{a}_i$, $\bar{f}_i$, $\mathcal{K}_{ij}$, θ_0, $\bar{A}$, $\bar{\beta}$, k_1, k_2 and $\mathcal{M}_j$ are given by (2.5), (2.10), (2.12), (2.16), (2.24), (2.25), (2.30), (2.32), (2.33), and (2.36) respectively.

3. The mesh and convergence

In order for the difference scheme (2.42)-(2.44) to be ε-uniform convergent, we use the Shishkin mesh. For a positive integer N, that can be divided by four, the interval $[0,l]$ is divided into three subintervals $[0,\sigma]$, $[\sigma, l-\sigma]$ and $[l-\sigma, l]$. Here σ is meant the transition point, which splits the fine and coarse parts of the mesh is acquired by receiving

$$\sigma = \min\left\{\frac{l}{4}, \left(\sqrt{\alpha}\right)^{-1}\sqrt{\varepsilon}\ln N\right\}.$$

The mesh is fine on $[0,\sigma]$ and $[l-\sigma, l]$ and coarse on $[\sigma, l]$. Therefore, if we indicate the step sizes in $[0,\sigma]$, $[\sigma, l-\sigma]$, and $[l-\sigma, l]$ by $h^{(1)}$ and $h^{(2)}$, separately, our Shishkin mesh can be phrased as

$$\overline{\omega}_N = \begin{cases} x_i = ih^{(1)}, & i = 0,1,...,\frac{N}{4}; & h^{(1)} = \frac{4\sigma}{N}; \\ x_i = \sigma + \left(i - \frac{N}{4}\right)h^{(2)}, & i = \frac{N}{4}+1,...,\frac{3N}{4}; & h^{(2)} = \frac{2(l-2\sigma)}{N}; \\ x_i = l - \sigma + \left(i - \frac{3N}{4}\right)h^{(1)}, & i = \frac{3N}{4}+1,...,N; & h^{(1)} = \frac{4\sigma}{N}. \end{cases}$$

Lemma 3.1 *For $a, f \in C^2[0,l]$ and $\frac{\partial^m K}{\partial x^m} \in C^2[0,l]^2, (m = 0,1,2)$. Then the truncation errors of the difference scheme (2.20) and (2.40) satisfy*

$$|R_i| \le CN^{-2}\ln N, \quad 1 \le i \le N-1, \tag{3.1}$$

$$|r| \le CN^{-2}\ln N. \tag{3.2}$$

Proof We estimate $R_i^{(1)}, R_i^{(2)}, R_i^{(3)}$ and $R_i^{(4)}$ individually. We will handle the case $\sigma = (\sqrt{\alpha})^{-1}\sqrt{\varepsilon}\ln N$. The case $\sigma = l/4$ can be analysed in the classic way. Here we will make the estimate on $[0,\sigma]$, $[\sigma, l-\sigma]$ and $[l-\sigma, l]$ separately.

First, for $R_i^{(2)}$. Since $f \in C^2[0,l]$, $h^{(1)}$ and $h^{(2)} \leq CN^{-1}$, $|x - x_{i+1}| \leq 2\hbar_i$ and $|x - x_i| \leq \max(h_i, h_{i+1})$, by then evidently from $R_i^{(2)}$:

$$
\begin{aligned}
\left|R_i^{(2)}\right| &\leq C\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} |(x - x_i)(x - x_{i+1})| f''(\xi_i(x))\varphi_i(x)\,dx \\
&\leq C\{\max(h_i, h_{i+1})\}^2 \\
&\leq CN^{-2}. \qquad (3.3)
\end{aligned}
$$

Secondly, for $R_i^{(3)}$ in the similar way as above, the boundedness of $\frac{\partial^2 K}{\partial x^2}$ and considering (2.1), from (2.17) it follows that

$$
\begin{aligned}
\left|R_i^{(3)}\right| &\leq \left|\frac{1}{2}\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^2 \varphi_i(x) \Bigg(\int\limits_0^l \frac{\partial^2}{\partial x^2} K\big(\xi_i(x), s\big) u(s) ds \Bigg) dx\right| \\
&\leq C\{\max(h_i, h_{i+1})\}^2 \\
&\leq CN^{-2}. \qquad (3.4)
\end{aligned}
$$

Thirdly, for $R_i^{(1)}$, since $|u(x)| \leq C$ and $a \in C^2[0,l]$, from (2.7), we get

$$
\begin{aligned}
\left|R_i^{(1)}\right| &\leq \left|\frac{1}{2}\chi_i \hbar_i^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)(x - x_{i+1}) a''\big(\xi_i(x)\big) u(x) \varphi_i(x) dx\right| \\
&+ \left|\chi_i \hbar_i^{-1} a_{x,i} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)\varphi_i(x) \Bigg(\int\limits_{x_i}^{x} u'(t) dt \Bigg) dx\right| \\
&\leq C\{\max(h_i, h_{i+1})\}^2 + C\max(h_i, h_{i+1}) \int\limits_{x_{i-1}}^{x_{i+1}} |u'(x)|\,dx \\
&\leq CN^{-1}\left\{ N^{-1} + \int\limits_{x_{i-1}}^{x_{i+1}} \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}} \right) dx \right\}. \qquad (3.5)
\end{aligned}
$$

The estimate of $R_i^{(1)}$ in the layer area $[0,\sigma]$ degrades to

$$
\begin{aligned}
\left|R_i^{(1)}\right| &\leq CN^{-1}\left(N^{-1} + \frac{4h^{(1)}}{\sqrt{\varepsilon}} \right) \\
&= CN^{-1}\left(N^{-1} + 4\left(\sqrt{\alpha}\right)^{-1} N^{-1}\ln N \right) \\
&\leq CN^{-2}\ln N, \qquad 1 \leq i \leq \frac{N}{4} - 1. \qquad (3.6)
\end{aligned}
$$

The same evaluation is acquired in the layer area $[l-\sigma, l]$ in the similarly.
For $\frac{N}{4}+1 \leq i \leq \frac{3N}{4}-1$ from (3.5) it follows that

$$
\begin{aligned}
\left|R_i^{(1)}\right| &\leq CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}\left(e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}-e^{-\frac{\sqrt{\alpha}x_{i+1}}{\sqrt{\varepsilon}}}\right)\right\} \\
&+ CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}\left(e^{-\frac{\sqrt{\alpha}(l-x_{i+1})}{\sqrt{\varepsilon}}}-e^{-\frac{\sqrt{\alpha}(l-x_{i-1})}{\sqrt{\varepsilon}}}\right)\right\} \\
&\leq CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}x_{i-1}}{\sqrt{\varepsilon}}}\left(1-e^{-\frac{2\sqrt{\alpha}h_i^{(2)}}{\sqrt{\varepsilon}}}\right)\right\} \\
&+ CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}(l-x_{i+1})}{\sqrt{\varepsilon}}}\left(1-e^{-\frac{2\sqrt{\alpha}h_i^{(2)}}{\sqrt{\varepsilon}}}\right)\right\} \\
&\leq CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}x_{\frac{N}{4}}}{\sqrt{\varepsilon}}}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}\left(l-x_{\frac{3N}{4}}\right)}{\sqrt{\varepsilon}}}\right\} \\
&\leq CN^{-2}. \qquad (3.7)
\end{aligned}
$$

It remains to estimate $R_i^{(1)}$ for $i=\frac{N}{4}$ and $i=\frac{3N}{4}$. For the $i=\frac{N}{4}$ inequality (3.5) degrades to

$$
\begin{aligned}
\left|R_{\frac{N}{4}}^{(1)}\right| &\leq CN^{-1}\left\{N^{-1}+\int\limits_{x_{\frac{N}{4}-1}}^{x_{\frac{N}{4}+1}}\frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)dx\right\} \\
&= CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}\left(e^{-\frac{\sqrt{\alpha}x_{\frac{N}{4}-1}}{\sqrt{\varepsilon}}}-e^{-\frac{\sqrt{\alpha}x_{\frac{N}{4}+1}}{\sqrt{\varepsilon}}}\right)\right\} \\
&+ CN^{-1}\left(\sqrt{\alpha}\right)^{-1}\left(e^{-\frac{\sqrt{\alpha}\left(l-x_{\frac{N}{4}+1}\right)}{\sqrt{\varepsilon}}}-e^{-\frac{\sqrt{\alpha}\left(l-x_{\frac{N}{4}-1}\right)}{\sqrt{\varepsilon}}}\right) \\
&= CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}x_{\frac{N}{4}-1}}{\sqrt{\varepsilon}}}\left(1-e^{-\frac{\sqrt{\alpha}\left(x_{\frac{N}{4}+1}-h^{(1)}-h^{(2)}\right)}{\sqrt{\varepsilon}}}\right)\right\} \\
&+ CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}\left(l-x_{\frac{N}{4}+1}\right)}{\sqrt{\varepsilon}}}\left(1-e^{-\frac{\sqrt{\alpha}\left(l-x_{\frac{N}{4}-1}-h^{(1)}-h^{(2)}\right)}{\sqrt{\varepsilon}}}\right)\right\} \\
&\leq CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}x_{\frac{N}{4}}}{\sqrt{\varepsilon}}}e^{\frac{\sqrt{\alpha}h^{(1)}}{\sqrt{\varepsilon}}}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}\sigma}{\sqrt{\varepsilon}}}e^{-\frac{\sqrt{\alpha}\left(l-h^{(2)}\right)}{\sqrt{\varepsilon}}}\right\} \\
&\leq CN^{-1}\left\{N^{-1}+\left(\sqrt{\alpha}\right)^{-1}e^{\frac{4\ln N}{N}}e^{-\frac{\sqrt{\alpha}x_{\frac{N}{4}}}{\sqrt{\varepsilon}}}+\left(\sqrt{\alpha}\right)^{-1}e^{-\frac{\sqrt{\alpha}\sigma}{\sqrt{\varepsilon}}}\right\} \\
&\leq CN^{-2}. \qquad (3.8)
\end{aligned}
$$

The same evaluation is acquired for $i=\frac{3N}{4}$ in the similarly.

Based on (3.6)-(3.8), it is possible to deduce the following inequality

$$\left|R_i^{(1)}\right| \leq CN^{-2}\ln N, \qquad 1 \leq i \leq N-1. \tag{3.9}$$

Finally, for $R_i^{(4)}$ from (2.19), by virtue of (2.2), we get

$$\begin{aligned}
\left|R_i^{(4)}\right| &\leq \left|\frac{1}{2}\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{d^2}{d\xi^2}\big(\mathcal{K}(x_i,\xi)u(\xi)\big)d\xi\right| \\
&\leq C\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\left(1+|u'(\xi)|+|u''(\xi)|\right)d\xi \\
&\leq C\left\{\sum_{j=1}^{N}h_j^3+\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi\right\}.
\end{aligned} \tag{3.10}$$

For the right side first term of (3.10), we get

$$\sum_{j=1}^{N}h_j^3=\frac{N}{2}\left|h^{(1)}\right|^3+\frac{N}{2}\left|h^{(2)}\right|^3 \leq CN^{-2}. \tag{3.11}$$

For the right side remaining term of (3.10) we can write

$$\begin{aligned}
&\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi \\
&=\sum_{j=1}^{\frac{N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi \\
&+\sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi \\
&+\sum_{j=\frac{3N}{4}+1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi.
\end{aligned} \tag{3.12}$$

For the right side first sum of (3.12), we get

$$\begin{aligned}
\sum_{j=1}^{\frac{N}{4}}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi &\leq \left|h^{(1)}\right|^2\int_0^{\sigma}\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi \\
&\leq \left|h^{(1)}\right|^2\frac{2\left(\sqrt{\alpha}\right)^{-1}}{\sqrt{\varepsilon}} \\
&\leq CN^{-2}\ln N.
\end{aligned} \tag{3.13}$$

The same evaluation is acquired for the right side third sum of (3.12) in the similarly.
Further, for the right side second sum of (3.12) after integration in parts, we have

$$\sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}} \int_{x_{j-1}}^{x_j} (x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi$$
$$=2\left(\sqrt{\alpha}\right)^{-1}\sum_{j=\frac{N}{4}+1}^{\frac{3N}{4}} \int_{x_{j-1}}^{x_j}\left(x_j-\xi-\frac{h^{(2)}}{2}\right)\frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi$$
$$\leq 2\left(\sqrt{\alpha}\right)^{-1}h^{(2)}\int_{\sigma}^{l-\sigma}\frac{1}{\sqrt{\varepsilon}}\left(e^{-\frac{\sqrt{\alpha}\xi}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-\xi)}{\sqrt{\varepsilon}}}\right)d\xi$$
$$=4\left(\sqrt{\alpha}\right)^{-2}h^{(2)}\left(e^{-\frac{\sqrt{\alpha}\sigma}{\sqrt{\varepsilon}}}-e^{-\frac{\sqrt{\alpha}(l-\sigma)}{\sqrt{\varepsilon}}}\right)$$
$$\leq 4\left(\sqrt{\alpha}\right)^{-2}h^{(2)}N^{-1}\leq CN^{-2}. \tag{3.14}$$

Based on (3.11)-(3.14), it is possible to deduce the following inequality

$$\left|R_i^{(4)}\right|\leq CN^{-2}\ln N, \qquad 1\leq i\leq N-1. \tag{3.15}$$

Substituting the estimates (3.3), (3.4), (3.9) and (3.15) in (2.21), we arrive at (3.1).
We now prove the inequality (3.2). First, we estimate $r^{(1)}$. Since $|u(x)|\leq C$ and $a\in C^2[0,l]$, from (2.28), we get

$$\begin{aligned}
\left|r^{(1)}\right| &\leq \left|\frac{1}{2}\chi_0^{-1}\int_0^{x_1} x(x-x_1)a''(\xi_0(x))u(x)\varphi_0(x)dx\right|+\left|\chi_0^{-1}a_{x,0}\int_0^{x_1}x\varphi_0(x)\left(\int_0^x u'(t)dt\right)dx\right|\\
&\leq Ch^{(1)}\left(h^{(1)}+\int_0^{x_1}\left|u'(x)\right|dx\right)\\
&\leq CN^{-1}\left(N^{-1}+\frac{1}{\sqrt{\varepsilon}}\int_0^{x_1}\left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}}+e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}}\right)dx\right)\\
&\leq CN^{-1}\left(N^{-1}+\frac{h^{(1)}}{\sqrt{\varepsilon}}\right)\\
&= CN^{-1}\left(N^{-1}+\left(\sqrt{\alpha}\right)^{-1}N^{-1}\ln N\right)\\
&\leq CN^{-2}\ln N.
\end{aligned} \tag{3.16}$$

Secondly for $r^{(2)}$, since $f \in C^2[0,l]$, from (2.34), we have

$$\begin{aligned}\left|r^{(2)}\right| &\leq \left|\frac{1}{2}\chi_0^{-1}\int\limits_0^{x_1} x(x-x_1)f''(\xi_0(x))\varphi_0(x)dx\right| \\ &\leq C\left|h^{(1)}\right|^2 \\ &\leq CN^{-2}. \end{aligned} \tag{3.17}$$

Thirdly, for $r^{(3)}$ in the same way as above, the boundedness of $\frac{\partial^2 K}{\partial x^2}$ and considering (2.1), from (2.37) it follows that

$$\begin{aligned}\left|r^{(3)}\right| &\leq \left|\frac{1}{2}\chi_0^{-1}\lambda\int\limits_0^{x_1} x^2\varphi_0(x)\left(\int\limits_0^{l}\frac{\partial^2}{\partial x^2}K(\xi_0(x),s)u(s)ds\right)dx\right| \\ &\leq C\left|h^{(1)}\right|^2 \\ &\leq CN^{-2}. \end{aligned} \tag{3.18}$$

Finally, for $r^{(4)}$ in the same manner as in (3.15) one can show that

$$\left|r^{(4)}\right| \leq CN^{-2}\ln N. \tag{3.19}$$

The inequalities (3.16)-(3.19) imply that

$$|r| \leq CN^{-2}\ln N.$$

Thus, the proof of lemma is completed. □

For the error $z_i = y_i - u_i$ considering (2.20) and (2.40), we get

$$-\varepsilon\delta^2 z_i + \bar{a}_i z_i + \lambda\sum_{j=0}^{N}\hbar_j\mathcal{K}_{ij}z_j = R_i, \qquad 1 \leq i \leq N-1, \tag{3.20}$$

$$-\sqrt{\varepsilon}\theta_0 z_{x,0} + \bar{\beta}z_0 + \lambda\sum_{j=0}^{N}\hbar_j\mathcal{M}_j z_j = r, \tag{3.21}$$

$$z_N = 0, \tag{3.22}$$

where R_i and $r^{(0)}$ are defined by (2.21) and (2.41). We note that since $a \in C^2[0,l]$, then exists a number $\bar{\alpha}$ such that for sufficiently large values of N will be $\bar{a}_i \geq \bar{\alpha} > 0$.

Theorem 3.2 *Let* $\frac{\partial^m K}{\partial x^m}, \frac{\partial^{m+1}K}{\partial x\partial s^m} \in C^2[0,l]^2\ (m=0,1,2)$, $a, f \in C^2[0,l]$ *and*

$$|\lambda| < \frac{1}{\sum\limits_{j=0}^{N}\hbar_j\left[(\bar{\alpha})^{-1}\max\limits_{1\leq i\leq N}|\mathcal{K}_{ij}| + \left(\bar{\beta}\right)^{-1}|\mathcal{M}_j|\right]}.$$

Then for the error of the scheme (2.42)-(2.44), we get

$$\|y-u\|_{\infty,\bar{\omega}_N} \le CN^{-2}\ln N.$$

Proof By applying the discrete maximum principle, from (3.20)-(3.22) we have

$$\begin{aligned}\|z\|_{\infty,\bar{\omega}_N} &\le (\bar{\alpha})^{-1}\left\|R-\lambda\sum_{j=0}^{N}\hbar_j\mathcal{K}_{ij}z_j\right\|_{\infty,\omega_N}+\left(\bar{\beta}\right)^{-1}\left|r-\lambda\sum_{j=0}^{N}\hbar_j\mathcal{M}_jz_j\right|\\ &\le (\bar{\alpha})^{-1}\|R\|_{\infty,\omega_N}+|\lambda|(\bar{\alpha})^{-1}\max_{1\le i\le N}\sum_{j=0}^{N}\hbar_j|\mathcal{K}_{ij}|\,\|z\|_{\infty,\bar{\omega}_N}+\left(\bar{\beta}\right)^{-1}|r|+|\lambda|\left(\bar{\beta}\right)^{-1}\sum_{j=0}^{N}\hbar_j|\mathcal{M}_j|\,|z_j|,\end{aligned}$$

hence

$$\|z\|_{\infty,\bar{\omega}_N} \le \frac{(\bar{\alpha})^{-1}\|R\|_{\infty,\omega_N}+\left(\bar{\beta}\right)^{-1}|r|}{1-|\lambda|\left((\bar{\alpha})^{-1}\max\limits_{1\le i\le N}\sum\limits_{j=0}^{N}\hbar_j|\mathcal{K}_{ij}|+\left(\bar{\beta}\right)^{-1}\sum\limits_{j=0}^{N}\hbar_j|\mathcal{M}_j|\right)}.$$

Thereby

$$\|z\|_{\infty,\bar{\omega}_N} \le C\|R\|_{\infty,\omega_N}.$$

This inequality together with (3.1) and (3.2) gives the desired result. □

4. Numerical results

Our particular example is

$$-\varepsilon u''+\left(2-e^{-x}\right)u+\frac{1}{2}\int_0^1\left(e^{xcos(\pi s)}-1\right)u(s)ds=\frac{1}{1+x},$$

$$-\sqrt{\varepsilon}u'(0)+2u(0)=1,\qquad u(1)=0.$$

The exact solution of this problem is unknown. Hereby, we use the double mesh principle. We define the maximum point-wise errors and the computed ε-uniform maximum point-wise errors as follows

$$e_\varepsilon^N=\max_i|y_i^{\varepsilon,N}-\tilde{y}_{2i}^{\varepsilon,2N}|_{\infty,\overline{\omega}_N},\quad e^N=\max_\varepsilon e_\varepsilon^N,$$

where $\tilde{y}_i^{\varepsilon,2N}$ is the approximate solution of the respective method on the mesh

$$\tilde{\omega}_{2N}=\{x_{i/2}:i=0,1,...,2N\}$$

with

$$x_{i+1/2}=\frac{x_i+x_{i+1}}{2}\qquad\text{for}\qquad i=0,1,...,N-1.$$

We also define the rates of convergence and computed ε-uniform the rates of convergence of the form

$$p_\varepsilon^N=\frac{\ln\left(e_\varepsilon^N/e_\varepsilon^{2N}\right)}{\ln 2},\quad p^N=\frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

Results in Table display that the rate of convergence of the difference scheme is substantially in accordance with the theoretical analysis.

Table . Maximum point-wise errors and the rates of convergence for different vales of ε and N.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^0	0.028066	0.008212	0.002228	0.000583	0.000149
	1.773	1.822	1.934	1.968	
2^{-2}	0,034153	0.009931	0.002776	0.000726	0.000185
	1.782	1.839	1.935	1.973	
2^{-4}	0.055712	0.016144	0.004497	0.001167	0.000294
	1.787	1.844	1.946	1.989	
2^{-6}	0.058525	0.016912	0.004688	0.001215	0.000305
	1.791	1.851	1.948	1.993	
2^{-8}	0.061625	0.017636	0.004865	0.001253	0.000314
	1.805	1.858	1.957	1.997	
e^N	0.061625	0.017636	0.004865	0.001253	0.000314
p^N	1.805	1.858	1.957	1.997	

5. Conclusion

A novel second order numerical approach for solving the second order Fredholm integro differential equation with boundary layers has been proposed. It has been done some estimates for the exact solution and its derivative before giving the numerical method. To solve the problem numerically, an exponential fitted finite difference approach on a nonequidistant mesh has been used. The difference method has been established by applying interpolating quadrature rules, with the remaining terms in integral form and including the basis functions. It has proved that the numerical scheme is almost second order convergent. The example is given which authorizes our proposed numerical approach. As expected, the accuracy increases with larger values of N. The presented method can be applied to different linear and nonlinear integro differential equations.

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