Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 225 – 241
© TÜBİTAK
doi:10.3906/mat-2109-15

New extension of Alexander and Libera integral operators

H. Özlem GÜNEY[1,*], **Shigeyoshi OWA**[2]
[1]Department of Mathematics, Faculty of Science, Dicle University, Diyarbakır, Turkey
[2]Honorary Professor "1 Decembrie 1918" , University Alba Iulia, Romania

Received: 30.09.2021 • **Accepted/Published Online:** 16.11.2021 • **Final Version:** 19.01.2022

Abstract: Let T be the class of analytic functions in the open unit disc $\mathbb{U}$ with $f(0) = 0$ and $f'(0) = 1$. For $f(z) \in T$, the Alexander integral operator $A_{-1}f(z)$, the Libera integral operator $L_{-1}f(z)$ and the Bernardi integral operator $B_{-1}f(z)$ were considered before. Using $A_{-1}f(z)$ and $L_{-1}f(z)$, a new integral operator $F_\lambda f(z)$ is considered. After discuss some properties of dominant for $F_\lambda f(z)$, another new integral operator $O_{-1}f(z)$ of $f(z) \in T$ is discussed. The object of the present paper is to discuss the dominant of new integral operators $F_\lambda f(z)$ and $O_{-1}f(z)$ concerning with some starlike functions and convex functions in $\mathbb{U}$.

Keywords: Analytic function, Alexander integral operator, Libera integral operator, Bernardi integral operator, dominant, new integral operator

1. Introduction

Let T be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Alexander [1] had defined the following the Alexander integral operator $A_{-1}f(z)$ given by

$$A_{-1}f(z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{k=2}^{\infty} \frac{a_k}{k} z^k \tag{1.2}$$

for $f(z) \in T$. Also, Libera [4] had defined

$$L_{-1}f(z) = \frac{2}{z} \int_0^z f(t)dt = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k \tag{1.3}$$

for $f(z) \in T$. Therefore, we said to be the Libera integral operator for $L_{-1}f(z)$. This the Libera integral operator is generalised as $B_{-1}f(z)$ which is said to be the Bernardi integral operator

$$B_{-1}f(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t)dt = z + \sum_{k=2}^{\infty} \frac{1+\gamma}{k+\gamma} a_k z^k \tag{1.4}$$

*Correspondence: ozlemg@dicle.edu.tr
2010 *AMS Mathematics Subject Classification:* 30C45

for $\gamma \in \mathbb{N} = \{1, 2, 3, \ldots\}$ by Bernardi [2]. More recently, Owa and Güney [7] have given new applications of the Bernardi integral operator.

The function $f(z) \in T$ is said to be starlike of order $\alpha \, (0 \leq \alpha < 1)$ in $\mathbb{U}$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{U}. \tag{1.5}$$

The class of starlike functions of order α in $\mathbb{U}$ is denoted by $S^*(\alpha)$. Also, the function $f(z) \in T$ is said to be convex of order $\alpha \, (0 \leq \alpha < 1)$ in $\mathbb{U}$ if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{U}. \tag{1.6}$$

The class of convex functions of order α in $\mathbb{U}$ is denoted by $K(\alpha)$. It follows from the above we know that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$. By Robertson [8], we have the extremal function for $S^*(\alpha)$ is

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{k=2}^{\infty} \frac{\prod_{j=2}^{k}(j - 2\alpha)}{(k-1)!} z^k \tag{1.7}$$

and the extremal function for $K(\alpha)$ is

$$f(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} = z + \sum_{k=2}^{\infty} \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} z^k; & \alpha \neq \frac{1}{2} \\ -log(1-z) = z + \sum_{k=2}^{\infty} \frac{1}{k} z^k & ; \quad \alpha = \frac{1}{2}. \end{cases} \tag{1.8}$$

Let us consider the new extension of the Alexander integral operator and the Libera integral operator by

$$\begin{aligned} F_\lambda f(z) &= \lambda A_{-1} f(z) + (1-\lambda) L_{-1} f(z) \qquad (1.9) \\ &= \int_0^z \left(\frac{\lambda}{t} + \frac{2(1-\lambda)}{z}\right) f(t) dt \\ &= z + \sum_{k=2}^{\infty} \left(\frac{\lambda}{k} + \frac{2(1-\lambda)}{k+1}\right) a_k z^k \end{aligned}$$

for $f(z) \in T$.

Example 1.1 *Let us consider the function*

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \in S^*(0) = S^*. \tag{1.10}$$

Then, we see that

$$F_\lambda f(z) = \lambda \int_0^z \frac{1}{(1-t)^2} dt + \frac{2(1-\lambda)}{z} \int_0^z \frac{t}{(1-t)^2} dt \tag{1.11}$$

$$= \left(\lambda + \frac{2(1-\lambda)}{z}\right)\int_0^z \frac{1}{(1-t)^2}dt - \frac{2(1-\lambda)}{z}\int_0^z \frac{1}{1-t}dt$$

$$= \left(\lambda + \frac{2(1-\lambda)}{z}\right)\left(\frac{1}{1-z} - 1\right) + \frac{2(1-\lambda)}{z}log(1-z)$$

$$= \frac{\lambda z + 2(1-\lambda)}{1-z} + \frac{2(1-\lambda)}{z}log(1-z)$$

$$= z + \sum_{k=2}^{\infty} \frac{(2-\lambda)k+\lambda}{k+1} z^k \in T.$$

This means that if $F_\lambda f(z)$ *is defined by*

$$F_\lambda f(z) = \frac{\lambda z + 2(1-\lambda)}{1-z} + \frac{2(1-\lambda)}{z}log(1-z) \tag{1.12}$$

for $f(z) \in T$, *then*

$$f(z) = \frac{z}{(1-z)^2} \in S^*. \tag{1.13}$$

Further, if we consider the function $f(z)$ *given by*

$$f(z) = \frac{z}{(1-z)} = z + \sum_{k=2}^{\infty} z^k \in K(0) = K, \tag{1.14}$$

then

$$F_\lambda f(z) = \left(\lambda + \frac{2(1-\lambda)}{z}\right)\int_0^z \frac{1}{1-t}dt - \frac{2(1-\lambda)}{z}\int_0^z dt \tag{1.15}$$

$$= -2(1-\lambda) - \left(\lambda + \frac{2(1-\lambda)}{z}\right) log(1-z)$$

$$= z + \sum_{k=2}^{\infty} \frac{(2-\lambda)k+\lambda}{k(k+1)} z^k \in T.$$

This gives us that if $F_\lambda f(z)$ *is defined by*

$$F_\lambda f(z) = -2(1-\lambda) - \left(\lambda + \frac{2(1-\lambda)}{z}\right) log(1-z) \tag{1.16}$$

for $f(z) \in T$, *then*

$$f(z) = \frac{z}{1-z} \in K. \tag{1.17}$$

Let a function $g(z) \in T$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathbb{U} \tag{1.18}$$

with $b_k \geq 0$. For $f(z)$ given by (1.1) and $g(z)$ given by (1.18), if

$$|a_k| \leq b_k, \quad (k = 2, 3, 4, ...), \tag{1.19}$$

then $f(z)$ is said to be dominanted by $g(z)$ (or $g(z)$ dominants $f(z)$), and we write

$$f(z) \ll g(z), \quad z \in \mathbb{U}. \tag{1.20}$$

Since $F_\lambda f(z)$ is given by (1.9) and the coefficients of $F_\lambda f(z)$ satisfy

$$\frac{\lambda}{k} + \frac{2(1-\lambda)}{k+1} < 1, \quad (k = 2, 3, 4, ...) \tag{1.21}$$

for $0 \leq \lambda \leq 1$, we have

$$F_\lambda f(z) \ll f(z), \quad z \in \mathbb{U} \tag{1.22}$$

if $a_k \geq 0$. We say that if $f(z) \in T$ with $a_k \geq 0$, then $A_{-1}f(z) \ll f(z)$, $L_{-1}f(z) \ll f(z)$, $B_{-1}f(z) \ll f(z)$ and $F_\lambda f(z) \ll f(z)$. Further, noting that $0 \leq \lambda \leq 1$ for $F_\lambda f(z)$ and $\gamma \in \mathbb{N}$ for $B_{-1}f(z)$, if $f(z) \in T$ with $a_k \geq 0$, then

$$A_{-1}f(z) \ll F_\lambda f(z) \ll L_{-1}f(z) \ll B_{-1}f(z) \ll f(z). \tag{1.23}$$

2. Some properties

First, we derive the following theorem.

Theorem 2.1 *Let $f(z)$ be in the class T. If*

$$|a_k| \leq \frac{k(k+1)\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!((2-\lambda)k+\lambda)}, \quad (k = 2, 3, 4, ...) \tag{2.1}$$

then

$$F_\lambda f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{2.2}$$

for $0 \leq \alpha < 1$ and if

$$|a_k| \leq \begin{cases} \frac{(k+1)\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!((2-\lambda)k+\lambda)} & ; \quad k = 2, 3, 4, ...; \alpha \neq \frac{1}{2} \\ \frac{k+1}{(2-\lambda)k+\lambda} & ; \quad k = 2, 3, 4, ...; \alpha = \frac{1}{2} \end{cases} \tag{2.3}$$

then

$$F_\lambda f(z) \ll \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & ; \quad \alpha \neq \frac{1}{2} \\ -log(1-z) & ; \quad \alpha = \frac{1}{2} \end{cases} \tag{2.4}$$

for $0 \leq \alpha < 1$.

Proof Using $F_\lambda f(z)$ in (1.9) and (1.7), we consider

$$\left(\frac{\lambda}{k}+\frac{2(1-\lambda)}{k+1}\right)|a_k| \leq \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!}, \quad (k=2,3,4,...). \tag{2.5}$$

We know that if $f(z)$ satisfies (2.1), then $F_\lambda f(z)$ satisfies the dominant (2.2), and if $f(z)$ satisfies (2.3), then $F_\lambda f(z)$ satisfies the dominant (2.4). □

Making $\lambda=0$ or $\lambda=1$ in Theorem 2.1, we have

Corollary 2.2 *If* $f(z)\in T$ *satisfies*

$$|a_k| \leq \frac{(k+1)\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!2}, \quad (k=2,3,4,...) \tag{2.6}$$

then

$$L_{-1}f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z\in\mathbb{U} \tag{2.7}$$

for $0\leq\alpha<1$ *and if*

$$|a_k| \leq \frac{k\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!}, \quad (k=2,3,4,...) \tag{2.8}$$

then

$$A_{-1}f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z\in\mathbb{U} \tag{2.9}$$

for $0\leq\alpha<1$. *Further, if* $f(z)$ *satisfies*

$$|a_k| \leq \begin{cases} \frac{(k+1)\prod_{j=2}^{k}(j-2\alpha)}{k!2} & ; \quad k=2,3,4,...;\alpha\neq\frac{1}{2} \\ \frac{k+1}{2k} & ; \quad k=2,3,4,...;\alpha=\frac{1}{2} \end{cases} \tag{2.10}$$

then

$$L_{-1}f(z) \ll \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & ; \quad \alpha\neq\frac{1}{2} \\ -log(1-z) & ; \quad \alpha=\frac{1}{2} \end{cases} \tag{2.11}$$

for $0\leq\alpha<1$, *and if*

$$|a_k| \leq \begin{cases} \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} & ; \quad k=2,3,4,...;\alpha\neq\frac{1}{2} \\ 1 & ; \quad k=2,3,4,...;\alpha=\frac{1}{2} \end{cases} \tag{2.12}$$

then

$$A_{-1}f(z) \ll \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & ; \quad \alpha\neq\frac{1}{2} \\ -log(1-z) & ; \quad \alpha=\frac{1}{2} \end{cases} \tag{2.13}$$

for $0 \leq \alpha < 1$.

Next, we derive the following theorem.

Theorem 2.3 *For* $f(z) \in T$,

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{2.14}$$

if and only if

$$F_\lambda f(z) \ll h_1(\lambda, \alpha; z) + h_2(\lambda, \alpha; z), \quad z \in \mathbb{U} \tag{2.15}$$

for $0 < \alpha < 1$ *and* $\alpha \neq \frac{1}{2}$ *where*

$$h_1(\lambda, \alpha; z) = \frac{\lambda}{1-2\alpha}\left(\frac{1}{(1-z)^{1-2\alpha}} - 1\right) \tag{2.16}$$

and

$$h_2(\lambda, \alpha; z) = \frac{1-\lambda}{\alpha(1-2\alpha)z}\left(\frac{1-(1-2\alpha)z}{(1-z)^{1-2\alpha}} - 1\right). \tag{2.17}$$

Proof Let the function $f(z) \in T$ satisfy the dominant (2.14). Then, it follows that

$$\lambda A_{-1} f(z) = \lambda \int_0^z \frac{f(t)}{t} dt \ll \lambda \int_0^z \frac{1}{(1-t)^{2(1-\alpha)}} dt \tag{2.18}$$

and

$$(1-\lambda) L_{-1} f(z) = \frac{2(1-\lambda)}{z} \int_0^z f(t) dt \ll \frac{2(1-\lambda)}{z} \int_0^z \frac{t}{(1-t)^{2(1-\alpha)}} dt. \tag{2.19}$$

The dominant (2.18) shows

$$\lambda A_{-1} f(z) \ll \frac{\lambda}{1-2\alpha}\left(\frac{1}{(1-z)^{1-2\alpha}} - 1\right) = h_1(\lambda, \alpha; z) \tag{2.20}$$

and

$$(1-\lambda) L_{-1} f(z) \ll \frac{1-\lambda}{\alpha(1-2\alpha)z}\left(\frac{1-(1-2\alpha)z}{(1-z)^{1-2\alpha}} - 1\right) = h_2(\lambda, \alpha; z). \tag{2.21}$$

Therefore, if $f(z)$ satisfies (2.14), then we have

$$F_\lambda f(z) = \lambda A_{-1} f(z) + (1-\lambda) L_{-1} f(z) \ll h_1(\lambda, \alpha; z) + h_2(\lambda, \alpha; z), \quad z \in \mathbb{U}. \tag{2.22}$$

Conversely, if $f(z)$ satisfies the dominant (2.15), then we can consider that

$$\lambda A_{-1} f(z) \ll h_1(\lambda, \alpha; z), \quad z \in \mathbb{U} \tag{2.23}$$

and

$$(1-\lambda) L_{-1} f(z) \ll h_2(\lambda, \alpha; z), \quad z \in \mathbb{U}. \tag{2.24}$$

The dominant (2.23) shows that

$$\frac{f(z)}{z} \ll \frac{1}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{2.25}$$

that is,

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U}. \tag{2.26}$$

Also the dominant (2.24) says

$$2\int_0^z f(t)dt \ll \frac{1}{\alpha(1-2\alpha)}\left(\frac{1-(1-2\alpha)z}{(1-z)^{1-2\alpha}}-1\right), \quad z \in \mathbb{U} \tag{2.27}$$

that is,

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U}. \tag{2.28}$$

This completes the proof of the theorem. □

Making $\lambda = 0$ or $\lambda = 1$ in Theorem 2.3, we derive

Corollary 2.4 *For $f(z) \in T$, $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$,*

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{2.29}$$

if and only if

$$L_{-1}f(z) \ll \frac{1}{\alpha(1-2\alpha)z}\left(\frac{1-(1-2\alpha)z}{(1-z)^{1-2\alpha}}-1\right), \quad z \in \mathbb{U} \tag{2.30}$$

or

$$A_{-1}f(z) \ll \frac{1}{1-2\alpha}\left(\frac{1}{(1-z)^{1-2\alpha}}-1\right), \quad z \in \mathbb{U}. \tag{2.31}$$

Theorem 2.5 *For $f(z) \in T$,*

$$f(z) \ll \frac{z}{1-z}, \quad z \in \mathbb{U} \tag{2.32}$$

if and only if

$$F_\lambda f(z) \ll \left(\lambda + \frac{2(1-\lambda)}{z}\right) log\left(\frac{1}{1-z}\right) - 2(1-\lambda), \quad z \in \mathbb{U}. \tag{2.33}$$

Proof Consider the dominant (2.32). Note that

$$\lambda A_{-1}f(z) = \lambda \int_0^z \frac{f(t)}{t}dt \ll \lambda \int_0^z \frac{1}{1-t}dt = \lambda log\left(\frac{1}{1-z}\right) \tag{2.34}$$

and

$$(1-\lambda)L_{-1}f(z) = \frac{2(1-\lambda)}{z}\int_0^z f(t)dt \ll \frac{2(1-\lambda)}{z}\int_0^z \frac{t}{1-t}dt = \frac{2(1-\lambda)}{z}log\left(\frac{1}{1-z}\right) - 2(1-\lambda). \tag{2.35}$$

Thus, we have

$$F_\lambda f(z) \ll \lambda log\left(\frac{1}{1-z}\right) + \frac{2(1-\lambda)}{z}log\left(\frac{1}{1-z}\right) - 2(1-\lambda) \tag{2.36}$$

$$= \left(\lambda + \frac{2(1-\lambda)}{z}\right) log\left(\frac{1}{1-z}\right) - 2(1-\lambda).$$

Conversely, we consider that the function $F_\lambda f(z)$ satisfies the dominant (2.33). Then we put

$$\lambda A_{-1} f(z) \ll \lambda log\left(\frac{1}{1-z}\right), \quad z \in \mathbb{U} \tag{2.37}$$

and

$$(1-\lambda)L_{-1} f(z) \ll \frac{2(1-\lambda)}{z}, \quad z \in \mathbb{U}. \tag{2.38}$$

Since (2.37) leads us

$$\frac{f(z)}{z} \ll \frac{d}{dz} log\left(\frac{1}{1-z}\right) = \frac{1}{1-z}, \tag{2.39}$$

we have

$$f(z) \ll \frac{z}{1-z}, \quad z \in \mathbb{U}. \tag{2.40}$$

Further, since (2.38) shows us

$$\int_0^z f(t)dt \ll log\left(\frac{1}{1-z}\right) - z, \tag{2.41}$$

we see

$$f(z) \ll \frac{1}{1-z} - 1 = \frac{z}{1-z}, \quad z \in \mathbb{U}. \tag{2.42}$$

□

Letting $\lambda = 0$ or $\lambda = 1$, we say

Corollary 2.6 *For* $f(z) \in T$,

$$f(z) \ll \frac{z}{1-z}, \quad z \in \mathbb{U} \tag{2.43}$$

if and only if

$$L_{-1} f(z) \ll \frac{2}{z} log\left(\frac{1}{1-z}\right) - 2, \quad z \in \mathbb{U} \tag{2.44}$$

or

$$A_{-1} f(z) \ll log\left(\frac{1}{1-z}\right), \quad z \in \mathbb{U}. \tag{2.45}$$

Next we consider the function

$$f(z) = \frac{z}{1-z^2} \in S^*. \tag{2.46}$$

This function $f(z)$ maps $\mathbb{U}$ onto the domain D which is given by

$$D = \left\{ w \in \mathbb{C} : w = \mathbb{C} - i\alpha, \ |\alpha| \geq \frac{1}{2} \right\}. \tag{2.47}$$

Theorem 2.7 *For* $f(z) \in T$,

$$f(z) \ll \frac{z}{1-z^2}, \quad z \in \mathbb{U} \tag{2.48}$$

if and only if

$$F_\lambda f(z) \ll \frac{\lambda}{2} log\left(\frac{1+z}{1-z}\right) + \frac{1-\lambda}{z} log\left(\frac{1}{1-z^2}\right), \quad z \in \mathbb{U}. \tag{2.49}$$

Proof Consider $f(z) \in T$ satisfies (2.48), then

$$\lambda A_{-1} f(z) = \lambda \int_0^z \frac{f(t)}{t} dt \ll \lambda \int_0^z \frac{1}{1-t^2} dt = \frac{\lambda}{2} log\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{U} \tag{2.50}$$

and

$$(1-\lambda) L_{-1} f(z) = \frac{2(1-\lambda)}{z} \int_0^z f(t) dt \ll \frac{2(1-\lambda)}{z} \int_0^z \frac{t}{1-t^2} dt = \frac{1-\lambda}{z} log\left(\frac{1}{1-z^2}\right), \quad z \in \mathbb{U}. \tag{2.51}$$

Therefore, we have the dominant (2.49).

Conversely, if the function $F_\lambda f(z)$ satisfies the dominant (2.49), then we consider

$$\lambda A_{-1} f(z) \ll \frac{\lambda}{2} log\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{U} \tag{2.52}$$

and

$$(1-\lambda) L_{-1} f(z) \ll \frac{1-\lambda}{z} log\left(\frac{1}{1-z^2}\right), \quad z \in \mathbb{U}. \tag{2.53}$$

Both of (2.52) and (2.53) implies that

$$f(z) \ll \frac{z}{1-z^2}, \quad z \in \mathbb{U}. \tag{2.54}$$

□

Taking $\lambda = 0$ or $\lambda = 1$ in the above theorem, we have

Corollary 2.8 *For* $f(z) \in T$,

$$f(z) \ll \frac{z}{1-z^2} \tag{2.55}$$

if and only if

$$L_{-1} f(z) \ll \frac{1}{z} log\left(\frac{1}{1-z^2}\right), \quad z \in \mathbb{U}. \tag{2.56}$$

or

$$A_{-1} f(z) \ll \frac{1}{2} log\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{U}. \tag{2.57}$$

For the Bernardi integral operator $B_{-1} f(z)$, we consider

Theorem 2.9 *Let* $f(z) \in T$ *and* $B_{-1}f(z)$ *be the Bernardi integral operator for* $\gamma = 2$. *Then*

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{2.58}$$

for $0 < \alpha < 1$ *and* $\alpha \neq \frac{1}{2}$ *if and only if*

$$B_{-1}f(z) \ll \frac{3}{z^2}\left\{\frac{1}{\alpha(4\alpha^2-1)} - (1-z)^{2\alpha-1}\left(\frac{1}{2\alpha-1} - \frac{1-z}{\alpha} + \frac{(1-z)^2}{2\alpha+1}\right)\right\}, \quad z \in \mathbb{U}. \tag{2.59}$$

Proof Note that if $f(z)$ satisfies (2.58), then

$$B_{-1}f(z) \ll \frac{3}{z^2}\int_0^z \frac{t^2}{(1-t)^{2(1-\alpha)}}dt \tag{2.60}$$

$$= \frac{3}{z^2}\int_{1-z}^1 \frac{(1-u)^2}{u^{2(1-\alpha)}}du \ ; (1-t=u)$$

$$= \frac{3}{z^2}\left\{\frac{1}{\alpha(4\alpha^2-1)} - (1-z)^{2\alpha-1}\left(\frac{1}{2\alpha-1} - \frac{1-z}{\alpha} + \frac{(1-z)^2}{2\alpha+1}\right)\right\}, \quad z \in \mathbb{U}.$$

Further, it is easy to see that if $B_{-1}f(z)$ satisfies the dominant (2.59), then $f(z)$ satisfies (2.58). □

3. New integral operator

For the Libera integral operator $L_{-1}f(z)$, we consider the following integral operator

$$O_{-1}f(z) = \frac{2}{z}\int_0^z tf'(t)dt = z + \sum_{k=2}^{\infty}\frac{2k}{k+1}a_k z^k \tag{3.1}$$

for $f(z) \in T$. Since

$$\frac{4}{3} \leq \frac{2k}{k+1} < 2, \quad (k = 2, 3, 4, ...) \tag{3.2}$$

if $a_k \geq 0$ $(k = 2, 3, 4, ...)$, then

$$f(z) \ll O_{-1}f(z), \quad z \in \mathbb{U}. \tag{3.3}$$

For this integral operator $O_{-1}f(z)$, we see

Theorem 3.1 *Let* $f(z)$ *be in the class* T. *If* $f(z)$ *satisfies*

$$|a_k| \leq \frac{(k+1)\prod_{j=2}^{k}(j-2\alpha)}{k!2}, \quad (k = 2, 3, 4, ...) \tag{3.4}$$

then

$$O_{-1}f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{3.5}$$

for $0 \le \alpha < 1$. *If* $f(z)$ *satisfies*

$$|a_k| \le \begin{cases} \frac{(k+1)\prod_{j=2}^{k}(j-2\alpha)}{k!2k} & ; \quad k = 2,3,4,...; \alpha \neq \frac{1}{2} \\ \frac{k+1}{2k^2} & ; \quad k = 2,3,4,...; \alpha = \frac{1}{2} \end{cases} \tag{3.6}$$

then

$$O_{-1}f(z) \ll \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & ; \quad \alpha \neq \frac{1}{2} \\ -log(1-z) & ; \quad \alpha = \frac{1}{2} \end{cases} \tag{3.7}$$

for $0 \le \alpha < 1$.

Proof Considering (1.7) and (3.1), we see the dominant (3.5). Using (1.8) and (3.1), we have (3.7). □

Next, we derive

Theorem 3.2 *For* $f(z) \in T$,

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U} \tag{3.8}$$

if and only if

$$O_{-1}f(z) \ll h(\alpha, z), \quad z \in \mathbb{U} \tag{3.9}$$

for $0 < \alpha < 1$ *and* $\alpha \neq \frac{1}{2}$ *where*

$$h(\alpha, z) = \frac{1}{2\alpha - 1}\left\{z - \frac{1}{\alpha(2\alpha+1)z} + \frac{(1+2\alpha z)(1-z)^{2\alpha}}{\alpha(2\alpha+1)z}\right\}. \tag{3.10}$$

Proof Let $f(z)$ satisfy the dominant (3.8). Then $f(z)$ satisfies

$$zf'(z) \ll \frac{z}{2\alpha - 1}\left(1 - (1-z)^{2\alpha-1}\right), \quad z \in \mathbb{U} \tag{3.11}$$

that is,

$$\begin{aligned} O_{-1}f(z) &\ll \frac{2}{(2\alpha-1)z}\int_0^z \left(t - t(1-t)^{2\alpha-1}\right) dt \\ &= \frac{2}{(2\alpha-1)z}\int_{1-z}^1 \left(1 - u - u^{2\alpha-1} + u^{2\alpha}\right) du\,;\ (1-t=u) \\ &= \frac{1}{2\alpha-1}\left\{z - \frac{1}{\alpha(2\alpha+1)z} + \frac{(1+2\alpha z)(1-z)^{2\alpha}}{\alpha(2\alpha+1)z}\right\}. \\ &= h(\alpha, z). \end{aligned} \tag{3.12}$$

Conversely, if the function $O_{-1}f(z)$ satisfies the dominant (3.9), we know

$$\frac{z}{2}O_{-1}f(z) \ll \frac{z}{2}h(\alpha, z), \quad z \in \mathbb{U} \tag{3.13}$$

and

$$zf'(z) \ll \frac{1}{2}\left(h(\alpha, z) + zh'(\alpha, z)\right), \quad z \in \mathbb{U}. \tag{3.14}$$

This shows us that

$$f(z) \ll \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{U}. \tag{3.15}$$

□

Next our result is

Theorem 3.3 *For* $f(z) \in T$,

$$f(z) \ll \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & ; \quad \alpha \neq \frac{1}{2} \\ \\ -log(1-z) & ; \quad \alpha = \frac{1}{2} \end{cases} \tag{3.16}$$

if and only if

$$O_{-1}f(z) \ll \begin{cases} \frac{1}{\alpha(2\alpha-1)z}\left(1 - \frac{1-(1-2\alpha)z}{(1-z)^{1-2\alpha}}\right) & ; \quad \alpha \neq \frac{1}{2} \\ \\ 2\left(\frac{1}{z}log\left(\frac{1}{1-z}\right) - 1\right) & ; \quad \alpha = \frac{1}{2} \end{cases} \tag{3.17}$$

for $0 < \alpha < 1$.

Proof Using the same manner in Theorem 3.2, we see that the dominant (3.16) leads

$$zf'(z) \ll \begin{cases} \frac{z}{(1-z)^{2(1-\alpha)}} & ; \quad \alpha \neq \frac{1}{2}, z \in \mathbb{U} \\ \frac{z}{1-z} & ; \quad \alpha = \frac{1}{2}, z \in \mathbb{U}. \end{cases} \tag{3.18}$$

It follows from (3.18) that

$$O_{-1}f(z) \ll \frac{2}{z}\int_0^z \frac{t}{(1-t)^{2(1-\alpha)}}dt \tag{3.19}$$

$$= \frac{1}{\alpha(2\alpha-1)z}\left(1 - \frac{1-(1-2\alpha)z}{(1-z)^{1-2\alpha}}\right), \quad z \in \mathbb{U}$$

for $\alpha \neq \frac{1}{2}$, and that

$$O_{-1}f(z) \ll \frac{2}{z}\int_0^z \frac{t}{1-t}dt \tag{3.20}$$

$$= 2\left(\frac{1}{z}log\left(\frac{1}{1-z}\right) - 1\right), \quad z \in \mathbb{U}$$

for $\alpha = \frac{1}{2}$. The converse is also clear. □

Further, we derive

Theorem 3.4 *For* $f(z) \in T$,

$$f(z) \ll \frac{1}{2} log\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{U} \tag{3.21}$$

if and only if

$$O_{-1}f(z) \ll \frac{1}{2} log\left(\frac{1}{1-z^2}\right), \quad z \in \mathbb{U}. \tag{3.22}$$

Proof By (3.21), we have

$$zf'(z) \ll \frac{1}{2}\left(\frac{z}{1+z} + \frac{z}{1-z}\right), \quad z \in \mathbb{U}. \tag{3.23}$$

This gives that

$$O_{-1}f(z) = \frac{2}{z}\int_0^z tf'(t)dt \ll \frac{1}{z}\int_0^z \left(\frac{t}{1+t} + \frac{t}{1-t}\right)dt = \frac{1}{z} log\left(\frac{1}{1-z^2}\right), \quad z \in \mathbb{U}. \tag{3.24}$$

Further, it is easy to see that $f(z)$ satisfies (3.21) if $O_{-1}f(z)$ satisfies (3.22). □

Remark 3.5 *The function*

$$f(z) = \frac{1}{2} log\left(\frac{1+z}{1-z}\right) = z + \sum_{k=2}^{\infty} \frac{1}{2k-1} z^{2k-1} \tag{3.25}$$

is convex in $\mathbb{U}$ *and maps* $\mathbb{U}$ *onto the strip region with* $-\frac{\pi}{4} < Imf(z) < \frac{\pi}{4}$.

Now, for m different boundary points z_s $(s = 1, 2, 3, ..., m)$ with $|z_s| = 1$, we consider

$$d_m = \frac{1}{m}\sum_{s=1}^{m} \frac{O_{-1}f(z_s)}{z_s}, \tag{3.26}$$

where $d_m \in e^{i\beta}O_{-1}f(\mathbb{U}), d_m \neq 1$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$. For such d_m, if $f(z) \in T$ satisfies

$$\left|\frac{e^{i\beta}\frac{O_{-1}f(z)}{z} - d_m}{e^{i\beta} - d_m} - 1\right| < \rho, \quad z \in \mathbb{U} \tag{3.27}$$

for some real $\rho > 0$, we say that $f(z)$ belongs to $B(d_m, \beta, \rho) \subset T$. It is easy to see that the condition

$$\left|\frac{O_{-1}f(z)}{z} - 1\right| < \rho\left|e^{i\beta} - d_m\right|, \quad z \in \mathbb{U} \tag{3.28}$$

is equivalent to (3.27). If we consider

$$f(z) = z + \frac{3}{4}\rho\left|e^{i\beta} - d_m\right| z^2, \quad z \in \mathbb{U} \tag{3.29}$$

then $f(z)$ satisfies (3.28).

To discuss our problem for $O_{-1}f(z)$, we have to recall here the following lemma by Miller and Mocanu [5, 6] (also, by Jack [3]).

Lemma 3.6 *Let the function* $w(z)$ *given by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \quad , \quad n \in \mathbb{N} \tag{3.30}$$

be analytic in $\mathbb{U}$ *with* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ *at a point* $z_0, (0 < |z_0| < 1)$ *then there exists a real number* $k \geq n$ *such that*

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \tag{3.31}$$

and

$$Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k. \tag{3.32}$$

Theorem 3.7 *If* $f(z) \in T$ *satisfies*

$$\left|\frac{z(O_{-1}f(z))'}{O_{-1}f(z)} - 1\right| < \frac{\left|e^{i\beta} - d_m\right| \rho}{1 + \left|e^{i\beta} - d_m\right| \rho}, \quad z \in \mathbb{U} \tag{3.33}$$

for some d_m *defined by (3.26) with* $d_m \neq 1$ *such that* $z_s \in \partial\mathbb{U}$ $(s = 1, 2, 3, ..., m)$*, and for some real* $\rho > 1$*, then*

$$\left|\frac{O_{-1}f(z)}{z} - 1\right| < \rho \left|e^{i\beta} - d_m\right|, \quad z \in \mathbb{U} \tag{3.34}$$

that is, $f(z) \in B(d_m, \beta, \rho)$*.*

Proof Define the function $w(z)$ by

$$w(z) = \frac{e^{i\beta} \frac{O_{-1}f(z)}{z} - d_m}{e^{i\beta} - d_m} - 1 \tag{3.35}$$

$$= \frac{e^{i\beta}}{e^{i\beta} - d_m} \left(\sum_{k=2}^{\infty} \frac{2k}{k+1} a_k z^{k-1}\right).$$

Then, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and

$$\frac{O_{-1}f(z)}{z} = 1 + (1 - e^{-i\beta} d_m) w(z). \tag{3.36}$$

It follows from (3.36) that

$$\frac{z(O_{-1}f(z))'}{O_{-1}f(z)} - 1 = \frac{(1 - e^{-i\beta} d_m) z w'(z)}{1 + (1 - e^{-i\beta} d_m) w(z)}. \tag{3.37}$$

This gives us that

$$\left|\frac{z(O_{-1}f(z))'}{O_{-1}f(z)} - 1\right| = \left|\frac{(1 - e^{-i\beta} d_m) z w'(z)}{1 + (1 - e^{-i\beta} d_m) w(z)}\right| < \frac{|e^{i\beta} - d_m| \rho}{1 + |e^{i\beta} - d_m| \rho}, \quad z \in \mathbb{U}. \tag{3.38}$$

We assume that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \tag{3.39}$$

Then, Lemma 3.6 shows that $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = kw(z_0), (k \geq 1)$. For such $z_0 \in \mathbb{U}$, we have

$$\begin{aligned} \left| \frac{z_0(O_{-1}f(z_0))'}{O_{-1}f(z_0)} - 1 \right| &= \left| \frac{(1 - e^{-i\beta}d_m)z_0 w'(z_0)}{1 + (1 - e^{-i\beta}d_m)w(z_0)} \right| \\ &= \left| \frac{(1 - e^{-i\beta}d_m)\rho}{1 + (1 - e^{-i\beta}d_m)\rho e^{i\theta}} \right| \\ &\geq \frac{|1 - e^{-i\beta}d_m|\rho}{1 + |1 - e^{-i\beta}d_m|\rho} \\ &= \frac{|e^{i\beta} - d_m|\rho}{1 + |e^{i\beta} - d_m|\rho}, \quad z \in \mathbb{U}. \end{aligned} \tag{3.40}$$

Since the above contradicts our condition (3.33), we say that there exists no $z_0, (0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. Letting $|w(z)| < \rho$ for all $z \in \mathbb{U}$, we know

$$|w(z)| = \left| \frac{e^{i\beta}\left(\frac{O_{-1}f(z)}{z} - 1\right)}{e^{i\beta} - d_m} \right| < \rho, \quad z \in \mathbb{U}. \tag{3.41}$$

This means that

$$\left| \frac{O_{-1}f(z)}{z} - 1 \right| < |e^{i\beta} - d_m|\rho, \quad z \in \mathbb{U}, \tag{3.42}$$

that is, that $f(z) \in B(d_m, \beta, \rho)$. □

Example 3.8 *Let us consider the function $f(z) \in T$ given by*

$$f(z) = z + a_2 z^2. \tag{3.43}$$

Then, $f(z)$ satisfies

$$\left| \frac{z(O_{-1}f(z))'}{O_{-1}f(z)} - 1 \right| = \left| \frac{\frac{8}{3}a_2 z}{1 + \frac{4}{3}a_2 z} \right| < \frac{\frac{8}{3}|a_2|}{1 - \frac{4}{3}|a_2|}, \quad z \in \mathbb{U}, \tag{3.44}$$

where $0 < |a_2| < \frac{3}{4}$. Consider five boundary points

$$z_1 = e^{-iarg(a_2)} \tag{3.45}$$

$$z_2 = e^{i\frac{\pi - 6arg(a_2)}{6}} \tag{3.46}$$

$$z_3 = e^{i\frac{\pi - 4arg(a_2)}{4}} \tag{3.47}$$

$$z_4 = e^{i\frac{\pi - 3arg(a_2)}{3}} \tag{3.48}$$

and

$$z_5 = e^{i\frac{\pi - 2arg(a_2)}{2}}. \tag{3.49}$$

For such points z_s $(s = 1, 2, 3, ..., m)$, *we see*

$$\frac{O_{-1}f(z_1)}{z_1} = 1 + \frac{4}{3}|a_2|, \tag{3.50}$$

$$\frac{O_{-1}f(z_2)}{z_2} = 1 + \frac{4}{3}|a_2|\frac{\sqrt{3}+i}{2}, \tag{3.51}$$

$$\frac{O_{-1}f(z_3)}{z_3} = 1 + \frac{4}{3}|a_2|\frac{\sqrt{2}(1+i)}{2}, \tag{3.52}$$

$$\frac{O_{-1}f(z_4)}{z_4} = 1 + \frac{4}{3}|a_2|\frac{1+\sqrt{3}i}{2}, \tag{3.53}$$

and

$$\frac{O_{-1}f(z_5)}{z_5} = 1 + \frac{4}{3}|a_2|i. \tag{3.54}$$

Therefore, we have

$$d_5 = \frac{1}{5}\sum_{s=1}^{5}\frac{O_{-1}f(z_s)}{z_s} = 1 + \frac{2(3+\sqrt{2}+\sqrt{3})(1+i)}{15}|a_2|. \tag{3.55}$$

This implies that

$$\left|1 - e^{-i\beta}d_5\right| = \frac{2\sqrt{2}(3+\sqrt{2}+\sqrt{3})}{15}|a_2| \tag{3.56}$$

for $\beta = 0$. *For such* d_5 *and* β, *we consider* $\rho > 1$ *with*

$$\frac{\frac{8}{3}|a_2|}{1 - \frac{4}{3}|a_2|} = \frac{|e^{i\beta} - d_5|\rho}{1 + |e^{i\beta} - d_5|\rho}. \tag{3.57}$$

This gives us that

$$\rho \geq \frac{8|a_2|}{3(1-4|a_2|)|e^{i\beta} - d_5|} > \frac{10\sqrt{2}}{3+\sqrt{2}+\sqrt{3}} > 1. \tag{3.58}$$

For such d_5 *and* $\rho > 1$, $f(z)$ *satisfies*

$$\left|\frac{O_{-1}f(z)}{z} - 1\right| < \frac{4}{3}|a_2| \leq \rho|e^{i\beta} - d_5| \quad , \quad z \in \mathbb{U}. \tag{3.59}$$

Finally, we consider two integral operators $O_{-1}f(z)$ and $L_{-1}f(z)$.

Theorem 3.9 *If* $f(z) \in T$ *satisfies*

$$Re\left(\frac{O_{-1}f(z)+z(O_{-1}f(z))'}{L_{-1}f(z)+z(L_{-1}f(z))'}\right) > \alpha, \quad , \quad z \in \mathbb{U}. \tag{3.60}$$

for $0 \leq \alpha < 1,$ *then* $f(z) \in S^*(\alpha)$. *If* $f(z) \in T$ *satisfies*

$$Re\left(\frac{2(O_{-1}f(z))'+z(O_{-1}f(z))''}{2(L_{-1}f(z))'+z(L_{-1}f(z))''}\right) > \alpha, \quad , \quad z \in \mathbb{U}. \tag{3.61}$$

for $0 \leq \alpha < 1,$ *then* $f(z) \in K(\alpha)$.

Proof We note that $O_{-1}f(z)$ satisfies

$$O_{-1}f(z)+z(O_{-1}f(z))' = 2zf'(z) \tag{3.62}$$

and $L_{-1}f(z)$ satisfies

$$L_{-1}f(z)+z(L_{-1}f(z))' = 2f(z). \tag{3.63}$$

Therefore, if $f(z)$ satisfies (3.60), then $f(z) \in S^*(\alpha)$.

Further, we know that

$$2(O_{-1}f(z))'+z(O_{-1}f(z))'' = 2(f'(z)+zf''(z)) \tag{3.64}$$

and

$$2(L_{-1}f(z))'+z(L_{-1}f(z))'' = 2f'(z). \tag{3.65}$$

Thus our condition (3.61) shows that $f(z) \in K(\alpha)$. □

References

[1] Alexander JW. Functions which map the interior of the unit circle upon simple regions. Annals of Mathematics, Second Series 1915 ;17: 12-22.

[2] Bernardi SD. Convex and starlike univalent functions. Transactions of the American Mathematical Society 1969; 135: 429-446.

[3] Jack IS. Functions starlike and convex of order α. Journal of the London Mathematical Society 1971; 3: 469-474.

[4] Libera RJ. Some classes of regular univalent functions. Proceedings of the American Mathematical Society 1965; 16: 755-758.

[5] Miller SS, Mocanu PT. Second order differential inequalities in the complex plane. Journal of Mathematical Analysis and Applications 1978; 65: 289-305.

[6] Miller SS, Mocanu PT. Differential Subordinations, Theory and Applications. New York: NY, USA:Marcel Dekker Inc., 2000.

[7] Owa S, Güney HÖ. New Applications of the Bernardi Integral Operator. Mathematics 2020; 8: 1-12.

[8] Robertson MS. On the theory of univalent functions. Annals of Mathematics 1936; 37: 374-408.