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Sober spaces

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Abstract: The goal of this paper is to introduce various forms of sober objects in a topological category and investigate the relationships among these various forms. Moreover, we characterize quasi-sober objects, T_0 objects and each of various forms of sober objects in some topological categories and give some invariance properties of them. Finally, we compare our results with some well-known results in the category of topological spaces.

Key words: Topological category, T_0 objects, T_1 objects, sober spaces, irreducible spaces.

1. Introduction

The sober spaces, which are one of the most important classes in the theory of non-Hausdorff topological spaces, were introduced in [12]. The usual notion of closedness and T_0-axiom of topology were generalized the topological categories [1, 17, 20] and they are being used in defining separation axioms $T_i, i = 2, 3, 4$, compactness and perfectness [1, 5].

In this paper, we introduce various forms of sober objects and quasi sober objects in topological categories. Moreover, we try to:

(i) find the relationships as well as interrelationships among these notions,

(ii) give the charactarization of these notions in the category of **PBorn** (resp. **CP** and **RRel**) of all prebornological (resp. pair and reflexive relation) spaces and show whether the subcategories $\mathbf{T_0'Sob}\mathcal{E}$ (resp. $\mathbf{T_0Sob}\mathcal{E}$, $\overline{\mathbf{T}}_\mathbf{0}\mathbf{Sob}\mathcal{E}$) of $\mathcal{E}$ consisting of all T_0' sober (resp. T_0 sober, $\overline{T}_0$ sober) objects of $\mathcal{E}$ are isomorphic or not, where $\mathcal{E} = \mathbf{PBorn}, \mathbf{CP}$ or $\mathbf{RRel}$,

(iii) examine some invariance properties of these various forms of sober objects in these topological categories,

(iv) compare our results with some results in the category of topological spaces.

2. Preliminaries

The category of prebornological spaces, **PBorn**, has as objects the pair $(X, \mathcal{F})$, where $\mathcal{F}$ is a family of subsets of X that is closed under finite union and contains all finite subsets of X and has morphisms $(X, \mathcal{F}) \to (Y, \mathcal{G})$ those functions $f : X \to Y$ such that $f(C) \in \mathcal{G}$ if $C \in \mathcal{F}$ [16]. **PBorn** is a topological category over **Set**, the category of all sets and functions [16].

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The category of pair, **CP**, has as objects the pair (X, Y), where $Y \subset X$ and has as morphisms $(X, Y) \to (X_1, Y_1)$ those functions $f : X \to X_1$ such that $f(Y) \subset Y_1$. This is a topological category [2].
The category of reflexive relation spaces (spatial graphs), **RRel**, has as objects the pair (B, R), where B is a set and R is reflexive relation on B, and as morphisms $(B, R) \to (B_1, R_1)$ those functions $f : B \to B_1$ such that if sRt, then $f(s)R_1f(t)$ for all $s, t \in B$ [9, 18].
A source $f_i : (B, R) \longrightarrow (B_i, R_i), i \in I$ is initial in **RRel** iff for all $s, t \in B$, sRt iff $f_i(s)R_if_i(t)$ for all $i \in I$ [9, 18].
An epimorphism $f : (B, R) \longrightarrow (B_1, R_1)$ is final in **RRel** iff for all $s, t \in B_1$, sR_1t holds in B_1 precisely when there exist $c, d \in B$, such that cRd and $f(c) = s$ and $f(d) = t$ [18].
RRel is a topological category.
Let B be a set, $p \in B$, and the wedge $B^2 \bigvee_\Delta B^2$ (resp. the infinity wedge $\bigvee_p^\infty B$) be two distinct copies of B^2 identified along the diagonal Δ (resp. taking countably many disjoint copies of B and identifying them at the point p) [1]. Define $A : B^2 \bigvee_\Delta B^2 \to B^3$ by

$$A((a,b)_i) = \begin{cases} (a,b,a), & i = 1 \\ (a,a,b), & i = 2 \end{cases},$$

$\nabla : B^2 \bigvee_\Delta B^2 \to B^2$ by

$$\triangledown((a,b)_i) = (a,b)$$

for $i = 1, 2$

$A_p^\infty : \bigvee_p^\infty B \to B^\infty$ by

$$A_p^\infty(a_i) = (p, ..., p, a, p, p, ...)$$

where a_i is in the i-th component of $\bigvee_p^\infty B$ and $B^\infty = B \times B \times ...$ is the countable cartesian product of B, and $\triangledown_p^\infty : \bigvee_p^\infty B \longrightarrow B$ by

$$\triangledown_p^\infty(a_i) = a$$

for all $i \in I$, where I is the index set $\{i : a_i$ is in the i-th component of $\bigvee_p^\infty B\}$ [1].

Definition 2.1 *(cf. [1, 17]) Let* $U : \mathcal{E} \to$ **Set** *be topological and* X *be an object in* $\mathcal{E}$ *with* $p \in U(X) = B$.
(1) *If the initial lift of the* U*-source*

$$\{A : B^2 \vee_\Delta B^2 \to U(X^3) = B^3 \quad \textit{and} \quad \nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$$

is discrete, then X *is called a* $\overline{T_0}$ *object, where* D *is the discrete functor.* **(2)** *If the initial lift of the* U*-source*

$$\{id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2 \quad \textit{and} \quad \nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$$

is discrete, then X *is called* T_0' *object, where* $(B^2 \bigvee_\Delta B^2)'$ *is the final lift of the* U*-sink*

$$\{q \circ i_1, q \circ i_2 : U(X^2) = B^2 \to B^2 \vee_\Delta B^2\},$$

where $i_k : B^2 \to B^2 \coprod B^2, k = 1,2$ *are the canonical injection maps and* $q : B^2 \coprod B^2 \to B^2 \bigvee_\Delta B^2$ *is the quotient map.* **(3)** *If* X *does not contain an indiscrete subspace with (at least) two points, then* X *is called a* T_0 *object.* **(4)** $\{p\}$ *is closed iff the initial lift of the* U*-source*

$$\{A_p^\infty : \vee_p^\infty B \to B^\infty = U(X^\infty) \quad and \quad \bigtriangledown_p^\infty : \vee_p^\infty B \to UD(B) = B\}$$

is discrete.
(5) $\emptyset \neq Z \subset B$ *is closed iff* $\{*\}$*, the image of* Z*, is closed in* X/Z*, where* X/Z *is the final lift of the epi* U*-sink*

$$Q : U(X) = B \to B/Z = (B\backslash Z) \cup \{*\},$$

identifying Z *with a point *.* **(6)** *If* $Z = B = \emptyset$*, then we define* Z *to be closed.*

Remark 2.2 *For the category* **Top** *of topological spaces and continuous functions, all of* $\overline{T_0}$*,* T_0'*, and* T_0 *are equivalent and they reduce to the usual* T_0 *separation axiom and the notion of closedness coincides with the usual closedness [1, 17, 20].*

3. Sober spaces

Let $\mathcal{E}$ be a topological category over **Set** and $X \in Ob(\mathcal{E})$ [18].

Definition 3.1

$$\boldsymbol{cl}(Z) = \bigcap\{U \subset X : Z \subset U \quad and \quad U \quad is \quad closed\}$$

is called the closure of a subobject Z *of* X*.*

It is shown that the notion of closedness forms closure operator [8] in some topological categories [5–7, 10, 11, 14, 19].
In Exercise 2.D of [9], let $\mathcal{M}$ be the class of all closed subobjects of an object of $\mathcal{E}$. If $\mathcal{M}$ is stable under pullback, then $\boldsymbol{cl}$ is an idempotent closure operator of $\mathcal{E}$.

Definition 3.2 *([7])* X *is said to be irreducible if* A, B *are closed subobjects of* X *and* $X = A \cup B$*, then* $A = X$ *or* $B = X$*.*

Definition 3.3 **(1)** X *is called quasi-sober if every nonempty irreducible closed subset of* X *is the closure of a point.*
(2) X *is called* $\overline{T_0}$ *sober if* X *is* $\overline{T_0}$ *and quasi-sober.*
(3) X *is called* T_0' *sober if* X *is* T_0' *and quasi-sober.*
(4) X *is called* T_0 *sober if* X *is* T_0 *and quasi-sober.*

Remark 3.4 *For* **Top** *all of* $\overline{T_0}$ *sober,* T_0' *sober, and* T_0 *sober are equivalent and they reduce to the usual sober [15] and quasi-sober reduces to the usual quasi-sober [13].*

Theorem 3.5 *If* X *is* $\overline{T_0}$ *sober, then* X *is* T_0' *sober.*

Proof If X is $\overline{T_0}$ sober, then in particular, X is $\overline{T_0}$ and by Theorem 3.2, of [3], X is T_0'. Hence, by Definition 3.3, X is T_0' sober. □

Recall that if the initial lift of the U-source

$$\{S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3 \quad \text{and} \quad \nabla : B^2 \vee_\Delta B^2 \to U(D(B^2)) = B^2\}$$

is discrete, then X is called a T_1 object and if the initial lift of the U-source

$$S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$$

and the initial lift of the U-source

$$A : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$$

coincide, then X is called a $Pre\overline{T}_2$ object, where S is the skewed axis map defined as

$$S((a,b)_i) = \begin{cases} (a,b,b), & \text{i=1} \\ (a,a,b), & \text{i=2} \end{cases}$$

[1]. If X is T_0' and $Pre\overline{T}_2$, then X is called KT_2 [4].

Theorem 3.6 *Let* $(X,\mathcal{F})$ *be a prebornological space.*
(A) **(1)** $(X,\mathcal{F})$ *is* T_0'*.*
(2) $(X,\mathcal{F})$ *is* $\overline{T_0}$*.*
(3) $(X,\mathcal{F})$ *is* T_1*.*
(B) *The following are equivalent:*
(1) $(X,\mathcal{F})$ *is quasi-sober.*
(2) $(X,\mathcal{F})$ *is* $\overline{T_0}$ *sober.*
(3) $(X,\mathcal{F})$ *is* T_0' *sober.*
(4) $(X,\mathcal{F})$ *is irreducible.*
(C) *The following are equivalent*

(1) $(X,\mathcal{F})$ *is* T_0*.*
(2) $(X,\mathcal{F})$ *is* T_0 *sober.*
(3) $X=\emptyset$ *or* X *is a one-point set.*

Proof **(A)** **(1)** Let $(X,\mathcal{F})$ be a prebornological space, $\mathcal{G}$ be the final prebornological structure on $X^2 \coprod X^2$ induced by the injection maps

$$i_1, i_2 : (X^2, \mathcal{F}^2) \longrightarrow X^2 \coprod X^2,$$

$\mathcal{H}$ be the final prebornological structure on $X^2 \bigvee_\Delta X^2$ induced by the quotient map

$$q : (X^2 \coprod X^2, \mathcal{G}) \longrightarrow X^2 \vee_\Delta X^2,$$

and $\overline{\mathcal{F}}$ be the initial structure on $X^2 \bigvee_\Delta X^2$ induced by

$$id : X^2 \vee_\Delta X^2 \longrightarrow (X^2 \vee_\Delta X^2, \mathcal{H}) \quad \text{and} \quad \bigtriangledown : X^2 \vee_\Delta X^2 \longrightarrow (X^2, \mathcal{D}\mathcal{F}),$$

where $\mathcal{DF}$ is discrete structure on X^2 i.e. $\mathcal{DF}$ is the set of all finite subsets of X^2 [16] and $\mathcal{F}^2$ is the product structure on X^2.
Let U be any element in $\overline{\mathcal{F}}$. In particular, $\bigtriangledown(U) \in \mathcal{DF}$,
If $\bigtriangledown(U) = \emptyset$, then it follows that $U = \emptyset$. Suppose that $\bigtriangledown(U) \neq \emptyset$. Since $\mathcal{DF}$ is discrete structure on X^2 and $\bigtriangledown(U) \in \mathcal{DF}$, $\bigtriangledown(U)$ is finite and consequently, U is finite. Hence, $\overline{\mathcal{F}}$ is discrete and by Definition 2.1, $(X, \mathcal{F})$ is T_0'.

(2) Let $\overline{\mathcal{F}}$ be the initial prebornological structure on $X^2 \bigvee_\Delta X^2$ induced by

$$A : X^2 \vee_\Delta X^2 \to (X^3, \mathcal{F}^3) \quad \text{and} \quad \bigtriangledown : X^2 \vee_\Delta X^2 \to (X^2, \mathcal{DF}),$$

where $\mathcal{F}^3$ is the product structure on X^3.
Let U be any element in $\overline{\mathcal{F}}$. Since $\overline{\mathcal{F}}$ is initial by [16], $\bigtriangledown(U) \in \mathcal{DF}$ and $\pi_i A(U) \in \mathcal{F}$, where $\pi_i : X^3 \to X$, $i = 1, 2, 3$ are the projection maps.
If $\bigtriangledown(U) = \emptyset$, then it follows that $U = \emptyset$. Suppose that $\bigtriangledown(U) \neq \emptyset$. Since $\mathcal{DF}$ is discrete structure on X^2 and $\bigtriangledown(U) \in \mathcal{DF}$, $\bigtriangledown(U)$ is finite and consequently, U is finite. Hence, $\overline{\mathcal{F}}$ is discrete and by Definition 2.1, $(X, \mathcal{F})$ is $\overline{T_0}$.
The proof for T_1 is similar.
(B) By Definition 3.3 and part (A), (1)-(3) are equivalent.
Suppose $(X, \mathcal{F})$ is quasi-sober and $X = M \cup N$, where M and N are closed subsets of X. By Theorem 3.9 of [2], $M = X$ or $M = \emptyset$ and $N = X$ or $N = \emptyset$. Hence, by Definition 3.2, $(X, \mathcal{F})$ is irreducible.
Suppose $(X, \mathcal{F})$ is irreducible and $\emptyset \neq M \subset X$ is irreducible closed. Since M is closed , by Theorem 3.9 of [2], $M = X$ and by Definition 3.1, $X = M = \boldsymbol{cl}\{a\}$ for some $a \in X$. Hence, by Definition 3.3, $(X, \mathcal{F})$ is quasi-sober. Hence, (1) and (4) are equivalent.
(C) (1) $\Rightarrow$ (2): Suppose $(X, \mathcal{F})$ is T_0 and M is $\emptyset \neq M \subset X$ is irreducible closed. Since M is closed, by Theorem 3.9 of [2], $M = X$ and by Definition 3.1, $M = X = \boldsymbol{cl}\{a\}$ for some $a \in X$. Hence, by Definition 3.2, $(X, \mathcal{F})$ is quasi-sober and by Definition 3.3, $(X, \mathcal{F})$ is T_0 sober.
(2) $\Rightarrow$ (3): Suppose $(X, \mathcal{F})$ is T_0 sober and $X \neq \emptyset$ and X is not a one-point set. Then, there exist distinct points c and d of X and by [16] $(\{c, d\}, P(\{c, d\}))$ is the indiscrete subspace of $(X, \mathcal{F})$, contradicting to $(X, \mathcal{F})$ is being T_0 sober. Hence, $X = \emptyset$ or X is a one-point set.
(3) $\Rightarrow$ (1): If $X = \emptyset$ or X is a one-point set, then by Definition 2.1, $(X, \mathcal{F})$ is T_0. □

Theorem 3.7 *Let* (X, Y) *be a pair space.*
(A) **(1)** (X, Y) *is* $\overline{T_0}$.
(2) (X, Y) *is* T_0'.

(3) (X, Y) *is* T_1.
(B) *The following are equivalent:*
(1) (X, Y) *is* $\overline{T_0}$ *sober.*
(2) (X, Y) *is* T_0' *sober.*
(3) (X, Y) *is quasi-sober.*
(4) *The nonempty proper irreducible closed subsets of* X *are exactly the one-point subsets.*

(C) *The following are equivalent:*
(1) (X,Y) *is* T_0.
(2) (X,Y) *is* T_0 *sober.*
(3) $X=\emptyset$ *or* X *is a one point set.*
(4) (X,Y) *is irreducible.*

Proof **(A) (1)** Note that $(X,\emptyset)$ is the discrete pair space [2]. Let (X,Y) be a pair space and V be the initial structure on $X^2 \vee_\Delta X^2$ induced by

$$A : X^2 \vee_\Delta X^2 \to (X^3, Y^3) \quad \text{and} \quad \bigtriangledown : X^2 \vee_\Delta X^2 \longrightarrow (X^2, \emptyset),$$

where Y^3 is the product structure on X^3 [2]. It follows that $\bigtriangledown(V) \subset \emptyset$. Hence, $V=\emptyset$ and by Definition 2.1, (X,Y) is $\overline{T_0}$.
(2) Let (X,Y) be a pair space, Z be the final structure on $X^2 \vee_\Delta X^2$ induced by

$$q \circ i_1, q \circ i_2 : (X^2, Y^2) \to X^2 \vee_\Delta X^2,$$

where the maps i_1, i_2, and q are defined in Section 2 and V be the initial structure on $X^2 \vee_\Delta X^2$ induced by

$$id : X^2 \vee_\Delta X^2 \to (X^2 \vee_\Delta X^2, Z) \quad \text{and} \quad \bigtriangledown : X^2 \vee_\Delta X^2 \to (X^2, \emptyset).$$

In particular, $\bigtriangledown(V) \subset \emptyset$ and consequently, $V=\emptyset$ and by Definition 2.1, (X,Y) is T_0'.
(3) The proof of Part (3) is similar to Part (1).
(B) By Part (A) and by Definition 3.3, (1)-(3) are equivalent.
(3) $\Rightarrow$ (4): (X,Y) is quasi-sober and $\emptyset \neq A \subset X$ is proper irreducible closed. Since (X,Y) is quasi-sober, then $A = \boldsymbol{cl}\{a\}$ for some $a \in X$. By Theorem 3.8 of [2], $\{a\}$ is closed. Thus, $\boldsymbol{cl}\{a\} = \{a\}$ and $A=\{a\}$.
(4) $\Rightarrow$ (3): Suppose the nonempty proper irreducible closed subsets of X are exactly the one-point subsets and $\emptyset \neq A \subset X$ is proper irreducible closed. Then by assumption, $A=\{a\}$ for some $a \in X$. By Theorem 3.8 of [2], $\{a\}$ is closed, we get $\boldsymbol{cl}(\{a\}) = \{a\} = A$. Hence, by Definition 3.3, (X,Y) is quasi-sober.

(C) Suppose (X,Y) is T_0 sober and $X \neq \emptyset$ and X is not a one point set. Then there exist distinct points s and t of X and by [2], $(\{s,t\},\{s,t\})$ is the indiscrete subspace of (X,Y) contradicting to (X,Y) is being T_0 sober. Hence, $X=\emptyset$ or X is a one-point set.
If $X=\emptyset$ or X is a one-point set, then by Definitions 2.1 and 3.3, (X,Y) is T_0 and quasi-sober. This shows Parts (1)-(3) are equivalent.

Suppose (X,Y) is irreducible and $X \neq \emptyset$ and X is not a one-point set. Then there exist distinct points s and t of X. By Theorem 3.8 of [2] both $\{t\}$ and $\{t\}^C$ are closed subsets of X and $X=\{t\}\cup\{t\}^C$ contradicting to (X,Y) is being irreducible.
If $X=\emptyset$ or X is a one-point set, then by Definition 3.2, (X,Y) is irreducible.
This shows Parts (3) and (4) are equivalent. □

Theorem 3.8 *Let* (B,R) *be a reflexive space and* $p \in B$.
(1) (B,R) *is* T_0'.

(2) *The following are equivalent:*

(a) (B,R) *is* $\overline{T_0}$.

(b) R *is antisymmetric.*

(c) (B,R) *is* T_0.

(3) **(a)** $\{p\}$ *is closed iff for each* $a \in B$, *if* aRp *and* pRa, *then* $a = p$.

(b) (B,R) *is* $\overline{T_0}$ *iff* $\{x\}$ *is closed for each* $x \in B$.

(4) *The following are equivalent:*

(a) $A \subset B$ *is closed.*

(b) *For each* $x \in B$ *if there exist* $c,d \in A$ *such that* xRc *and* dRx, *then* $x \in A$.

(c) A *is upward-closed or downward-closed.*

(5) (B,R) *is quasi-sober if and only if* (B,R) *is* T_0' *sober.*

(6) *The following are equivalent:*

(a) (B,R) *is* $\overline{T_0}$ *sober.*

(b) (B,R) *is* T_0 *sober.*

(c) $\{x\}$ *is closed for each* $x \in B$ *and the nonempty proper irreducible closed subsets of* B *are exactly the one-point subsets.*

Proof **(1)** Let (B,R) be a reflexive space, $(B^2 \bigvee_\Delta B^2)'$ be the final lift of

$$q \circ i_1, q \circ i_2 : (B^2, R^2) \to B^2 \vee_\Delta B^2,$$

where R^2 is the product relation on B^2, and R_1 be the initial structure on the wedge $B^2 \bigvee_\Delta B^2$ induced by

$$id : B^2 \vee_\Delta B^2 \to (B^2 \vee_\Delta B^2)' \quad \text{and} \quad \nabla : B^2 \vee_\Delta B^2 \to (B^2, N)$$

where N is discrete relation on B^2. We have to show that R_1 is discrete, i.e. for any points s and t of the wedge $B^2 \bigvee_\Delta B^2$ if sR_1t, then $s = t$. If sR_1t, then, in particular, $\nabla(s) = (a,b) = \nabla(t)$ for some $a,b \in B$, and consequently, s and t have the form. $(a,b)_1$ and $(a,b)_2$. If $s = (a,b)_1$ (resp. $(a,b)_2$) and $t = (a,b)_2$ (resp. $(a,b)_1$), then $i_k((a,b)) = (a,b)_1 = s, i_k((a,b)) = (a,b)_2 = t$ for k=1 or 2, which implies s and t must lie in the same component of the wedge $B^2 \bigvee_\Delta B^2$ which means $s = t$ for all s and t. Thus, R_1 is discrete and by 2.1, (B,R) is T_0'.

(2) $(a) \Rightarrow (b)$: Suppose (B,R) is $\overline{T}_0$ and for any $x,y \in B$, xRy and yRx. We show that $x = y$. Assume $x \neq y$. Let $s = (x,y)_1$ and $t = (x,y)_2$. Note that

$$\pi_1 A(s) R \pi_1 A(t) = xRx,$$

$$\pi_2 A(s) R \pi_2 A(t) = yRx,$$

$$\pi_3 A(s) R \pi_3 A(t) = xRy,$$

and

$$\nabla(s) = (x,y) = \nabla(t)$$

. Since (B,R) is $\overline{T}_0$, by Definition 2.1, $s=t$. Thus, $x=y$, i.e. R is antisymmetric.

$(b)\Rightarrow(c)$: Suppose that R is antisymmetric and A is any subset of B with at least two points. Let $c,d\in A$ with $c\neq d$, $D=\{c,d\}$, and R_D be the initial relation on D induced by the inclusion map $i:D\subset B$. If $R_D=D^2$, the indiscrete reflexive relation on D^2, then, in particular, $cR_Dd=cRd$ and $dR_Dc=dRc$. Since R is antisymmetric, $c=d$, a contradiction. Hence, a reflexive space (B,R) cannot contain an indiscrete subspace with at least two points. Thus, by Definition 2.1, (B,R) is T_0.

$(c)\Rightarrow(a)$: Suppose (B,R) is T_0. Let R^3 is the product relation on B^3 and R_1 be the initial structure on the wedge $B^2\bigvee_\Delta B^2$ induced by

$$A:B^2\vee_\Delta B^2\to(B^3,R^3)\quad\text{and}\quad\nabla:B^2\vee_\Delta B^2\to(B^2,B^2\times B^2).$$

We need to show that R_1 is discrete. If sR_1t for any points s and t of the wedge $B^2\bigvee_\Delta B^2$, then $\pi_1A(s)R\pi_1A(t)$, $\pi_2A(s)R\pi_2A(t)$, $\pi_3A(s)R\pi_3A(t)$, and $\nabla(s)=\nabla(t)$. Since $\nabla(s)=\nabla(t)$, s and t have the form: $(x,y)_1$ and $(x,y)_2$ for some x and y in B.

If $s=(x,y)_1$ (resp. $(x,y)_2$) and $t=(x,y)_2$ ((resp. $(x,y)_1$)), then

$$\pi_1A(s)R\pi_1A(t)=xRx,$$

$$\pi_2A(s)R\pi_2A(t)=yRx\quad(\text{resp.}xRy),$$

$$\pi_3A(s)R\pi_3A(t)=xRy\quad(\text{resp.}\quad yRx),$$

and

$$\nabla(s)=(x,y)=\nabla(t).$$

Since xRy and yRx, $x=y$. Otherwise, $(\{x,y\},\{x,y\}\times\{x,y\})$ is the indiscrete subspace of (B,R), which contradicts to (B,R) is being T_0. Hence, $s=t$ and R_1 is discrete, i.e. (B,R) is $\overline{T_0}$. This completes the proof of Part (2).

(3) (**a**): Suppose $\{p\}$ is closed and for each $a\in B$ aRp and pRa. Note that

$$\nabla_p^\infty(a,p,p,...)=a=\nabla_p^\infty(p,a,p,...),$$

$$\pi_1A_p^\infty(a,p,p,...)R(p,a,p,...)=aRp,$$

$$\pi_2A_p^\infty(a,p,p,...)R(p,a,p,...)=pRa,$$

and

$$\pi_iA_p^\infty(a,p,p,...)R(p,a,p,...)=pRp$$

for all $i\geq 3$, where $\pi_i:B^\infty\to B$ are the projection maps, for all $i\in I$. Since $\{p\}$ is closed, by Definition 2.1, $(a,p,p,...)=(p,a,p,...)$ which implies $a=p$.

Conversely, suppose for each $a\in B$, if aRp and pRa, then $a=p$. Let s and t be any points in $\vee_p^\infty B$ with $\pi_iA_p^\infty(s)R\pi_iA_p^\infty(t)$ for all i, and $\nabla_p^\infty(s)=\nabla_p^\infty(t)$. It follows $s=a_k$ and $t=a_n$ for some k and n. Note that

$$\pi_kA_p^\infty(s)R\pi_kA_p^\infty(t)=aRp,$$

$$\pi_n A_p^\infty(s) R \pi_n A_p^\infty(t) = pRa,$$

and

$$\pi_i A_p^\infty(s) R \pi_i A_p^\infty(t) = pRp$$

for all $i \neq k$. By the assumption, $a = p$, and consequently, $s = a_k = a_n = t$. Thus, by Definition 2.1, $\{p\}$ is closed.

(**b**): Combine Part (2) and Part 3(a).

(4) $(a) \Rightarrow (b)$: Suppose A is closed and for each $x \in B$ $\exists c, d \in A$ such that xRc and dRx. Note that $Q(x)R_1(Q(c) = *)$ and $(* = Q(d))R_1Q(x)$, where R_1 is the final structure on B/A that is induced by $Q : B \to B/A$. Since $\{*\}$ is closed, by (3)(a), $Q(x) = *$ which means $x \in A$.

$(b) \Rightarrow (c)$: Assume (b) holds. If for each $x \in B$ there exists $c \in A$ such that xRc (resp. cRx), then, by assumption, $x \in A$ and so, A is downward-closed (resp. upward-closed) [9, 15].

$(c) \Rightarrow (a)$: Suppose A is upward-closed or downward-closed. Assume yR_1* and $*R_1y$ for some $y \in B/A$. It follows that there exist $x \in B$ and $c, d \in A$ such that xRc and cRx with $Q(x) = y$ and $Q(c) = * = Q(d)$.

If xRc, then $x \in A$ since A is downward-closed. Thus, $y = Q(x) = *$.

If dRx, then $x \in A$ since A is upward-closed. Thus, $y = Q(x) = *$ and by Part (3)(a), $\{*\}$ is closed. Hence, by Definition 2.1, A is closed.

(5) We get the proof from Part (1) and Definition 3.3.

(6) By Part (2), (a) and (b) are equivalent.

$(b) \Rightarrow (c)$: Suppose (B, R) is T_0 sober. In particular, by Parts (2) and (3), $\{x\}$ is closed for each $x \in B$. If $\emptyset \neq A \subset B$ is proper irreducible closed, then $A = \boldsymbol{cl}\{a\}$ for some $a \in B$ since (B, R) is quasi-sober. Since $\{a\}$ is closed, we have $\{a\} = \boldsymbol{cl}\{a\} = A$.

$(c) \Rightarrow (b)$: Suppose that $\{x\}$ is closed for each $x \in B$ and every nonempty proper irreducible closed subset of B is a one-point set. By Parts (2) and (3), (B, R) is T_0.

If $\emptyset \neq A \subset B$ is proper irreducible closed subset of B, then by assumption, $A = \{a\}$ for some $a \in B$. But since $\{a\}$ is closed, $\boldsymbol{cl}(\{a\}) = \{a\} = A$. Hence, by Definition 3.3, (B, R) is quasi-sober and consequently, (B, R) is T_0 sober. □

Let $\mathbf{T_0'}\mathcal{E}$ (resp. $\mathbf{T_0}\mathcal{E}$, $\overline{\mathbf{T}}_\mathbf{0}\mathcal{E}$, $\mathbf{T_1}\mathcal{E}$) be the full subcategory of $\mathcal{E}$ consisting of all T_0' (resp. T_0, $\overline{T}_0$, T_1) objects of $\mathcal{E}$. Let $\mathbf{T_0'Sob}\mathcal{E}$ (resp. $\mathbf{T_0Sob}\mathcal{E}$, $\overline{\mathbf{T}}_\mathbf{0}\mathbf{Sob}\mathcal{E}$) be the full subcategory of $\mathcal{E}$ consisting of all T_0' sober (resp. T_0 sober, $\overline{T}_0$ sober) objects of $\mathcal{E}$.
By Theorem 3.6, we get:

Theorem 3.9 **(1)** *The categories* $\mathbf{T_0'PBorn}$*,* $\overline{\mathbf{T}}_\mathbf{0}\mathbf{PBorn}$*, and* $\mathbf{T_1PBorn}$ *are pairwise isomorphic.*
(2) *The categories* $\mathbf{T_0'SobPBorn}$*,* $\overline{\mathbf{T}}_\mathbf{0}\mathbf{SobPBorn}$ *and* **QSobPBorn** *are pairwise isomorphic, where* **QSobPBorn** *is the full subcategory of* **PBorn** *consisting of all quasi-sober prebornological spaces.*

(3) *The categories* $\mathbf{T_0PBorn}$ *and* $\mathbf{T_0SobPBorn}$ *are isomorphic.*

By Theorem 3.7, we get:

Theorem 3.10 **(1)** *The categories* $\mathbf{T_0'CP}$, $\mathbf{\overline{T}_0CP}$, *and* $\mathbf{T_1CP}$ *are pairwise isomorphic.*

(2) *The categories* $\mathbf{T_0'SobCP}$, $\mathbf{\overline{T}_0SobCP}$, *and* $\mathbf{QSobCP}$ *are pairwise isomorphic, where* $\mathbf{QSobCP}$ *is the full subcategory of* $\mathbf{CP}$ *consisting of all quasi-sober pair spaces.*

(3) *The categories* $\mathbf{T_0CP}$ *and* $\mathbf{T_0SobCP}$ *are isomorphic.*

Theorem 3.11 **(1)** *The categories* $\mathbf{RRel}$ *and* $\mathbf{T_0'RRel}$ *are isomorphic.*

(2) *The categories* $\mathbf{T_0RRel}$ *and* $\mathbf{\overline{T}_0RRel}$ *are isomorphic.*

(3) *The categories* $\mathbf{T_0'SobRRel}$ *and* $\mathbf{QSobRRel}$ *are isomorphic, where* $\mathbf{QSobRRel}$ *is the full subcategory of* $\mathbf{RRel}$ *consisting of all quasi-sober reflexive spaces.*

(4) *The categories* $\mathbf{T_0SobRRel}$ *and* $\mathbf{\overline{T}_0SobRRel}$ *are isomorphic.*

Proof It follows from Theorem 3.8. □

We now state some well-known results [15] in **Top**.

Theorem 3.12 **(1)** *Every* T_2 *topological space is sober.*

(2) *Sobriety is between* T_0 *and* T_2.

(3) *Being* T_1 *and being sober are incomparable properties.*

(4) *Being* T_0 *and being quasi-sober are incomparable properties.*

(5) *Every irreducible and* T_2 *is a one-point space.*

(6) X *is* T_1 *iff* $\{x\}$ *is closed for each* $x \in X$.

(7) *Let* X *be a topological space,* $S(X)$ *be the set of all irreducible closed subsets of* X, *and the map* $f : X \to S(X)$ *be defined as* $f(x) = \boldsymbol{cl}\{x\}$.

(i) X *is sober iff* f *is bijective.*

(ii) X *is* T_0 *iff* f *is injective.*

(iii) X *is quasi-sober iff* f *is surjective.*

Theorem 3.13 *Let* (X, d) *be an extended pseudo-quasi-semi metric space.*

(1) (X, d) *is quasi-sober iff* (X, d) *is* T_0' *sober,*

(2) (X, d) *is* $\overline{T_0}$ *sober iff* $\{x\}$ *is closed for each* $x \in X$ *and the nonempty proper irreducible closed subsets of* X *are exactly the one-point subsets,*

(3) (X, d) *is* T_0 *sober iff* (X, d) *is a quasi-sober and an extended quasi-semi metric space.*

Proof **(1)** By Theorem 3.5 of [14], (X, d) is T_0' and by Definition 3.3, the result follows.

(2) Suppose that (X, d) is $\overline{T_0}$ sober. By Theorem 3.5 of [14] and Theorem 3.2 of [7], $\{x\}$ is closed for each $x \in X$. If A is any nonempty proper irreducible closed subset of X, then by Definition 3.3, $A = cl\{a\}$ for some $a \in X$ and Theorem 3.5 of [14], $cl\{a\} = \{a\}$. Hence, $A = \{a\}$.

Suppose that $\{x\}$ is closed for each $x \in X$ and every nonempty proper irreducible closed subset of X is a one-point set. By Theorem 3.5 of [14] and Theorem 3.2 of [7], (X,d) is $\overline{T_0}$. If A is any nonempty proper irreducible closed subset of X, then by the assumption, $A = \{a\} = cl\{a\}$ for some $a \in X$. Hence, by Definition 3.3, (X,d) is quasi-sober and $\overline{T_0}$ sober.

(3) By Theorem 3.3 of [14], an extended pseudo-quasi-semi metric space (X,d) is T_0 iff d is pseudo. The result follows from this fact and Definition 3.3. □

We can infer the following results:

(1) By Theorems 3.9-3.11, the categories $\mathbf{T_0'PBorn}$, $\mathbf{\overline{T}_0PBorn}$, $\mathbf{T_1PBorn}$, $\mathbf{T_0'CP}$, $\mathbf{\overline{T}_0CP}$, $\mathbf{T_1CP}$, $\mathbf{T_0'RRel}$, $\mathbf{T_0'SobPBorn}$, $\mathbf{\overline{T}_0SobPBorn}$, and **QSobPBorn** have all limits and colimits.

(2) By Theorems 3.3 and 3.25 of [14], if the extended pseudo-quasi-semi metric space (X,d) is T_1, then each subset of X is closed and if (X,d) is a nonempty irreducible T_1, then (X,d) must be a one-point space. Every irreducible and KT_2 may not be a one-point space. For example, let $X = \{a,b\}$ and d be defined as $d(a,a) = 0 = d(b,b), d(a,b) = 2 = d(b,a)$. By Theorem 3.13 of [14], (X,d) is KT_2 and by Theorem 3.19 of [14], (X,d) is irreducible but (X,d) is not a one-point space. This shows Theorem 3.12(5) does not hold in the category **pqsMet** of all extended pseudo-quasi-semi metric spaces and nonexpansive maps. Let $X = \{a,b\}$ and d be defined as $d(a,a) = 0 = d(b,b), d(a,b) = \infty$ and $d(b,a) = 1$. By Theorem 3.4 of [7], both $\{a\}$ and $\{b\}$ are closed but by Theorem 3.3 of [14], (X,d) is not T_1. This shows Theorem 3.12(6) does not hold in **pqsMet**. If (R,d) is the indiscrete extended pseudo-quasi-semi metric space, i.e. $d(a,b) = 0$ for all $a,b \in R$, where R is the set of reel numbers, then by Theorem 3.13 of [14] and Theorem 3.13, (R,d) is both T_0 sober and KT_2 but by Theorem 3.4 of [14], (R,d) is neither $\overline{T}_0$ sober nor $\overline{T}_0$. This shows Theorem 3.12(2) is not valid in **pqsMet**.

(3) By Theorem 3.6, a T_0 sober prebornological space is a quasi-sober, T_0' sober, and $\overline{T}_0$ sober. The prebornological space $(R, P(R))$ is quasi-sober, T_0' sober, and $\overline{T}_0$ sober but it is not T_0 sober, where R is the set of reel numbers. Moreover, By Theorem 3.6, all of quasi-sober prebornological spaces, T_0' sober prebornological spaces, and $\overline{T}_0$ sober prebornological spaces are equivalent. By Theorem 3.6, every T_0 is $\overline{T}_0$ sober but the reverse implication may not be true. This shows Theorem 3.12(2) does not hold in **PBorn**.

(4) By Theorem 3.9 of [2] and Definition 3.1, $\boldsymbol{cl} = \iota$, the trivial closure operator [9] of **PBorn**.

(5) By Theorem 3.7, each of a quasi-sober pair space, T_0' sober pair space, T_0 sober pair space, and $\overline{T}_0$ sober pair space is T_1. The pair space (R,N) is T_1 but it none of quasi-sober, T_0' sober, T_0 sober, and $\overline{T}_0$ sober, where N is the set of natural numbers. Moreover, by Theorem 3.7, T_0 implies T_0' and $\overline{T}_0$ and the pair space (R,N) is T_0' and $\overline{T}_0$ but it is not T_0.

(6) By Theorem 3.8 of [2] and Definition 3.1, $\boldsymbol{cl} = \delta$, the discrete closure operator [9] of **CP**.

(7) By Theorems 3.3 and 3.4 of [14], the indiscrete extended pseudo-quasi-semi metric space (R,d) is both T_0 sober and quasi-sober, (R,d) is neither T_1 nor $\overline{T}_0$. By Theorem 3.7, the pair space (R,N) is both $\overline{T}_0$ and T_1 but it is neither T_0 sober nor quasi-sober. By Theorem 3.6, being T_1 (resp. $\overline{T}_0$) and being $\overline{T}_0$ sober (resp. quasi-sober) are the same. This shows Theorem 3.12(3) and (4) do not hold in **PBorn**.

(8) By Theorem 3.6, a nonempty prebornological space $(X,\mathcal{F})$ is $\overline{T}_0$ but $f : X \to S(X)$ is not injective, where $S(X)$ is the set of all irreducible closed subsets of X. This shows Theorem 3.12(7) does not hold in an arbitrary topological category.

(9) By Theorem 3.8, $\overline{T}_0$ sober=T_0 sober (resp. $\overline{T}_0$=T_0) implies T_0' sober (resp. T_0'). Let B be an infinite set. Then by Theorem 3.8, the indiscrete reflexive space (B, B^2) is T_0' sober and quasi sober but it is neither T_0 sober nor T_0. Moreover, a reflexive space (B, R) is T_0 iff $\{x\}$ is closed for each $x \in B$.

(10) In arbitrary topological category, by Theorem 3.5, every $\overline{T_0}$ sober object is T_0' sober. By Theorem 3.13 and Parts (2) and (3), there is no implication between T_0 sober and $\overline{T}_0$ sober.

References

[1] Baran M. Separation properties. Indian Journal Pure Applied Mathematics 1991; 23 (5): 333-341.

[2] Baran M. The notion of closedness in topological categories. Commentationes Mathematicae Universitatis Carolinae 1993; 34 (2): 383-395.

[3] Baran M, Altindis H. T_0-objects in topological categories, Kuwait Journal of Science 1995; (22): 123-127.

[4] Baran M. T_2-objects in categories of filter and local filter convergence spaces. Turkish Journal of Mathematics 1996; 159-168.

[5] Baran M. Compactness, perfectness, separation, minimality and closedness with respect to closure operators. Applied Categorical Structures 2002; (10): 403-415.

[6] Baran M, Kula S, Baran TM, Qasim M. Closure operators in semi uniform convergence spaces. Filomat 2016; (30): 131-140.

[7] Baran TM. Closedness, separation and connectedness in pseudo-quasi-semi metric spaces. Filomat 2020; (34): 14, 4757-4766. doi: 10.2298/FIL2014757B

[8] Giuli E, Dikranjan D. Closure operators I. Topology and its Applications 1987; (27): 129-143.

[9] Tholen W, Dikranjan D. Categorical structure of closure operators. Kluwer Academic Publishers, Dordrecht, 1995.

[10] Erciyes A, Baran TM, Qasim M. Closure operators constant filter convergence spaces. Konuralp Journal of Mathematics 2020; 185-191.

[11] Erciyes A, Baran TM. T_4, Urysohn lemma, and Tietze extension theorem for constant filter convergence spaces. Turkish Journal of Mathematics 2021; 843-855.

[12] Dieudonné JA, Grothendieck A. Eléments de géométrie algébrique, Springer, New York, 1971 (in French).

[13] Hong SS. Extensive subcategory of the category of T_0-spaces. Canadian Journal Mathematics 1975; (27): 318-331.

[14] Kula M, Baran TM. Separation axioms, Urysohn's lemma and Tietze extension theorem for extended pseudo-quasi-semi metric spaces. Filomat 2022. to appear.

[15] Larrecq JG. Non-Hausdorff Topology and Domain Theory, Cambridge University Press, 2013.

[16] Mielke MV. Convenient categories for internal singular algebraic topology. Illinois Journal of Mathematics 1983.

[17] Marny Th. Rechts-Bikategoriestrukturen in topologischen Kategorien, Dissertation, Freie Universität Berlin, 1973 (in German).

[18] Preuss G. Theory of Topological Structures, An Approach to topological Categories, D. Reidel Publ. Co., Dordrecht, 1988.

[19] Qasim M, Baran M, Abughalwa H. Closure operator in convergence approach spaces. Turkish Journal of Mathematics 2021; 139-152.

[20] Weck-Schwarz S. T_0-objects and separated objects in topological categories. Quaestiones Mathematicae 1991; (14): 315-325.