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Commutativity degree of crossed modules

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Abstract: In this work, we define the notion of commutativity degree of crossed modules and find some bounds on commutativity degree for special types of crossed modules. Also, we give a function for finding commutativity degree of crossed modules in GAP and classify crossed modules by using this function.

Key words: Commutativity degree, crossed module, isoclinic crossed module, GAP

1. Introduction

The notion of commutativity degree of groups was introduced by Gallagher [13] in 1970. The probability that two elements of a group commute is called the commutativity degree of the group. Finding the commutativity degree of a finite group is equivalent to finding the number of conjugacy classes of the group or to finding the number of irreducible characters of the group. This relates commutativity degree to many areas of group theory; there are many questions and a long history of results, concerning the relationship between irreducible characters of group and group-theoretic properties of the group.

A crossed module (S, R, ∂) is a group homomorphism $S \xrightarrow{\partial} R$ with an action of R on S satisfying certain conditions. MacLane and Whitehead, [24], show that crossed modules modeled homotopy 2-types. Later crossed modules had been considered as "2-dimensional groups", [8, 9]. We refer to [7, 21, 22] for detail and comprehensive research of crossed modules. On the other hand, the concept of isoclinism of crossed modules was defined in [23]. A shared package XMod was described in [1]. Also, one can find several algorithms about these notions in [2, 6, 11].

Crossed modules of groups provide a simultaneous generalization of the concepts of groups. Thus, it is interest to investigate generalizations of groups theoretic concepts and structures to crossed modules. In this paper, we generalize the notion of commutativity degree, obtain some bounds for special crossed modules and show that two isoclinic crossed modules have same commutativity degree. Also, we have developed new functions for GAP which used to compute commutative degree of a group and a crossed module.

2. Crossed modules

In this section we recall some needed material about crossed modules. We refer to [7, 19–22].

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A crossed module is a group homomorphism such that

$$\partial : S \longrightarrow R$$

with an action R on S written $(r,s) \mapsto {}^{r}s$, for $r \in R, s \in S$ satisfying the following conditions:

$$\begin{array}{llll} 1) & \partial({}^{r}s) & = & r\partial(s)r^{-1}, \\ 2) & {}^{\partial(s)}s' & = & ss's^{-1}, \end{array}$$

for all $r \in R$, $s, s' \in S$.

Examples of crossed modules:

(1) A conjugation crossed module is an inclusion of a normal subgroup $N \trianglelefteq R$, where R acts on N by conjugation.

(2) A zero boundary crossed module has a R-module as source and $\partial = 0$.

(3) Any homomorphism $\partial : S \longrightarrow R$, with S Abelian and $im\partial$ in the center of R, provides a crossed module with R acting trivially on S.

(4) The direct product of $\partial_1 : S_1 \longrightarrow R_1$ and $\partial_2 : S_2 \longrightarrow R_2$ is $\partial_1 \times \ \partial_2 : S_1 \times S_2 \longrightarrow R_1 \times R_2$ with direct product action ${}^{(r_1,r_2)}(s_1,s_2) = ({}^{r_1}s_1, {}^{r_2}s_2)$.

A morphism between two crossed modules $\partial : S \longrightarrow R$ and $\partial' : S' \longrightarrow R'$ is a pair (α, β) of group homomorphisms $\alpha : S \longrightarrow S'$, $\beta : R \longrightarrow R'$, such that $\beta\partial = \partial'\alpha$ and $\alpha({}^{r}s) = {}^{\beta(r)}\alpha(s)$, for all $r \in R$, $s \in S$. Consequently we have the category **XMod** whose objects are the crossed modules and its morphisms are the morphisms of crossed modules.

A crossed module $\partial' : S' \longrightarrow R'$ is a *subcrossed module* of a crossed module $\partial : S \longrightarrow R$ if S', R' are subgroups of S, R, respectively, $\partial' = \partial|_{S'}$ and the action of R' on S' is induced by the action of R on S. Also, a subcrossed module $\partial' : S' \longrightarrow R'$ of a crossed module $\partial : S \longrightarrow R$ is *normal* if R' is a normal subgroup of R, ${}^{r}s' \in S'$ and ${}^{r'}ss^{-1} \in S'$, for all $r \in R$, $r' \in R'$, $s \in S$, $s' \in S'$. So, we have the *quotient crossed module* $S/S' \longrightarrow R/R'$ with the induced boundary map and action.

Now, we recall some notions about crossed modules from [21].

Let $\partial : S \longrightarrow R$ be a crossed module. Then the *center* of the crossed module $S \xrightarrow{\partial} R$, $Z(S,R,\partial)$, is defined as the normal subcrossed module

$$S^R \xrightarrow{\partial|} St_R(S) \cap Z(R)$$

where

$$S^R = \{s \in S : \ {}^{r}s = s, \text{ for all } r \in R\},$$

$St_R(S) = \{r \in R : \ {}^{r}s = s, \text{ for all } s \in S\}$.

Any crossed module which coincides with its center is called *Abelian*.

Let $\partial : S \longrightarrow R$ be a crossed module. The *commutator subcrossed module* of $S \xrightarrow{\partial} R$ is defined by

$$D_R(S) \xrightarrow{\partial|} [R,R]$$

where $D_R(S)$ is the subgroup generated by $\{{}^{r}ss^{-1} : s \in S, \ r \in R\}$ and $[R,R]$ is the commutator subgroup of R. Shortly, we show that commutator of (S,R,∂) with $(S,R,\partial)'$.

A crossed module $S \xrightarrow{\partial} R$ is called *finite* if S and R are finite groups.

The order of a finite crossed module is defined as the pair $[m,n]$ where m, n are the orders of S, R, respectively.

Let $H \xrightarrow{\partial'} K$ (here shortly show (H,K)) be a subcrossed module of the crossed module $S \xrightarrow{\partial} R$ (here shortly show (S,R)). Suppose that there is a finite sequence $(H,K)^i : (H^i \longrightarrow K^i)_{0\leq i\leq n}$ of the subcrossed modules of (S,R) such that

$$(H,K) = (H,K)^0 \trianglelefteq (H,K)^1 \trianglelefteq \ldots \trianglelefteq (H,K)^{n-1} \trianglelefteq (H,K)^n = (S,R).$$

This will be called a *series* of length n from (H,K) to (S,R). The subcrossed modules $(H,K)^0$, $(H,K)^1, \ldots,$ $(H,K)^n$ are called the *terms* of the series and quotient crossed modules $(H,K)^i/(H,K)^{i-1}$, $i = 1,\ldots,n,$ are called the *factors* of the series. A series 1 to (S,R) is shortly called a *series* of (S,R). A series is called *central* if all factors are central. (S,R) is called *nilpotent* if it has a series all of whose factors are central factors of (S,R).

Let $S \xrightarrow{\partial} R$ be nilpotent. Then, for any central series

$$1 = (S,R)^0 \trianglelefteq (S,R)^1 \trianglelefteq \cdots \trianglelefteq (S,R)^r = R$$

of (S,R), we have

$$\Gamma_{r-i+1}((S,R)) \leq (S,R)^i \leq \xi_i((S,R)),$$

$i = 0,1,\ldots,r$ where $\Gamma_1((S,R)) = ((S,R))$, $\xi_0((S,R)) = 1$ and

$$\Gamma_n((S,R)) = [\Gamma_{n-1}((S,R)),(S,R)],\ n > 1$$

$\xi_n(R)/\xi_{n-1}(R) = \xi(R/\xi_{n-1}(R))$, $n > 0$.
Furthermore, the least integer c such that $\Gamma_{c+1}((S,R)) = 1$ is equal to the least integer c such that $\xi_c((S,R)) = (S,R)$. The integer c is called the *nilpotency class* of the crossed module (S,R).

3. Commutativity degree of crossed modules

In this section, we introduce two dimensional version of commutativity degree and give fundamental idea can be found in [17]. Another investigation of that is given in [18]. Before this, we recall commutativity degree of finite groups and give some bounds for this commutativity degree. For details, see [10, 14, 16–18].

3.1. Commutativity degree of groups

Let G be a finite group. The probability that two elements of G commute is called the *commutativity degree* and denoted by $d(G)$.

Formally, the commutativity degree $d(G)$ of G is defined by

$$d(G) = \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

Obviously, G is Abelian if and only if $d(G) = 1$; furthermore, the following results are known:

1. $d(G) = \frac{k(G)}{|G|}$, where $k(G)$ is the number of distinct conjugacy classes of G.

2. Let p be a prime. If G is non-Abelian with $|G| = p^3$ then $d(G) = \frac{p^2+p-1}{p^3}$.

3. If $d(G) > \frac{1}{2}$, then G is nilpotent.

4. Let G and H be 2 isoclinic finite groups, then $d(G) = d(H)$.

3.2. Bounds on commutativity degree of crossed modules

Let S and R be two finite groups. What is the probability that a random chosen pair of crossed modules written between S and R commute? This probability is called the commutativity degree of the crossed module, (S, R, ∂).

Definition 3.1 *Let (S, R, ∂) be a finite crossed module. The commutativity degree of (S, R, ∂) is defined by*

$$d(S, R, \partial) = \left[\frac{|\{(r, s) \in R \times S \mid {}^{r}s = s\}|}{|R|\,|S|}, \frac{|\{(r, r') \in R \times R \mid rr' = r'r\}|}{|R|\,|R|} \right].$$

Example 3.2 *Let $D_8 = \langle a, b \mid a^2, b^2, (ab)^4 \rangle$ be the dihedral group of order 8, and let $c = [a, b] = (ab)^2$ so that $a^b = ac$ and $b^a = bc$. (The standard permutation representation is given by $a = (1,2)(3,4), b = (1,3), ab = (1,2,3,4), c = (1,3)(2,4)$.) Define $C_4 = \langle (1,2,3,4) \rangle$ is the subgroup of D_8. Then, (C_4, D_8, i) conjugation crossed module and*

$$d(C_4, D_8, i) = \left[\frac{3}{4}, \frac{5}{8} \right].$$

Theorem 3.3 *Let (S, R, ∂) be finite crossed module. Then the commutativity degree of (S, R, ∂) is*

$$d(S, R, \partial) = \left[\frac{|Orb(S, R)|}{|S|}, \frac{k(R)}{|R|} \right]$$

where $k(R)$ is the number of distinct conjugacy classes of R and $Orb(S, R)$ is the number of orbits of R on S.

Proof Let $\{[x_1], [x_2], \ldots, [x_k]\}$ be the set of distinct conjugacy classes of R such that $k(R) = k$ and $|Orb(S, R)|$ be the number of orbits of R on S. It is well known that we can write

$$|\{(r, r') \in R \times R \mid rr' = r'r\}| = k(R)\,|R|\,.$$

So, we get

$$\frac{|\{(r, r') \in R \times R \mid rr' = r'r\}|}{|R|\,|R|} = \frac{k(R)\,|R|}{|R|\,|R|} = \frac{k(R)}{|R|}.$$

On the other hand, from the Burnside Lemma, we can write

$$|\{(r, s) \in R \times S \mid {}^{r}s = s\}| = |R|\,|Orb(S, R)|$$

and we have

$$\frac{|\{(r, s) \in R \times S \mid {}^{r}s = s\}|}{|R|\,|S|} = \frac{|R|\,|Orb(S, R)|}{|R|\,|S|} = \frac{|Orb(S, R)|}{|S|}.$$

□

Proposition 3.4 *Let* (S,R,∂) *be finite crossed module. Then*

$$(S,R,\partial) \text{ is Abelian if and only if } d(S,R,\partial)=[1,1].$$

Proof We know that if (S,R,∂) is Abelian, then R is Abelian and the action of R on S is trivial. So, we have $|\{(r,s)\in R\times S \mid {}^{r}s=s\}| = |R\times S| = |R|\,|S|$ and $|\{(r,r')\in R\times R \mid rr'=r'r\}| = |R\times R|$, that is $d(S,R,\partial)=[1,1]$. □

Theorem 3.5 *If* (S,R,∂) *and* (S',R',∂') *are two finite crossed modules, then*

$$d((S,R,\partial)\times(S',R',\partial')) = d(S,R,\partial)\times d(S',R',\partial').$$

Proof $d((S,R,\partial)\times(S',R',\partial')) = d(S\times S', R\times R', \widetilde{\partial})$

$$= \left[\tfrac{1}{|\mathcal{R}||\mathcal{S}|}\left|\{(a,b)\in\mathcal{R}\times\mathcal{S} \mid {}^{a}b=b\ \}\right|, \tfrac{1}{|\mathcal{R}||\mathcal{R}|}\left|\{(a_1,a_1')\in\mathcal{R}\times\mathcal{R} \mid a_1a_1'=a_1'a_1\}\right|\right]$$

$$= \left[\tfrac{1}{|R||R'||S||S'|}\left|\{(r,s)\in R\times S \mid {}^{r}s=s\}\right|\left|\left\{(r',s')\in R'\times S' \mid {}^{r'}s'=s'\right\}\right|,\right.$$

$$\left.\tfrac{1}{|R||R'||R||R'|}\left|\{(r_1,r_1')\in R\times R \mid r_1r_1'=r_1'r_1\}\right|\left|\{(r_2,r_2')\in R'\times R' \mid r_2r_2'=r_2'r_2\}\right|\right]$$

$$= \left[\tfrac{1}{|R||S|}\left|\{(r,s)\in R\times S \mid {}^{r}s=s\}\right|, \tfrac{1}{|R||R|}\left|\{(r_1,r_1')\in R\times R \mid r_1r_1'=r_1'r_1\}\right|\right]\times$$

$$\left[\tfrac{1}{|R'||S'|}\left|\left\{(r',s')\in R'\times S' \mid {}^{r'}s'=s'\right\}\right|, \tfrac{1}{|R'||R'|}\left|\{(r_2,r_2')\in R'\times R' \mid r_2r_2'=r_2'r_2\}\right|\right]$$

$$= d(S,R,\partial)\times d(S',R',\partial')$$

where $\mathcal{R}=R\times R'$, $\mathcal{S}=S\times S'$, $a=(r,r')$, $b=(s,s')$, $a_1=(r_1,r_2)$, $a_1'=(r_1',r_2')$. □

Now, we will find some useful bounds on commutativity degree for special types of crossed modules. (non-Abelian, simply connected, aspherical and nilpotent)

Theorem 3.6 *If* (S,R,∂) *is a finite crossed module and* $|(S,R,\partial)/Z(S,R,\partial)|=[l,m]$, *then*

$$d(S,R,\partial) \geq \left[\frac{2l-1}{l^2}\frac{|S|}{|R|}, \frac{2m-1}{m^2}\right].$$

Proof Let $|(S,R,\partial)/Z(S,R,\partial)|=[l,m]$. The set of pairs $\{\{(r,s)\in R\times S \mid {}^{r}s=s\}, \{(r,r')\in R\times R \mid rr'=r'r\}\}$ contains 2 copies of $Z(S,R,\partial)$ so that

$[|\{(r,s)\in R\times S \mid {}^{r}s=s\}|, |\{(r,r')\in R\times R \mid rr'=r'r\}|] \geq |(S,R,\partial)\times Z(S,R,\partial)| + |Z(S,R,\partial)\times(S,R,\partial)| - |Z(S,R,\partial)\times Z(S,R,\partial)|$. Then,

$$d(S,R,\partial) = \left[\frac{|\{(r,s)\in R\times S \mid {}^{r}s=s\}|}{|R||S|}, \frac{\left|\{(r,r')\in R\times R \mid rr'=r'r\}\right|}{|R||R|}\right]$$

$$\geq \frac{|(S,R,\partial)\times Z(S,R,\partial)| + |Z(S,R,\partial)\times(S,R,\partial)| - |Z(S,R,\partial)\times Z(S,R,\partial)|}{[|R||S|, |R||R|]}$$

$$= \frac{\left[\left|S\times S^R\right|,\left|R\times A\right|\right]+\left[\left|S^R\times S\right|,\left|A\times R\right|\right]-\left[\left|S^R\times S^R\right|,\left|A\times A\right|\right]}{\left[|R||S|,|R||R|\right]}$$

$$= \frac{|S|\left|S^R\right|+\left|S^R\right||S|-\left|S^R\right|\left|S^R\right|,|R||A|+|A||R|-|A||A|}{\left[|R||S|,|R||R|\right]}$$

$$= \left[\frac{l\left|S^R\right|^2+\left|S^R\right|^2 l-\left|S^R\right|^2}{|R||S|}, \frac{m|A|^2+|A|^2 m-|A|^2}{|R||R|}\right]$$

$$= \left[\frac{(2l-1)\left|S^R\right|^2}{lm\left|S^R\right||A|}, \frac{(2m-1)|A|^2}{m^2|A|^2}\right]$$

$$= \left[\frac{2l-1}{l^2}\frac{|S|}{|R|}, \frac{2m-1}{m^2}\right]$$

where $A = St_R(S)\cap Z(R)$. □

Theorem 3.7 *If (S,R,∂) is a finite crossed module and $[q,p]$ is the smallest prime pairs dividing $|(S,R,\partial)/Z(S,R,\partial)|$, then*

$$d(S,R,\partial)\leq\left[\frac{q^2+q-1}{q^3},\frac{p^2+p-1}{p^3}\right].$$

Proof Let (S,R,∂) be a finite crossed module and $[q,p]$ be the smallest prime pairs dividing $|(S,R,\partial)/Z(S,R,\partial)|$. Then q is the smallest prime dividing $\left|S/S^R\right|$ and p is the smallest prime dividing $|R/St_R(S)\cap Z(R)|$. It is well known that when p is the smallest prime dividing $|R/Z(R)|$, then $d(R)\leq\frac{p^2+p-1}{p^3}$. If p is the smallest prime dividing $|R/St_R(S)\cap Z(R)|$, then p is the smallest prime dividing $|R/Z(R)|$, too. So we can write $d(R)\leq\frac{p^2+p-1}{p^3}$.

On the other hand, we have that q is the smallest prime dividing $\left|S/S^R\right|$. Also, the class equation yields the bound

$$|S|\geq\left|S^R\right|+q(|Orb(S,R)|-\left|S^R\right|).$$

Solving for $|Orb(S,R)|$ yields

$$|Orb(S,R)|\leq\frac{|S|+(q-1)\left|S^R\right|}{q}.$$

Then since q is the smallest prime dividing $\left|S/S^R\right|$, we have $|S|=ql\left|S^R\right|$ and also we can write

$$\frac{|Orb(S,R)|}{|S|}\leq\frac{|S|+(q-1)\left|S^R\right|}{q\,|S|}=\frac{(q-1)\left|S^R\right|+ql\left|S^R\right|}{q^2l\,|S^R|}=\frac{(q-1)+ql}{q^2l}.$$

Next consider the ratio

$$\begin{aligned}\frac{(q-1)+ql}{q^2l}\Big/\frac{q-1+q^2}{q^3} &= \frac{(q-1+ql)q^3}{q^2l(q-1+q^2)}\\ &= \frac{(q-1+ql)q}{l(q-1+q^2)}\\ &= \frac{q^2l+q(q-1)}{q^2l+l(q-1)}\\ &\leq 1.\end{aligned}$$

Since this ratio is less than or equal to 1 and $\frac{|Orb(S,R)|}{|S|}$ is less than or equal to the numerator $\frac{(q-1)+ql}{q^2 l}$, it follows that $\frac{|Orb(S,R)|}{|S|}$ is less than or equal to denominator as well. So we have $\frac{|Orb(S,R)|}{|S|} \leq \frac{q-1+q^2}{q^3}$. That is,

$$d(S,R,\partial) \leq \left[\frac{q^2+q-1}{q^3}, \frac{p^2+p-1}{p^3}\right].$$

□

Corollary 3.8 *Let q and p be primes. If (S,R,∂) is non-Abelian crossed module with $|(S,R,\partial)| = [q^3, p^3]$ then*

$$d(S,R,\partial) = \left[\frac{q^2+q-1}{q^3}, \frac{p^2+p-1}{p^3}\right].$$

Theorem 3.9 *If (S,R,∂) is a finite crossed module, then*

(i) $d(S,R,\partial) \leq \left[\frac{1}{4}\left(1+\frac{3}{|D_R(S)|}\right), \frac{1}{4}\left(1+\frac{3}{|[R,R]|}\right)\right]$ *and*

(ii) $\left[\frac{1}{|D_R(S)|}, \frac{1}{|[R,R]|}\right] \leq d(S,R,\partial)$.

Proof (i) We can write the degree equation in the form

$$|S| = [S : D_R(S)] + \sum_{i=[S:D_R(S)]+1}^{|Orb(S,R)|} (n_i)^2$$

with each $n_i \geq 2$. It follows that

$$|S| \geq [S : D_R(S)] + 4(|Orb(S,R)| - [S : D_R(S)]).$$

Solving for $|Orb(S,R)|$,

$$|Orb(S,R)| \leq \frac{1}{4}(|S| + 3[S : D_R(S)]).$$

So we can write

$$\frac{|Orb(S,R)|}{|S|} \leq \frac{1}{4}\left(1+\frac{3}{|D_R(S)|}\right).$$

On the other hand, we know that

$$d(R) \leq \frac{1}{4}\left(1+\frac{3}{|[R,R]|}\right)$$

by group theory. That is

$$d(S,R,\partial) \leq \left[\frac{1}{4}\left(1+\frac{3}{|D_R(S)|}\right), \frac{1}{4}\left(1+\frac{3}{|[R,R]|}\right)\right].$$

(ii) Since $[S : D_R(S)] < |Orb(S,R)|$, we have

$$\frac{|S|}{|D_R(S)|\,|S|} \leq \frac{|Orb(S,R)|}{|S|}.$$

Also since $[R:[R,R]] < k(R),$ we have

$$\frac{1}{[R,R]} \leq \frac{k(R)}{|R|} = d(R).$$

That is

$$\left[\frac{1}{|D_R(S)|}, \frac{1}{|[R,R]|}\right] \leq d(S,R,\partial).$$

□

Theorem 3.10 *Let (S,R,∂) be a finite simply connected and aspherical crossed module. If $|(S,R,\partial)'| = [2,2],$ then $(S,R,\partial)' \subseteq Z(S,R,\partial).$*

Proof As $|(S,R,\partial)'| = \left|(D_R(S),[R,R],\widetilde{\partial})\right| = [2,2],$ we have $|D_R(S)| = 2$ and $|[R,R]| = 2.$ It is well known that if $|[R,R]| = 2,$ then $[R,R] \subseteq Z(R).$ Since (S,R,∂) is simply connected, we have $S^R = Z(S)$ and $D_R(S) = [S,S].$ On the other hand, since (S,R,∂) is aspherical we have $Z(R) \subseteq St_R(S).$ So we can write $|D_R(S)| = 2 \Longrightarrow |[S,S]| = 2$ and

$$D_R(S) = [S,S] \subseteq Z(S) = S^R.$$

On the other hand, since $[R,R] \subseteq Z(R) \subseteq St_R(S),$ we have

$$(S,R,\partial)' \subseteq Z(S,R,\partial).$$

□

Theorem 3.11 *(i) If (S,R,∂) is a crossed module such that $d(S,R,\partial) > \left[\frac{1}{2},\frac{1}{2}\right],$ then $|D_R(S)| < 3$ and $[R,R] < 3.$*

(ii) If (S,R,∂) is a simply connected and aspherical crossed module such that $d(S,R,\partial) > \left[\frac{1}{2},\frac{1}{2}\right],$ then (S,R,∂) is nilpotent.

Proof (i) Suppose that $d(S,R,\partial) > \left[\frac{1}{2},\frac{1}{2}\right].$ By the upper degree equation bound on $d(S,R,\partial),$ we get

$$\left[\frac{1}{2},\frac{1}{2}\right] < d(S,R,\partial) \leq \left[\frac{1}{4}\left(1+\frac{3}{|D_R(S)|}\right), \frac{1}{4}\left(1+\frac{3}{|[R,R]|}\right)\right].$$

Solving this equation for $|D_R(S)|$ yields $|D_R(S)| < 3$ and solving this equation for $|[R,R]|$ yields $[R,R] < 3.$

(ii) If $|(S,R,\partial)'| = [1,1],$ then (S,R,∂) is trivial. Hence (S,R,∂) is Abelian and nilpotent. Suppose $|(S,R,\partial)'| = [2,2].$ Then from above theorem $(S,R,\partial)' \subseteq Z(S,R,\partial).$ So, $(S,R,\partial)/Z(S,R,\partial)$ is Abelian. That is (S,R,∂) is nilpotent. □

In the following theorem, we will discuss bounds on the commutativity degree of crossed module in terms of the commutativity degree of a normal subcrossed module. From here, we will obtain that the commutativity degree of a crossed module is less than or equal to the commutativity degree of any of its normal subcrossed module.

Theorem 3.12 *Let* (S,R,∂) *be a finite crossed module and* (S',R',∂') *be a normal subcrossed module of* (S,R,∂)*. We have*

(i) $d(S,R,\partial) \leq d(S',R',\partial')d((S,R,\partial)/(S',R',\partial'))$.

(ii) $d(S,R,\partial) = d((S,R,\partial)/(S',R',\partial'))$ *if and only if* (S',R',∂') *is Abelian.*

(iii) If (S',R',∂') *is Abelian and one has equality, then* $(S',R',\partial') \subseteq Z(S,R,\partial)$.

Proof (i) Let (S',R',∂') be a normal subcrossed module of (S,R,∂). It is well known that $d(R) \leq d(R')d(R/R')$, where R' is a normal subgroup of R. That is, we must show that

$$\frac{|Orb(S,R)|}{|S|} \leq \frac{|Orb(S',R')|}{|S'|}\frac{|Orb(S/S',R/R')|}{|S/S'|}.$$

It is clear that $\frac{S^r R'}{R'} \subseteq (R/R')^{rR'}$. Therefore, from the Burnside Lemma, we can write

$$\begin{aligned}
|R||Orb(S,R)| &= \sum_{r\in R}|S^r| \\
&= \sum_{A\in R/R'}\sum_{r\in A}|S^r| \\
&= \sum_{A\in R/R'}\sum_{r\in A}\left|\frac{S^r R'}{R'}\right||S'^{r}| \\
&\leq \sum_{A\in R/R'}\sum_{r\in A}\left|(S/S')^A\right||S'^{r}| \\
&= \sum_{A\in R/R'}\left|(S/S')^A\right|\sum_{r\in A}|\{s'\in S'|{}^r s'=s'\}| \\
&= \sum_{A\in R/R'}\left|(S/S')^A\right|\sum_{r'\in R'}\left|S^{r'}\cap A\right|.
\end{aligned}$$

Let us suppose $S^{r'}\cap A \neq \varnothing$ and let $x_0 \in S^{r'}\cap A$; then $A = R'x_0$ whence

$$\begin{aligned}
S^{r'}\cap A &= S^{r'}\cap R'x_0 \\
&= (S^{r'}\cap R')(x_0) \\
&= S'^{r'}(x_0).
\end{aligned}$$

Therefore $S^{r'}\cap A$ is either empty, or a left coset of $S'^{r'}$; in both cases one has

$$|S^{r'}\cap A| \leq |S'^{r'}|$$

and

$$\begin{aligned} |R||Orb(S,R)| &\leq \sum_{A\in R/R'} \left|(S/S')^A\right| \sum_{r'\in R'} |S'^{r'}| \\ &= |R/R'||Orb(S/S',R/R')||R'||Orb(S',R')| \\ &= |R||Orb(S/S',R/R')||Orb(S',R')|. \end{aligned}$$

Hence $\frac{|Orb(S,R)|}{|S|} \leq \frac{|Orb(S',R')|}{|S'|}\frac{|Orb(S/S',R/R')|}{|S/S'|}$. That is, $d(S,R,\partial) \leq d(S',R',\partial')d((S,R,\partial)/(S',R',\partial'))$.

(ii) If (S',R',∂') is Abelian, then R' is Abelian and the action of R' on S' is trivial. We have

$$d((S,R,\partial)/(S',R',\partial')) = \left(\frac{|Orb(S/S',R/R')|}{|S/S'|}, d(R/R')\right).$$

It is well known that if R' is Abelian, then $d(R) = d(R/R')$. So, we must show that

$$\frac{|Orb(S/S',R/R')|}{|S/S'|} = \frac{|Orb(S,R)|}{|S|}.$$

Since the action of R' on S' is trivial, we have $|\{(r',s') \in R' \times S' | {}^{r}s' = s'\}| = |R'||S'|$. On the other hand, we know that

$$\left|\left\{(r',s') \in R' \times S' |{}^{r'}s' = s'\right\}\right| = |R'||Orb(S',R')|.$$

So, we can write $|Orb(S',R')| = |S'|$. That is,

$$\frac{|Orb(S/S',R/R')|}{|S/S'|} = \frac{|Orb(S,R)|}{|S|}.$$

So, we have $d(S,R,\partial) = d((S,R,\partial)/(S',R',\partial'))$.

Similarly, if $d(S,R,\partial) = d((S,R,\partial)/(S',R',\partial'))$, then it is clear that (S',R',∂') is Abelian.

(iii) Let $d(S,R,\partial) = d((S,R,\partial)/(S',R',\partial'))$ and (S',R',∂') is Abelian. Since $d(R) = d(R/R')$ and R' is Abelian, we have $R' \subseteq Z(R)$. So, it also satisfy that $R' \subseteq St_R(S)$ since the action of R' on S' is trivial. That is, $R' \subseteq St_R(S) \cap Z(R)$.

On the other hand, since the action of R' on S' is trivial, we have ${}^{r'}s' = s'$ for all $r' \in R'$. So, $s' \in S^R$, i.e $S' \subseteq S^R$. □

Corollary 3.13 *If (S',R',∂') is a subcrossed module of (S,R,∂) and $d(S,R,\partial) = d(S',R',\partial')$ then (S',R',∂') is normal subcrossed module and $(S,R,\partial)/(S',R',\partial')$ is Abelian crossed module.*

3.3. Commutativity degree of isoclinic crossed modules

First, we will recall the definition of isoclinic crossed modules and give some examples. For more detail properties of this notion, see [23].

The crossed modules $S \xrightarrow{\partial} R$ and $S' \xrightarrow{\partial'} R'$ are *isoclinic* if there exist isomorphisms

$$(\eta_1, \eta_0) : (\overline{S} \xrightarrow{\overline{\partial}} \overline{R}) \longrightarrow (\overline{S'} \xrightarrow{\overline{\partial'}} \overline{R'})$$

and

$$(\xi_1, \xi_0) : (D_R(S) \xrightarrow{\partial|} [R, R]) \longrightarrow (D_{R'}(S') \xrightarrow{\partial'|} [R', R'])$$

such that the diagrams

$$\begin{tikzcd} \overline{G_1} \times \overline{G_0} \arrow[r, "c_1"] \arrow[d, "\eta_1 \times \eta_0"'] & D_{G_0}(G_1) \arrow[d, "\xi_1"] \\ \overline{H_1} \times \overline{H_0} \arrow[r, "{c_1}'"'] & D_{H_0}(H_1) \end{tikzcd} \tag{3.1}$$

and

$$\begin{tikzcd} \overline{G_0} \times \overline{G_0} \arrow[r, "c_0"] \arrow[d, "\eta_0 \times \eta_0"'] & [G_0, G_0] \arrow[d, "\xi_0"] \\ \overline{H_0} \times \overline{H_0} \arrow[r, "{c_0}'"'] & [H_0, H_0] \end{tikzcd} \tag{3.2}$$

are commutative where c_1, c_1' defined by $c_1(sS^R, r(St_R(S) \cap Z(R))) = {}^r s s^{-1}$, $c_1'(s'S'^{R'}, r'(St_{R'}(S') \cap Z(R'))) = {}^{r'} s' s'^{-1}$, for all $s \in S$, $r \in R$, $s' \in S'$, $r' \in R'$ and c_0, c_0' defined by $c_0(r(St_R(S) \cap Z(R)),\ r'(St_R(S) \cap Z(R'))) = [r, r']$, $c_0'(k(St_{R'}(S') \cap Z(R')), k'(St_{R'}(S') \cap Z(R'))) = [k, k']$, for all $r, r' \in R$ and $k, k' \in R'$. Here $\overline{S} = S/S^R$, $\overline{R} = R/St_R(S) \cap Z(R)$, $\overline{S'} = S'/S'^{R'}$ and $\overline{R'} = R'/St_{R'}(S') \cap Z(R')$.

The pair $((\eta_1, \eta_0), (\xi_1, \xi_0))$ is called an *isoclinism* from $S \xrightarrow{\partial} R$ to $S' \xrightarrow{\partial'} R'$.

Example 3.14 *(1) All Abelian crossed modules are isoclinic.*

(2) Let M and N be isoclinic groups. Then $M \xrightarrow{id} M$ is isoclinic to $N \xrightarrow{id} N$.

(3) Let M be a group and let N be a normal subgroup of M with $NZ(M) = M$. Then $N \xhookrightarrow{inc.} M$ is isoclinic to $M \xrightarrow{id} M$.

Now, we will show that isoclinic crossed modules have same commutativity degrees.

Theorem 3.15 *Let (S, R, ∂) and (S', R', ∂') be two isoclinic finite crossed modules, then $d(S, R, \partial) = d(S', R', \partial')$.*

Proof Suppose that (S, R, ∂) and (S', R', ∂') are isoclinic. Then S is isoclinic to S' and R is isoclinic to R'. So, $d(S) = d(S')$ and $d(R) = d(R')$, by group theory. On the other hand, we have

$$\frac{|S \times R|}{|S^R \times A|} \frac{|\{(r,s) \in R \times S|^r s = s\}|}{|R \times S|} = \frac{1}{|S^R \times A|} |\{(r, s) \in R \times S|^r s = s\}|$$

$$= \frac{1}{|S^R \times A|} \left|\{(r, s) \in R \times S | c_1(sS^R, r(A)) = 1\}\right|$$

$$= \left|\left\{(\alpha, \beta) \in \frac{S}{S^R} \times \frac{R}{A} | \xi_1(c_1(\alpha, \beta)) = 1\right\}\right| (\because \xi_1 \text{ iso.})$$

$$= \left|\left\{(\alpha,\beta) \in \tfrac{S}{S^R} \times \tfrac{R}{A} \middle| c_1'(\eta_1 \times \eta_0)(\alpha,\beta) = 1\right\}\right| \; (\because \, com.diag)$$

$$= \left|\left\{(\alpha,\beta) \in \tfrac{S}{S^R} \times \tfrac{R}{A} \middle| c_1'(\eta_1(\alpha),\eta_0(\beta)) = 1\right\}\right|$$

$$= \left|\left\{(\gamma,\delta) \in \tfrac{S'}{S'^{R'}} \times \tfrac{R'}{A'} \middle| c_1'(\gamma,\delta) = 1\right\}\right| \; (\because \, \eta_1, \eta_0 \text{ iso.})$$

$$= \frac{1}{|S'^{R'} \times A'|} \left|\left\{(r',s') \in R' \times S' \middle| {}^{r'}s' = s'\right\}\right|$$

where $A = St_R(S) \cap Z(R)$ and $A' = St_{R'}(S') \cap Z(R')$. Since $S/S^R \cong S'/S'^{R'}$ and $R/A \cong R'/A'$, we can write

$$\frac{|S \times R|}{|S^R \times A|} = \frac{|S' \times R'|}{|S'^{R'} \times A'|}.$$

That is $d(S,R,\partial) = d(S',R',\partial')$. □

Corollary 3.16 *Let (S',R',∂') be a subcrossed module of (S,R,∂) and $(S,R,\partial) = (S',R',\partial')Z(S,R,\partial)$. Then $d(S,R,\partial) = d(S',R',\partial')$.*

Proof It is clear from the proposition 4 of [23]. □

Corollary 3.17 *Let (S,R,∂) be a finite crossed module such that $(S,R,\partial)' \cap Z(S,R,\partial) = (1,1,id)$; then there is a finite crossed module $(H,K,\widetilde{\partial})$ such that $d(S,R,\partial) = d(H,K,\widetilde{\partial})$, $(H,K,\widetilde{\partial})' \simeq (S,R,\partial)'$ and $Z(H,K,\widetilde{\partial}) = (1,1,id)$.*

Proof By Proposition 3 of [23], there exists a finite crossed module $(H,K,\widetilde{\partial})$ isoclinic to (S,R,∂) and such that $Z(H,K,\widetilde{\partial}) \subseteq (H,K,\widetilde{\partial})'$. Then we have

$$\begin{aligned} \left|Z(H,K,\widetilde{\partial})\right| &= \left|Z(H,K,\widetilde{\partial}) \cap (H,K,\widetilde{\partial})'\right| \\ &= |Z(S,R,\partial) \cap (S,R,\partial)'| \\ &= (1,1,id) = [1,1] \end{aligned}$$

from the hypothesis on (S,R,∂), i.e,

$$Z(H,K,\widetilde{\partial}) = (1,1,id).$$

On the other hand, the isoclinism between $(H,K,\widetilde{\partial})$ and (S,R,∂) implies $(H,K,\widetilde{\partial})' \simeq (S,R,\partial)'$ by definition, and $d(H,K,\widetilde{\partial}) = d(S,R,\partial)$ by Theorem 3.15. □

3.4. Hyper order commutativity degree

We can define the "n-th commutativity degree" of a finite crossed module (S,R,∂).

Definition 3.18 *Let (S,R,∂) be a finite crossed module. The n-th commutativity degree of (S,R,∂) is defined by*

$$d_n(S,R,\partial) = \left[\begin{array}{c} \frac{\left|\{(r_1,r_2,\ldots,r_n,s_1,s_2,\ldots s_n) \in R^n \times S^n \,|\, {}^{(r_1,r_2,\ldots,r_n)}(s_1,s_2,\ldots,s_n) = (s_1,s_2,\ldots,s_n)\}\right|}{|R|^n |S|^n}, \\ \frac{\left|\left\{(r_1,r_2,\ldots,r_n,r_{n+1}) \in R^{n+1} \,|\, r_i r_j = r_j r_i \text{ where } i,j \in \{1,2,\ldots,n+1\}\right\}\right|}{|R|^{n+1}} \end{array} \right].$$

Clearly, $d_0(S,R,\partial) = [1,1]$, $d_1(S,R,\partial) = d(S,R,\partial)$.

Theorem 3.19 *If (S,R,∂) and (S',R',∂') are isoclinic, then for $n \in \mathbb{N}$*

$$d_n(S,R,\partial) = d_n(S',R',\partial').$$

Proof It is similar to the proof of Theorem 3.15. □

4. Computer implementation

GAP (Groups, Algorithms, Programming [12]) is the leading symbolic computation system for solving computational discrete algebra problems. Symbolic computation has underpinned several key advances in Mathematics and Computer Science, for example, in number theory and coding theory (see [5]). The system consists of a library of implementations of mathematical structures: groups, vector spaces, modules, algebras, graphs, codes, designs, etc., plus databases of groups of small order, character tables, etc. The system has world wide usage in the area of education and scientific research.

The XMod package for GAP contains functions for computing with crossed modules, cat^1-groups and their morphisms, and was first described in [1]. Moreover, The XMod package has many functions for isoclinism classes of groups and crossed modules and some family invariants. A GAP package XModAlg [3] was written to compute cat^1-algebras and crossed modules of algebras, as described in [4].

On the other hand, we implement the functions **CommutativeDegreeOfGroup** and **CommutativeDegreeOfXMod** which used to compute commutative degree of a group and a crossed module, respectively.

The following GAP session illustrates the use of these functions.

```
gap> Q16 := QuaternionGroup(IsPermGroup,16);
Group([ (1,9,5,13)(2,16,6,12)(3,15,7,11)(4,14,8,10),
(1,2,3,4,5,6,7,8)(9,10,11,12,13,14,15,16) ])
gap> C8 := Subgroup(Q16, [Q16.2]);
Group([ (1,2,3,4,5,6,7,8)(9,10,11,12,13,14,15,16) ])
gap> StructureDescription(C8);
"C8"
gap> CM := XModByNormalSubgroup(Q16,C8);
[Group( [ ( 1, 2, 3, 4, 5, 6, 7, 8)( 9,10,11,12,13,14,15,16) ] )->
Group([ ( 1, 9, 5,13)( 2,16, 6,12)( 3,15, 7,11)( 4,14, 8,10),
( 1, 2, 3, 4, 5, 6, 7, 8)( 9,10,11,12,13,14,15,16) ] )]
gap> CommutativeDegreeOfGroup(Q16);
7/16
gap> CommutativeDegreeOfXMod(CM);
[ 5/8, 7/16 ]
```

Following GAP session shows that two crossed modules in different isoclinism families would be a same commutative degree.

```
gap> all := AllXModsFromData();;
gap> Length(all);
2692
gap> CM1 := all[413];
[Group( [ f2, f4 ] )->Group( [ f1, <identity> of ..., f2*f3*f4 ] )]
```

```
gap> CM2 := all[2381];
[Group( [ f3, f6 ] )->Group( [ f1, f2*f6, <identity> of ..., f4*f6 ] )]
gap> CommutativeDegreeOfXMod(CM1);
[ 3/4, 5/8 ]
gap> CommutativeDegreeOfXMod(CM2);
[ 3/4, 5/8 ]
```

Following GAP session shows that the numbers of all crossed modules of order $[4, 8]$ are 336, which give rise 59 isomorphism classes and 5 isoclinism families. The commutative degree of the crossed modules in the same isoclinism family are equal, as expected.

```
gap> all48 := AllXMods([4,8]);;
gap> Length(all48);
336
gap> iso_all48 := AllXModsUpToIsomorphism(all48);;
gap> Length(iso_all48);
59
gap> f_1 := IsoclinicXModFamily(iso_all48[1],iso_all48);
[ 1, 3, 4, 6, 9, 10, 12, 27, 29, 31, 33, 35, 38, 40, 43, 55, 57, 59 ]
gap> f_2 := IsoclinicXModFamily(iso_all48[2],iso_all48);
[ 2, 5, 7, 8, 11, 13, 14, 28, 30, 32, 34, 36, 37, 39, 41, 42, 56, 58 ]
gap> f_3 := IsoclinicXModFamily(iso_all48[15],iso_all48);
[ 15, 18, 22, 25, 44, 47, 51, 53 ]
gap> f_4 := IsoclinicXModFamily(iso_all48[16],iso_all48);
[ 16, 17, 19, 20, 23, 26, 45, 46, 48, 49, 52, 54 ]
gap> f_5 := IsoclinicXModFamily(iso_all48[21],iso_all48);
[ 21, 24, 50 ]
gap> CommutativeDegreeOfXMod(iso_all48[1]);
[ 1, 1 ]
gap> ForAll(f_1, i -> CommutativeDegreeOfXMod(iso_all48[i]) = [1,1]);
true
gap> CommutativeDegreeOfXMod(iso_all48[2]);
[ 3/4, 1 ]
gap> ForAll(f_2, i -> CommutativeDegreeOfXMod(iso_all48[i]) = [3/4,1]);
true
gap> CommutativeDegreeOfXMod(iso_all48[15]);
[ 1, 5/8 ]
gap> ForAll(f_3, i -> CommutativeDegreeOfXMod(iso_all48[i]) = [1,5/8]);
true
gap> CommutativeDegreeOfXMod(iso_all48[16]);
[ 3/4, 5/8 ]
gap> ForAll(f_4, i -> CommutativeDegreeOfXMod(iso_all48[i]) = [3/4,5/8]);
true
gap> CommutativeDegreeOfXMod(iso_all48[21]);
[ 3/4, 5/8 ]
gap> ForAll(f_5, i -> CommutativeDegreeOfXMod(iso_all48[i]) = [3/4,5/8]);
true
```

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