Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 47 – 58
© TÜBİTAK
doi:10.3906/mat-2102-102

Hyperelastic curves in 3−dimensional lightlike cone

Sümeyra Tuğçe KAĞIZMAN[1], **Ahmet YÜCESAN**[2,*]
[1]Akören Ali Aşık Multi Program Anatolian High School, Konya, Turkey
[2]Department of Mathematics, Süleyman Demirel University, Isparta, Turkey

Received: 22.02.2021 • **Accepted/Published Online:** 17.12.2021 • **Final Version:** 19.01.2022

Abstract: We study hyperelastic curves known as a generalization of elastic curves in 3−dimensional lightlike cone which is a degenerate hypersurface in Minkowski 4−space as critical points of the cone curvature energy functional constructed with the $r-$th power of the cone curvature depending on the given boundary conditions for the natural number $r \geq 2$. We derive the Euler-Lagrange equations for the critical points of this functional that is namely the hyperelastic curves and solve completely the Euler-Lagrange equations by quadratures. Then, we construct Killing vector fields along the hyperelastic curves. Lastly, we give explicitly the hyperelastic curves by integral according to the selected cylindrical coordinate systems in 3−dimensional lightlike cone using these Killing vector fields.

Key words: Hyperelastic curves, Euler-Lagrange equations, lightlike cone.

1. Introduction

If a material turns back into its original form after the force that is applied on the material is removed, that specific material is called an elastic material. If a material has elasticity (so it is an elastic material) it absolutely has an equilibrium position and when it is in the state of equilibrium, it has minimum energy. Due to this situation, studying on elastic materials is an attraction. Elastic curve (elastica) is one of the most popular topics on the calculus of variations in a space, surface or manifold and it is defined as the curve that provides a suitable variation condition for interpolation problems. According to the idealization of D. Bernoulli, the elastic curves minimize the total square curvature $\int_0^\ell \kappa^2 ds$ between same length curves and first-order boundary data. Geodesics on a plane or a sphere and eight curve in Euclidean space are clear examples of elastic curves (see, [5, 6, 13, 22]). Like many problems investigated by mathematicians of today, studying elastic curves originated in a physical situation. Euler and Bernoulli's elastic curve idea has turned this physical problem into a mathematical one. Study of the bending energy of a physical elastic rod was modified by investigating the total square curvature of a regular curve. Total square curvature functional has appeared as a beneficial quantity that will be useful in geodesic research and closed thin elastic rod is generally used as a model for DNA molecule [8, 23]. So far, many researchers have written articles on elastic curves in Euclidean and non-Euclidean spaces. Singer has obtained the Euler-Lagrange equations of elastic curves in 3−dimensional Euclidean space and solved differential equations by using Jacobi elliptic functions. Moreover, he generalized the variational problem of elastic curves in Riemann manifold with constant sectional curvature [22]. Langer and Singer have

*Correspondence: ahmetyucesan@sdu.edu.tr
2010 *AMS Mathematics Subject Classification:* 53C50, 49Q20, 53Z05

studied closed elastic curves on manifolds with constant sectional curvature. In addition, Brunnet and Crouch have given equations for the main invariants of spherical elastic curves. They solved differential equations by using the Jacobi elliptic functions and presented a classification of the basic forms of elastic curves [5]. The system of differential equations for elastic curves has been studied and developed by many geometricians in non-Euclidean spaces.

Although Minkowski geometry and Euclidean geometry have some parallel features, different results can be found for curves or surfaces in Minkowski geometry due to the metric structure. Moreover, Minkowski geometry is more complex than Euclidean geometry because while Euclidean geometry includes only spherical geometry, Minkowski geometry includes both spherical and hyperbolic geometry. In addition, there are three curves called spacelike, timelike and lightlike (null) and two surfaces called degenerate and nondegenerate in Minkowski space [18, 19]. $n-$dimensional lightlike cone is a degenerate hypersurface that includes all lightlike (null) vectors in Minkowski $(n+1)-$space and the lightlike cone only has spacelike curves [14, 19]. In Minkowski $3-$space, elastic curves on hyperquadratics (de Sitter $2-$space and pseudo-hyperbolic $2-$space) and $2-$dimensional lightlike cone have been studied previously [20, 24, 27]. A complete description of timelike relativistic elastica, nongeodesic spacetime curves that solve the Euler-Lagrange equations for a Lagrangian that depends on the square of the acceleration of the curve as well as its Lorentzian length have been given [7]. Elastic curves and the generalization of elastic curves have been studied both in Euclidean and non-Euclidean spaces [8–11, 21, 25]. One of these generalizations is hyperelastic curves (or $r-$elastic curves). Hyperelastic curves have been defined as critical points of the functional $\int_0^\ell (\kappa^r+\lambda)ds$ for the natural number $r \geq 2$ [1–3]. If $\lambda = 0$, then critical points of this functional are called free hyperelastic curves (or free $r-$elastic curves). Free hyperelastic curves are used in the reduction method in construction of Chen-Willmore submanifolds [1–4]. In Minkowski $4-$space, elastic curves in $3-$dimensional lightlike cone have been studied by the second author [28], but hyperelastic curves have not been studied. Therefore, it has become attraction to work on the hyperelastic curves in $3-$dimensional lightlike cone which is a special hypersurface. With this motive, hyperelastic curves in $3-$dimensional lightlike cone are characterized. To this end, we derive the Euler-Lagrange equations for the hyperelastic curves and solve completely the Euler-Lagrange equations by quadratures. Finally, we give explicitly the hyperelastic curves by integral according to the selected cylindrical coordinate systems in $3-$dimensional lightlike cone with the help of Killing vector fields constructed along the hyperelastic curves.

2. Hyperelastic curves in $3-$dimensional lightlike cone

In this section, we derive a system of differential equations that characterize hyperelastic curves in $3-$dimensional lightlike cone given by

$$\boldsymbol{Q}^3 = \{p \in \mathbb{R}^4_1 \setminus \{0\} :< p, p >= 0\}$$

in Minkowski $4-$space $\mathbb{R}^4_1$ and solve this system of differential equations. Therefore, we describe hyperelastic curves in $3-$dimensional lightlike cone $\boldsymbol{Q}^3$. When studying on curves in Euclidean and Minkowski spaces, Frenet frame is usually used. However, Frenet frame is not suitable when studying on curves in $\boldsymbol{Q}^3$. In this situation, a different frame known as cone Frenet frame (or asymptotic orthonormal frame) defined by Liu is used. We assume that

$$\begin{array}{rccc} \gamma: & I \subset \mathbb{R} & \longrightarrow & \boldsymbol{Q}^3 \subset \mathbb{R}^4_1, \\ & s & \longrightarrow & \gamma(s) \end{array}$$

is a spacelike curve parametrized by the arc length parameter s in $\boldsymbol{Q}^3$. Then, the curve γ has a spacelike unit tangent vector

$$T(s) = \gamma'(s) = \frac{d\gamma(s)}{ds}.$$

Then we can select null normal vector field

$$N(s) = -\gamma^{''}(s) - \frac{1}{2} < \gamma^{''}(s), \gamma^{''}(s) > \gamma(s)$$

satisfying the following conditions

$$\begin{array}{l} \langle\gamma(s),\gamma(s)\rangle = \langle N(s),N(s)\rangle = \langle\gamma(s),T(s)\rangle = \langle\gamma(s),B(s)\rangle = 0 \\ \langle T(s),N(s)\rangle = \langle N(s),B(s)\rangle = \langle T(s),B(s)\rangle = 0 \\ \langle\gamma(s),N(s)\rangle = \langle T(s),T(s)\rangle = \langle B(s),B(s)\rangle = 1. \end{array}$$

Therefore, the frame field $\{\gamma, T, N, B\}$ is called cone Frenet frame (or an asymptotic orthonormal frame) along the curve γ in $\boldsymbol{Q}^3$ and it has derivative equations given by

$$\begin{array}{ll} \gamma'(s) & = T(s) \\ T^{'}(s) & = \kappa(s)\gamma(s) - N(s) \\ N'(s) & = -\kappa(s)T(s) - \tau(s)B(s) \\ B'(s) & = \tau(s)\gamma(s), \end{array} \tag{2.1}$$

where

$$\kappa(s) = -\frac{1}{2}\langle\gamma''(s),\gamma''(s)\rangle$$

and

$$\tau(s) = \sqrt{\langle\gamma'''(s),\gamma'''(s)\rangle - \langle\gamma''(s),\gamma''(s)\rangle^2},$$

are the (first) cone curvature and the cone torsion (or second cone curvature) of the curve γ, respectively [14–17].

Since the hyperelastic curve problem is a variational problem, it is natural to ask how the variation is defined. We suppose that $\gamma : I \to \boldsymbol{Q}^3$ is a spacelike curve parametrized by an arbitrary parameter t in $\boldsymbol{Q}^3$. $V = V(t)$, T and v denote tangent vector, unit tangent vector to the curve γ and the speed of γ, respectively. Here the speed of γ is given by

$$v(t) = \|V(t)\| = \left|< \gamma^{'}(t), \gamma^{'}(t) >\right|^{\frac{1}{2}}.$$

A variation of the curve γ is defined by

$$\begin{array}{llll} \gamma : & (-\varepsilon,\varepsilon)\times I & \longrightarrow & \boldsymbol{Q}^3 \subset \mathbb{R}^4_1 \\ & (w,t) & \longrightarrow & \gamma(w,t) = \gamma_w(t) \end{array}$$

such that $\gamma(0,t) = \gamma(t)$ and the variation vector field associated with variation is

$$W = W(t) = (\partial\gamma/\partial w)(0,t)$$

along γ. Hence we can write $V = V(w,t) = (\partial\gamma/\partial t)(w,t)$, $W = W(w,t)$, $T = T(w,t)$ and $v = v(w,t)$. Considering the arc length parameter as s, we can write $\gamma(s)$, $\kappa(w,s)$, etc., where $s \in [0,\ell]$ and ℓ are the arc lengths of γ.

Now, we give a lemma that will be used to obtain Euler-Lagrange equations for hyperelastic curves with the equations defined above.

Lemma 2.1 *Let γ be a spacelike curve in $\boldsymbol{Q}^3$ and $\gamma(w,t)$ be a variation with a variation vector field $W = W(w,t)$. Then, we have the following formulas:*
i) $< W(w,t), \gamma(w,t) >= 0,$
ii) $[V,W] = 0,$
iii) $g = - < \nabla_T W, T >$ *such that* $W(v) = -gv,$
iv) $[W,T] = gT,$
v) $W(\kappa) = 4g\kappa - < \nabla_T^2 W, \kappa\gamma - N >,$
vi) $W\left(\tau^2\right) = 6g\tau^2 - 8g\kappa^2 + < \nabla_T W, 2\kappa\kappa'\gamma - 8\kappa^2 T - 6\kappa' N - 4\kappa\tau B >$
$+ < \nabla_T^2 W, 6\kappa' T > + < \nabla_T^3 W, 2\kappa'\gamma + 2\tau B >.$

Proof *i)* Since $\gamma(w,t)$ is the curve in $\boldsymbol{Q}^3$, we have

$$< \gamma(w,t), \gamma(w,t) >= 0.$$

If we take the partial derivative of both sides with respect to w, we obtain

$$< W(w,t), \gamma(w,t) >= 0.$$

ii) For a variation $\gamma(w,t)$**,** we use the equation

$$[\frac{\partial}{\partial w}, \frac{\partial}{\partial t}] = \frac{\partial}{\partial w}(\frac{\partial}{\partial t}) - \frac{\partial}{\partial t}(\frac{\partial}{\partial w}) = 0$$

and the tangent map $d\gamma$, we find

$$[V(w,t), W(w,t)] \quad = 0$$

iii) Substituting $V(w,t) = v(w,t)T(w,t)$ into

$$[W,V] = W(V) - V(W) = 0,$$

we get

$$W(vT) - vT(W) = W(v)T + vW(T) - v\nabla_T W = 0. \tag{2.2}$$

Both sides of (2.2) are scalar multiplied by T and by using $< T,T >= 1$, we calculate

$$W(v) < T,T > +v < W(T), T > -v < \nabla_T W, T >= 0.$$

Since $< W(T), T >= 0$, we get

$$W(v) = -gv,$$

where $g = - < \nabla_T W, T >$.

$iv)$ From (2.2), we have

$$\begin{aligned} 0 &= W(v)T + vW(T) - v\nabla_T W \\ &= W(v)T + v[W,T]. \end{aligned}$$

So, we obtain

$$[W,T] = gT. \tag{2.3}$$

$v)$ Let γ be a spacelike curve parametrized by the arc length parameter s without breaking generality. Take into consideration definition of the cone curvature and by using the equations (2.1) and (2.3), we obtain

$$\begin{aligned} W(\kappa) &= -\tfrac{1}{2}W(<\nabla_T T, \nabla_T T>) \\ &= 4g\kappa - <\nabla_T\nabla_T W, \kappa(s)\gamma(s) - N> . \end{aligned}$$

$vi)$ Considering the definitions of the cone torsion and the cone curvature, and by using equations $iv)$ and $v)$, we have

$$\begin{aligned} W\left(\tau^2\right) &= W\left(\langle TTT, TTT\rangle\right) - 4W(\kappa^2) \\ &= 2 < W(TTT), TTT\rangle - 32g\kappa^2 + 8\kappa < \nabla_T\nabla_T W, \kappa(s)\gamma(s) - N > . \end{aligned}$$

Since

$$\begin{aligned} \langle W(TTT), TTT\rangle &= 6g\tau^2 + 24g\kappa^2 + < \nabla_T W, 2\kappa^{'}\kappa\gamma - 8\kappa^2 T - 6\kappa^{'}N - 4\kappa\tau B > \\ &\quad + < \nabla_T\nabla_T W, -8\kappa^2\gamma + 6\kappa^{'}T + 8\kappa N > \\ &\quad + < \nabla_T\nabla_T\nabla_T W, 2\kappa^{'}\gamma + 2\tau B >, \end{aligned}$$

we see that

$$\begin{aligned} W(\tau^2) &= 6g\tau^2 - 8g\kappa^2 + < \nabla_T W, 2\kappa\kappa^{'}\gamma - 8\kappa^2 T - 6\kappa^{'}N - 4\kappa\tau B > \\ &\quad + < \nabla_T\nabla_T W, 6\kappa^{'}T > + < \nabla_T\nabla_T\nabla_T W, 2\kappa^{'}\gamma + 2\tau B > . \end{aligned}$$

□

2.1. Euler-Lagrange equations for hyperelastic curves

In this subsection we derive the Euler-Lagrange equations characterizing the critical points of the cone curvature energy functional under some boundary conditions.

Let $\gamma : [0, \ell] \to \boldsymbol{Q}^3$ be a spacelike curve parametrized by arc length s, $0 < s < \ell$, with the cone curvature function $\kappa(s)$ in $\boldsymbol{Q}^3$. A hyperelastic curve γ is an extremal of the cone curvature energy functional defined by

$$\mathcal{F}(\gamma_w) = \int_0^\ell (\kappa^r + \lambda) ds$$

which stands in the family of all curves having the same initial point and initial direction as γ, where $r \geq 2$ is a natural number and ℓ is the arclength of $\gamma_w(t) = \gamma(w,t)$.

Assume that γ is a critical point of the functional $F(\gamma_w)$. Then for a variation γ_w associated with a variation vector field W along γ, we compute

$$\begin{aligned} \frac{d}{dw}\mathcal{F}(\gamma_w)|_{w=0} &= \frac{d}{dw}\int_0^\ell (\kappa^r + \lambda) ds|_{w=0} \\ &= \frac{d}{dw}\int_I (\kappa^r + \lambda) v dt|_{w=0} \\ &= \int_I [r\kappa^{r-1}W(\kappa) + (\kappa^r + \lambda)W(v)]dt|_{w=0} \\ &= \int_0^\ell \left(r\kappa^{r-1}\left(-4 < \nabla_T W, T > \kappa - < \nabla_T\nabla_T W, \kappa\gamma - N >\right)\right. \\ &\quad \left. + (\kappa^r + \lambda) < \nabla_T W, T >\right) ds. \end{aligned}$$

By using a standard argument involving some integrations by parts, we get the first variation formula of the cone curvature energy functional as follows

$$0 = \int_0^\ell < \mathcal{E}, W > ds + \mathcal{B}(\gamma, W)|_0^\ell ,$$

where the boundary term is given by

$$\begin{aligned}\mathcal{B}(\gamma, W) =& < W, ((1-2r)\kappa^r + \lambda)T + r(r-1)\kappa^{r-2}\kappa^{'} N + r\kappa^{r-1}\tau B > \\ &+ < \nabla_T W, r\kappa^{r-1}(\kappa\gamma - N) >\end{aligned}$$

and

$$\begin{aligned}\mathcal{E} =& (r(r-1)\kappa^{r-3}(\kappa\kappa^{''} + (r-2)(\kappa^{'})^2) + (1-2r)\kappa^r + \lambda)N \\ &- r\kappa^{r-2}(2(r-1)\kappa^{'}\tau + \kappa\tau^{'})B.\end{aligned}$$

Then we obtain the Euler-Lagrange equations

$$(r(r-1)\kappa^{r-3}(\kappa\kappa^{''} + (r-2)(\kappa^{'})^2) + (1-2r)\kappa^r + \lambda) = 0, \tag{2.4}$$

$$r\kappa^{r-2}(2(r-1)\kappa^{'}\tau + \kappa\tau^{'}) = 0. \tag{2.5}$$

Therefore, we define hyperelastic curve in $\boldsymbol{Q}^3$ as follows:

Definition 2.2 *A regular unit speed spacelike curve in $\boldsymbol{Q}^3$ is called a hyperelastic curve if it satisfies the Euler-Lagrange equations (2.4) and (2.5).*

2.2. Killing vector fields along hyperelastic curves

The purpose of this subsection is to find out explicit solutions of a hyperelastic curve in $\mathbf{Q}^3$. As it can be seen in the equation (2.5), the cone torsion τ is a function of the cone curvature κ. Therefore, solutions will be searched according to the cone curvature κ.

Firstly, let the cone curvature κ be a constant. From (2.5) we known the cone torsion τ is a constant too. So, (2.1) becomes a system of linear ordinary differential equations with constant coefficients and this system of equations can be solved directly.

Now, suppose that the cone curvature κ is not constant and $\kappa^{r-1} \neq 0$. From (2.5) we get

$$(r\kappa^{r-1})^2\tau = C_1, \tag{2.6}$$

where C_1 is a constant. Furthermore we assume that $\kappa^{r-2}\kappa^{'} \neq 0$. Both sides of the equation (2.4) are multiplied by $r(r-1)\kappa^{r-2}\kappa^{'}$ and if the resulting equation integrated, then

$$\left(r(r-1)\kappa^{r-2}\kappa^{'}\right)^2 + 2r\kappa^{r-1}(\kappa^r + \lambda) - 2r^2\kappa^{2r-1} = C_2 \tag{2.7}$$

is obtained, where C_2 is a constant. Hence, we can express the cone curvature $\kappa(s)$ by quadratures

$$\pm \int \frac{r(r-1)\kappa^{r-2}d\kappa}{\sqrt{C_2 - 2r\kappa^{r-1}(\kappa^r + \lambda) + 2r^2\kappa^{2r-1}}} = \int ds.$$

Thus, we have the following theorem.

Theorem 2.3 *The Euler-Lagrange equations (2.4) and (2.5) in $\boldsymbol{Q}^3$ can be completely solved by quadratures.*

In order to solve the derivative equations of cone Frenet frame (2.1) for a hyperelastic curve in $\boldsymbol{Q}^3$, we define the concept of Killing vector field along the curve in $\boldsymbol{Q}^3$. This concept was introduced by Langer and Singer [13] for the purpose of integrating the structural equations of the elastic curves by quadratures in a system of cylindrical coordinates.

Definition 2.4 *Let γ be a regular curve in $\boldsymbol{Q}^3$. If a vector field W along γ annihilates v, κ and τ, that is $W(v) = W(\kappa) = W(\tau) = 0$, then it is called a Killing.*

We suppose that W is Killing vector field along hyperelastic curve γ having the form

$$W = f_1(s)\gamma(s) + f_2(s)T(s) + f_3(s)N(s) + f_4(s)B(s).$$

From Lemma 2.1, the functions $f_1(s)$, $f_2(s)$, $f_3(s)$ and $f_4(s)$ must satisfy the following equations:

$$f_3(s) = 0, \quad f_1(s) + f_2'(s) = 0,$$

$$f_2'''(s) - 2\kappa(s)f_2'(s) - \kappa'(s)f_2(s) - 2\tau(s)f_4'(s) - \tau'(s)f_4(s) = 0$$

and

$$f_4'''(s) - 2\kappa(s)f_4'(s) - \kappa'(s)f_4(s) + 2\tau(s)f_2'(s) + \tau'(s)f_2(s) = 0.$$

Taking into consideration these equations and the Euler-Lagrange equations (2.4) and (2.5), we obtain Killing vector fields

$$P_1 = -r(r-1)\kappa^{r-2}\kappa'\gamma + r\kappa^{r-1}T$$

and

$$P_2 = r\kappa^{r-1}B$$

along hyperelastic curve γ. The equations $W(v) = W(\kappa) = W(\tau) = 0$ constitute a linear system whose solution space is $6-$dimension. This dimension agrees with the dimension of the isometry group of $\boldsymbol{Q}^3$. Thus a Killing vector field along hyperelastic curve γ can extend to a Killing vector field in $\boldsymbol{Q}^3$ at least locally.

The hyperelastic curve is invariant along any Killing vector field W, that is $\mathcal{B}(\gamma, W)$ is constant along γ. Also,

$$< P_1, P_1 >=< P_2, P_2 >= r^2\kappa^{2r-2}$$

and

$$< P_1, P_2 >= 0.$$

So, the following theorem gives a characterization of hyperelastic curves in $\boldsymbol{Q}^3$:

Theorem 2.5 *If γ is a hyperelastic curve in $\boldsymbol{Q}^3$, then γ satisfies the first integral equation*

$$-(r(r-1)\kappa^{r-2}\kappa')^2 - 2r\kappa^{r-1}(\kappa + \lambda) + 2\kappa < P_i, P_i >= -C_2$$

for $i \in \{1, 2\}$, where C_2 is a constant.

2.3. Cylindrical coordinates and representation by means of quadratures

We use the Killing fields P_1 and P_2 to construct a system of cylindrical coordinates in $\boldsymbol{Q}^3$ whether C_2 is different from zero and not. Then, we express the hyperelastic curves in $\boldsymbol{Q}^3$ by means of quadratures.

Case 1. $C_2 \neq 0$. We consider a system of cylindrical coordinates in $\boldsymbol{Q}^3$ given by

$$X(\rho,\theta,\varphi) = (\sin\rho\cos\theta, \sin\rho\sin\theta, \sinh\varphi\sin\rho, \cosh\varphi\sin\rho)$$

for $\rho \in (0, \frac{\pi}{2})$, $\theta \in (0, 2\pi)$ and $\varphi \in \mathbb{R}$. Then metric is

$$ds^2 = \sin^2\rho d\theta^2 + \sin^2\rho d\varphi^2.$$

The vector fields $X_\theta(\rho,\theta,\varphi)$ and $X_\varphi(\rho,\theta,\varphi)$ are Killing vector fields. By using a suitable Lorentzian transformation in $\boldsymbol{Q}^3$, we may assume

$$P_1 = aX_\theta + bX_\varphi, \quad P_2 = bX_\theta - aX_\varphi$$

for different constants a and b. We get

$$X_\theta = \frac{1}{a^2+b^2}(aP_1 + bP_2), \quad X_\varphi = \frac{1}{a^2+b^2}(bP_1 - aP_2).$$

Since

$$< P_1, P_1 >= r^2\kappa^{2r-2} = (a^2+b^2)\sin^2\rho,$$

we find

$$\sin\rho = \frac{r\kappa^{r-1}}{\sqrt{a^2+b^2}}. \tag{2.8}$$

If we can express the curve in terms of the local coordinates as $\gamma(s) = X(\rho(s), \theta(s), \varphi(s))$, then

$$T(s) = \gamma'(s) = \rho^{'}(s)X_\rho + \theta^{'}(s)X_\theta + \varphi^{'}(s)X_\varphi.$$

The inner product of P_i and T for $i \in \{1,2\}$ and the unit property of $T(s)$, we have

$$\theta^{'}(s) = \frac{a}{r\kappa^{r-1}}, \quad \varphi^{'}(s) = \frac{b}{r\kappa^{r-1}}. \tag{2.9}$$

Calculating the cone curvature of $\gamma(s) = X(\rho(s), \theta(s), \varphi(s))$ gives

$$\begin{aligned}\kappa \; &= \{((\theta^{'})^2 + (\varphi^{'})^2)\rho^{''} - 2(\theta^{'}\theta^{''} + \varphi^{'}\varphi^{''})\rho^{'}\}\cos\rho\sin\rho \\ &-(\rho^{'})^2((\theta^{'})^2 + (\varphi^{'})^2)\cos^2\rho - \tfrac{1}{2}\{(\theta^{''})^2 + (\varphi^{''})^2 + (\theta^{'})^4 - (\varphi^{'})^4\}\sin^2\rho \\ &-(\rho^{'})^2((\theta^{'})^2 + (\varphi^{'})^2).\end{aligned}$$

From (2.4), (2.7), (2.8), (2.9) and

$$\begin{aligned}&\theta^{''}(s) = \tfrac{-a(r-1)\kappa^{'}}{r\kappa^r}, \quad \varphi^{''}(s) = \tfrac{-b(r-1)\kappa^{'}}{r\kappa^r}, \\ &\rho^{'}(s) = \tfrac{r(r-1)\kappa^{r-2}\kappa^{'}}{\sqrt{a^2+b^2-r^2\kappa^{2r-2}}}, \\ &\rho^{''}(s) = \tfrac{(a^2+b^2-r^2\kappa^{2r-2})r(r-1)\kappa^{r-3}[(r-2)(\kappa^{'})^2+\kappa\kappa^{''}]+r^3(r-1)^2\kappa^{3r-5}(\kappa^{'})^2}{(a^2+b^2-r^2\kappa^{2r-2})^{3/2}},\end{aligned} \tag{2.10}$$

we obtain

$$C_2 = b^2 - a^2 \neq 0. \tag{2.11}$$

By calculating the cone torsion of $\gamma(s) = X(\rho(s), \theta(s), \varphi(s))$, then we find

$$\begin{aligned}
\tau^2 + 4\kappa^2 \quad &= \{-6\rho^{'''}(\theta^{'}\theta^{''} + \varphi^{'}\varphi^{''}) + 6\theta^{'''}(\theta^{'}\rho^{''} + \theta^{''}\rho^{'}) + 6\varphi^{'''}(\varphi^{'}\rho^{''} + \varphi^{''}\rho^{'}) \\
&+18(\rho^{'})^2\rho^{''}((\theta^{'})^2 + (\varphi^{'})^2) + 6\theta^{'}\theta^{''}(3(\theta^{'})^2\rho^{'} + (\rho^{'})^3) \\
&+6(\rho^{''}\theta^{'} + \theta^{''}\rho^{'})(3(\rho^{'})^2\theta^{'} + (\theta^{'})^3) - 6(\rho^{''}\varphi^{'} + \varphi^{''}\rho^{'})(3(\rho^{'})^2\varphi^{'} - (\varphi^{'})^3) \\
&-6\varphi^{'}\varphi^{''}(3(\varphi^{'})^2\rho^{'} - (\rho^{'})^3)\}\sin\rho\cos\rho + \{-6\rho^{'''}\rho^{'}((\theta^{'})^2 + (\varphi^{'})^2) \\
&+9(\theta^{'}\rho^{''} + \theta^{''}\rho^{'})^2 + 9(\varphi^{'}\rho^{''} + \varphi^{''}\rho^{'})^2 + 9(\rho^{'})^2((\theta^{'})^4 - (\varphi^{'})^4) \\
&+6(\rho^{'})^4((\theta^{'})^2 + (\varphi^{'})^2)\}\cos^2\rho + \{(\theta^{'''})^2 + (\varphi^{'''})^2 - 2\theta^{'}\theta^{'''}(3(\rho^{'})^2 + (\theta^{'})^2) \\
&-2\varphi^{'}\varphi^{'''}(3(\rho^{'})^2 - (\varphi^{'})^2) + 9(\theta^{'})^2(\theta^{''})^2 - 9(\varphi^{'})^2(\varphi^{''})^2 \\
&+9(\rho^{'})^4((\theta^{'})^2 + (\varphi^{'})^2) + 18\rho^{'}\rho^{''}(\theta^{'}\theta^{''} + \varphi^{'}\varphi^{''}) + 6(\rho^{'})^2((\theta^{'})^4 - (\varphi^{'})^4) \\
&+(\theta^{'})^6 + (\varphi^{'})^6\}\sin^2\rho.
\end{aligned}$$

Here (2.4), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11) and

$$\begin{aligned}
\theta^{'''}(s) &= \tfrac{a}{r\kappa^{r+1}}\{2(r-1)^2(\kappa^{'})^2 - (r-1)((r-2)(\kappa^{'})^2 + \kappa\kappa^{''})\}, \\
\varphi^{'''}(s) &= \tfrac{b}{r\kappa^{r+1}}\{2(r-1)^2(\kappa^{'})^2 - (r-1)((r-2)(\kappa^{'})^2 + \kappa\kappa^{''})\}, \\
\rho^{'''}(s) &= \tfrac{r(r-1)\kappa^{r-4}[(r-2)\kappa^{'}\{(r-3)(\kappa^{'})^2+3\kappa\kappa^{''}\}+\kappa^2\kappa^{'''}]}{\sqrt{a^2+b^2-r^2\kappa^{2r-2}}} + \tfrac{3r^5(r-1)^3\kappa^{5r-8}(\kappa^{'})^3}{(a^2+b^2-r^2\kappa^{2r-2})^{5/2}} \\
&\quad + \tfrac{r^3(r-1)^2\kappa^{2r-3}(\kappa^{'})\{(r-1)\kappa^{r-3}\kappa^{'}+3(r-2)\kappa^{r-3}(\kappa^{'})^2+\kappa^{r-2}\kappa^{''}\}}{(a^2+b^2-r^2\kappa^{2r-2})^{3/2}}
\end{aligned}$$

are used,

$$C_1^2 = a^2b^2 > 0 \tag{2.12}$$

is obtained. Combining (2.11) and (2.12), we get

$$a^2 = \frac{\sqrt{C_2^2 + 4C_1^2} - C_2}{2}, \; b^2 = \frac{2C_1}{\sqrt{C_2^2 + 4C_1^2} - C_2}.$$

Therefore, the hyperelastic curve $\gamma(s) = X(\rho(s), \theta(s), \varphi(s))$ can be represented in the cylindrical coordinates around plane spanned by P_1 and P_2 as follows:

$$\begin{aligned}
\rho^{'}(s) &= \tfrac{r(r-1)\kappa^{r-2}\kappa^{'}}{\sqrt{\tfrac{\sqrt{C_2^2+4C_1^2}-C_2}{2} + \tfrac{2C_1}{\sqrt{C_2^2+4C_1^2}-C_2} - r^2\kappa^{2r-2}}}, \qquad \theta^{'}(s) = \tfrac{\sqrt{\sqrt{C_2^2+4C_1^2}-C_2}}{\sqrt{2}r\kappa^{r-1}}, \\
\varphi^{'}(s) &= \tfrac{\sqrt{2C_1}}{r\kappa^{r-1}\sqrt{\sqrt{C_2^2+4C_1^2}-C_2}}.
\end{aligned}$$

Case 2. $C_2 = 0$. We choose a system of cylindrical coordinates in $\boldsymbol{Q}^3$ given by

$$X(\rho, \theta, \varphi) = (\rho\varphi, \rho\theta, \frac{\rho\varphi^2}{2} - \frac{\rho}{2}(1 - \theta^2), \frac{\rho\varphi^2}{2} + \frac{\rho}{2}(1 + \theta^2))$$

for $\rho > 0$ and $\theta, \varphi \in \mathbb{R}$. The metric is

$$ds^2 = \rho^2 d\theta^2 + \rho^2 d\varphi^2.$$

Then the vector fields $X_\theta(\rho,\theta,\varphi)=(0,\rho,\rho\theta,\rho\theta)$ and $X_\varphi(\rho,\theta,\varphi)=(\rho,0,\rho\varphi,\rho\varphi)$ are Killing vector fields. By using a suitable Lorentzian transformation in $\boldsymbol{Q}^3$, we may assume

$$P_1 = X_\theta + X_\varphi, \qquad P_2 = X_\theta - X_\varphi.$$

Then we have

$$X_\theta = \frac{1}{2}(P_1+P_2), \quad X_\varphi = \frac{1}{2}(P_1-P_2).$$

Since $<P_1,P_1>= r^2\kappa^{2r-2} = 2\rho^2$, we obtain

$$\rho = \frac{r}{\sqrt{2}}\kappa^{r-1}. \tag{2.13}$$

We can express the curve in terms of the local coordinates as $\gamma(s) = X(\rho(s),\theta(s),\varphi(s))$, then

$$T(s) = \gamma'(s) = \rho^{'}(s)X_\rho + \theta^{'}(s)X_\theta + \varphi^{'}(s)X_\varphi.$$

The inner product of P_i and T for $i\in\{1,2\}$ gives

$$\theta^{'}(s)+\varphi^{'}(s) = \frac{2}{r\kappa^{r-1}}$$

and

$$\theta^{'}(s)-\varphi^{'}(s)=0.$$

The unit property of $T(s)$ yields

$$\theta^{'}(s)^2+\varphi^{'}(s)^2 = \frac{2}{r^2\kappa^{2r-2}}.$$

Then

$$\theta^{'}(s)=\varphi^{'}(s)=\frac{1}{r\kappa^{r-1}}. \tag{2.14}$$

Calculating the cone curvature of $\gamma(s) = X(\rho(s),\theta(s),\varphi(s))$ gives

$$\begin{aligned}\kappa = {}& \rho\rho^{''}((\theta^{'})^2+(\varphi^{'})^2) - \tfrac{1}{2}\rho^2((\theta^{''})^2+(\varphi^{''})^2) - 2\rho\rho^{'}(\theta^{'}\theta^{''}+\varphi^{'}\varphi^{''})\\ & - 2(\rho^{'})^2((\theta^{'})^2+(\varphi^{'})^2).\end{aligned}$$

By using (2.4), (2.7), (2.13), (2.14) and

$$\begin{aligned}\theta^{''}(s) &= \varphi^{''}(s) = \tfrac{-(r-1)\kappa^{'}}{r\kappa^r},\\ \rho^{'}(s) &= \tfrac{r(r-1)\kappa^{r-2}\kappa^{'}}{\sqrt{2}},\\ \rho^{''}(s) &= \tfrac{r(r-1)\kappa^{r-3}[(r-2)(\kappa^{'})^2+\kappa\kappa^{''}]}{\sqrt{2}},\end{aligned} \tag{2.15}$$

we get $C_2=0$. Furthermore, by calculating the cone torsion of $\gamma(s) = X(\rho(s),\theta(s),\varphi(s))$, we get

$$\begin{aligned}\tau^2+4\kappa^2 \quad &= -6\rho\rho^{'''}(\theta^{'}\theta^{''}+\varphi^{'}\varphi^{''}) - 6\rho^{'}\rho^{'''}((\theta^{'})^2+(\varphi^{'})^2) + \rho^2((\theta^{'''})^2+(\varphi^{'''})^2)\\ &\quad +6\rho\theta^{'''}(\rho^{''}\theta^{'}+\rho^{'}\theta^{''}) + 6\rho\varphi^{'''}(\rho^{''}\varphi^{'}+\rho^{'}\varphi^{''}) + 9(\rho^{''}\theta^{'}+\rho^{'}\theta^{''})^2\\ &\quad +9(\rho^{''}\varphi^{'}+\rho^{'}\varphi^{''})^2.\end{aligned}$$

From (2.4), (2.6), (2.7), (2.13), (2.14), (2.15),

$$\theta^{'''}(s) = \varphi^{'''}(s) = \tfrac{-(r-1)}{r\kappa^{r+1}}(\kappa\kappa^{''} - r(\kappa^{'})^2),$$

and

$$\rho^{'''}(s) = \tfrac{r(r-1)}{\sqrt{2}}\kappa^{r-4}[(r-2)(r-3)(\kappa^{'})^3 + 2(r-1)\kappa\kappa^{'}\kappa^{''} + \kappa^2\kappa^{'''}],$$

we obtain $C_1 = 0$, that is $\tau = 0$. Taking into consideration (2.13) and (2.14), the hyperelastic curve $\gamma(s) = X(\rho(s), \theta(s), \varphi(s))$ can be described in the cylindrical coordinates around plane spanned by P_1 and P_2.

Hence, we can give the following theorem:

Theorem 2.6 *A hyperelastic curve in* $\boldsymbol{Q}^3$ *can be expressed by integral explicitly.*

References

[1] Arroyo, J, Garay, OJ, Barros, M. Closed free hyperelastic curves in the hyperbolic plane and Chen-Willmore rotational hypersurfaces. Israel Journal of Mathematics 2003; 138: 171-187.

[2] Arroyo, J, Garay, OJ, Mencia, JJ. Closed free hyperelastic curves in real space forms. In: Proceeding of the XII Fall Workshop on Geometry and Physics; Coimbra, Portugal; 2003. pp. 1-13.

[3] Arroyo, J, Garay, OJ, Mencia, JJ. A note on closed hyperelastica in S^3. Soochow Journal of Mathematics 2004; 30 (3): 269-282.

[4] Barros, M, Ferrandez, A, Lucas, P, Merono, MA. Willmore tori and Willmore-Chen submanifolds in pseudo-Riemannian spaces. Journal of Geometry and Physics 1998; 28: 45-46. doi.org/10.1016/S0393-0440(98)00010-2

[5] Brunnett, G. A new characterization of plane elastica. Mathematical Methods in Computer Aided Geometric Design II, Academic Press, Boston, MA, 1992; 43-56.

[6] Brunnett, G, Crouch, PE, Elastic curves on the sphere. Advances in Computational Mathematics 1994; 2 (1): 23-40.

[7] Dereli, T, Hartley, DH, Önder, M, Tucker, RW. Relativistic elastica. Physics Letters B 1990; 252 (4): 601-604. doi.org/10.1016/0370-2693(90)90491-N

[8] Huang R. A note on the p-elastica a constant sectional curvature manifold. Journal of Geometry and Physics 2004; 49: 343-349. doi.org/10.1016/S0393-0440(03)00107-4

[9] Huang, R, Shang, D. Generalized elastic curves in the Lorentz flat space L^4. Applied Mathematics and Mechanics, 2009; 30 (9): 1193-1200.

[10] Huang, R, Liao, C, Shang, D. Generalized elastica in anti-de sitter space H_1^3. Chinese Quarterly Journal of Mathematics 2011; 26 (2): 311-316.

[11] Huang, R, Junyan, Y. Generalized elastica on $2-$dimensional de sitter space S_1^2. International Journal of Geometric Methods in Modern Physics 2016; 13 (4): 1650047,7. doi.org/10.1142/S021988781650047X

[12] Kağızman, ST. Hyperelastic curves in $3-$dimensional lightlike cone. MSc, Süleyman Demirel University, Isparta, Turkey, 2021.

[13] Langer, J, Singer, DA. The total squared curvature of closed curves. Journal of Differential Geometry 1984; 20: 1-22. doi: 10.4310/jdg/1214438990

[14] Liu, H. Curves in the lightlike cone. Beitrräge zur Algebra und Geometrie. 2004; 45 (1): 291-303.

[15] Liu, H, Meng, Q. Representation formulas of curves in a two- and three-dimensional lightlike cone. Results in Mathematics 2011; 59: 437-451. doi: 10.1007/s00025-011-0108-y

[16] Liu, H, Meng, Q. Erratum to: Representation formulas of curves in a two- and three-dimensional lightlike cone. Results in Mathematics 2012; 61: 423-424. doi: 10.1007/s00025-012-0257-7

[17] Liu, H. Erratum: Curves in the lightlike cone, Beitrräge zur Algebra und Geometrie 2013; 54 (1): 469-470. doi:10.1007/s13366-012-0134-2

[18] Lopez R. Differential geometry of curves and surfaces in Lorentz-Minkowski space. International Electronic Journal of Geometry 2014; 7 (1): 44-107. doi: 10.36890/iejg.594497

[19] O'Neill B. Semi-Riemannian Geometry with Applications to Relativity. New York, USA: Academic Press, 1993.

[20] Oral, M. Elastic curves on hyperquadrics in Minkowski 3- space. Msc, Süleyman Demirel University, Isparta, Turkey, 2010.

[21] Sager, I, Abazeri, N, Ekmekci, N, Yaylı, Y. The classical elastic curves in Lorentz-Minkowski space. International Journal of Contemporary Mathematical Sciences 2011; 6 (7): 309-320.

[22] Singer, D. Lectures on elastic curves and rods. In: AIP Conference Proceedings; 1002 American Institute of Physics, Melville, New York, USA; 2008. pp. 3-32.

[23] Steinberg, DH. Elastic curves in hyperbolic space. PhD, Case Western Reserve University, Cleveland, Ohio, USA, 1995.

[24] Tükel, GÖ, Yücesan, A. Elastic curves in a two-dimensional lightlike cone. International Electronic Journal of Geometry 2015; 8 (2): 1-8. doi:10.36890/iejg.592272

[25] Tükel, GÖ, Huang, R, Yücesan, A. A note on hyperelastic curves. Acta Universitatis Apulensis 2020; 63: 1-9. doi:10.17114/j.aua.2020.63.01

[26] Weinstock, R. Calculus of Variations with Applications to Physics and Engineering. Mineola, NY, USA: Dover Publications Inc, 1974.

[27] Yücesan, A, Oral, M. Elastica on $2-$dimensional anti-de sitter space. International Journal of Geometric Methods in Modern Physics 2011; 8 (1): 107-113. doi:10.1142/S0219887811005002

[28] Yücesan, A. Elastica in a $3-$dimensional lightlike cone. In: International Conference on Advancements in Mathematical Sciences; Antalya, Turkey; 2015.