Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 71 – 86
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doi:10.3906/mat-2105-6

Continuation value computation using Malliavin calculus under general volatility stochastic process for American option pricing

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Received: 03.05.2021 • **Accepted/Published Online:** 25.10.2021 • **Final Version:** 19.01.2022

Abstract: American options represent an important financial instrument but are notoriously difficult to price, especially when the volatility is not constant. We explore the conditions required to apply Malliavin calculus to price American options when the volatility follows a general stochastic differential process, and develop the expressions to compute the continuation value at any time before the expiration date, given the current asset price and volatility. The developed methodology can then be applied to price American options.

Key words: Conditional expectation, Malliavin calculus, stochastic volatility, pricing American option

1. Introduction

Since its invention in 1978, Malliavin calculus [21] has been used in various mathematical fields. In theoretical research, it has been investigated for instance in Lie algebra [19], probability [8, 9], stochastic analysis [20]. Recently Laukkarinen [18] combined Malliavin calculus with fractional derivatives. Malliavin calculus has also been successfully applied to solve practical problems, for example in mechanics [14], and has received a special interest in finance due to its ability to address some stochastic differential equations two decades ago [11, 12].

For instance, Benhamou [5] studies the estimation of conditional expectations using Malliavin calculus and illustrate the computational efficiency of this method. Bally et al. [4], Bouchard et al. [6], Caramellino and Zanette [7] considered the problem of pricing and hedging American options, combining Malliavin Calculus and Monte Carlo, and Abbas-Turki and Lapeyre [1] proposed nonparametric variance and bias reduction methods. Kharrat [17] developed an expression of the continuation value for American options generated by the Heston model. Mallivin calculus has also been successfully applied the other option problems. Mancino and Sanfelici [22] proposed for instance a methodology to compute the Malliavin weight for the Delta hedging under a local volatility model, and Saporito [24] constructed an approximation for the price of path-dependent derivatives under the multiscale stochastic volatility models. Yamada [28] provided an approximation scheme for multidimensional Stratonovich stochastic differential equations using Malliavin calculus, and applied it to the SABR model. Yamada and Yamamoto [29] then constructed a second-order discretization scheme that they applied to price European option under the SABR model, while Alòs and Shiraya [3] studied volatility

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2010 *AMS Mathematics Subject Classification:* 91Gxx, 60Gxx

swaps and European options. Alòs and Lorite [2] have recently reviewed the application of Malliavin calculus in finance, but ignored American option pricing. American options are notoriously more difficult to price than European options as the latter can only be exercised at the expiration date T while the former can be exercised at any time t before and including the expiration date. Bally et al. [4] developed a representation formula for the continuation value of American option using Malliavin calculus, assuming constant volatility. The constant volatility hypothesis is however unrealistic [16], and the volatility dynamics is fundamental in the development of trading strategies for hedging and arbitrage. A lot of efforts have therefore been devoted to option pricing under stochastic volatility over the last decade (see for instance Fouque et al. [10] and Pascucci [23], subsection 10.5.3, and references within) to capture the reality of the financial markets. Assumptions on the stochastic volatility process remains however restrictive, and the main contribution of this paper is to extend previous results to more general stochastic volatility processes.

In a financial context, let X_t represent the asset price at time $t \in [0,T]$. V_t is the volatility at time t supposed to be stochastic and P_t represents the derivative value, r is the constant drift, W^S and W^V are two correlated standard Wiener processes, where $W_t^S = \sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ and $W_t^V = B_t^2$, with B^1 and B^2 being two independent Brownian motions, and $\rho \in [-1,1]$ is the correlation coefficient. The parameters θ_V, k_V and η represent the long-run average variance, the rate of mean reversion, and volatility of volatility, respectively, and α is a positive real number. In this paper, we aim to elaborate and compute theoretically the value of the conditional expectation

$$\mathbb{E}\left[P_t(X_t,V_t) \,|\, (X_l,V_l)\right] \tag{1.1}$$

for any $0 \le l < t$, using Malliavin calculus, where V_t is generated by a general stochastic differential process (GSDP)

$$dV_t = k_V(\theta_V - V_t)dt + \eta V_t^{\alpha} dW_t^V, \tag{1.2}$$

and X is generated by the geometric Brownian process

$$dX_t = rX_t dt + X_t\sqrt{V_t}dW_t^S, \tag{1.3}$$

or, by integration

$$X_t = X_l exp\left(\int_l^t r ds + \int_l^t \sqrt{V_s}dW_s^S - \frac{1}{2}\int_l^t V_s ds\right). \tag{1.4}$$

(1.1) can be used for instance to price and hedge options and convertible bonds, and in actuarial calculations. This is in particular the case when pricing American options, (1.1) give the continuation value at time t if the option is not exercised. Here, the continuation value at time t is defined as the expected optimal profit that can be obtained by the option holder, accounting for the uncertain environment. The continuation value is used to price the option contract. The process (1.2) can be seen as a generalization of popular processes, as in particular, if $\alpha = 0$, we have the Stein and Stein model [26]; if W_t^S and W_t^V are independent, and if $\alpha = 0.5$, we get the Heston model [15] and the SABR model [13] if we additionally impose $r = 0$ in (1.3) and $k_V = 0$ in (1.2). Similarly, we can consider a stochastic interest rate and a constant volatility as in the Vasicek Model [27] when $\alpha = 0.5$.

(1.2) is also a direct extension of the process considered by Kharrat [17], where α is fixed to 0.5. Kharrat derived an expression of the conditional expectation (1.1) using Malliavin calculus, and performed some numerical experiments to illustrate the practical interest of the approach. We extend these results to the case where α is not restricted to the value 0.5. The outline of this paper is as follows. In the second section, we introduce some preliminary concepts and the key tools required to obtain our results. The conditional expectation expression is derived in the third section in the unidimensional case. We conclude in the fourth section and discuss avenues for future research.

2. Preliminaries

We first introduce some notations and definitions that will be used to obtain our results. The presentation closely follows Pascucci [23], Chapter 16, to which we refer the reader for more details. The maturity, or expiration date, of the financial instrument to price is denoted by T, normalized to 1 for simplicity. We also divide the horizon in 2^n dyadic intervals,

$$I_n^k =]t_n^{k-1}, t_n^k], \; t_n^k = \frac{k}{2^n},$$

for $k = 1, \dots, 2^n$, $n \in \mathbb{N}$, and $t_n^0 = 0$. We also denote by $x_n = (x_n^1, \dots, x_n^{2^n})$ a point in $\mathbb{R}^{2^n}$, and for every $t \in]0, T]$, let $k_n(t)$ be the only element $k \in \{1, \dots, 2^n\}$ such that $t \in I_n^k$. We first introduce the notion of simple functionals.

Definition 2.1 *Given $n \in \mathbb{N}$, the family of simple n-th order functionals is defined by*

$$\mathcal{S}_n := \left\{ \varphi(\Delta_n) \,|\, \varphi \in C_{pol}^{\infty}(\mathbb{R}^{2^n}; \mathbb{R}) \right\}$$

where C_{pol}^{∞} is the family of infinitely differentiable functions which, together with their derivatives of any order, have at most a polynomial growth, and $\Delta_n := \left(\Delta_n^1, \dots, \Delta_n^{2^n}\right)$ is the vector of Brownian increments $\Delta_n^k = W_{t_n^k} - W_{t_n^{k-1}}$, $k = 1, \dots, 2^n$.

We are now in position to formally define the Malliavin derivative.

Definition 2.2 *For every $X = \varphi(\Delta_n) \in \mathcal{S}$, the stochastic (or Malliavin) derivative of X at time t is defined by the $k_n(t)$-th partial derivative of $\varphi(\Delta_n)$:*

$$D_t X := \frac{\partial \varphi}{\partial x_n^{k_n(t)}}(\Delta_n),$$

and we denote by DX the associated stochastic process on $[0, T]$.

We can also construct the space of Malliavin differentiable variables and the family of the n-th order simple processes in the following definitions.

Definition 2.3 *The space $\mathbb{D}^{1,2}$ of the Malliavin-differentiable random variables is the closure of $\mathcal{S}$ with respect to the norm $\|.\|_{1,2}$, defined as*

$$\|X\|_{1,2} = \sqrt{\mathbb{E}[X^2]} + \sqrt{\int_0^T (D_s X)^2 ds}$$

In other words, $X \in \mathbb{D}^{1,2}$ *if and only if, there exists a sequence* (X_n), $n \in \mathbb{N}$, *in* S *such that* X_n *converges in distribution to a square integrable random variable* X *as* $n \to \infty$, *the limit* $\lim_{n\to\infty} DX_n$ *exists and is square integrable.*

Definition 2.4 *The family* $\mathcal{P}_n$, $n \in \mathbb{N}$, *of the* n*-th order simple processes consists of the processes* U *of the form*

$$U_t = \sum_{k=1}^{2^n} \varphi_k(\Delta_n) \mathbf{1}_{I_n^k}(t) = \varphi_{k_n(t)}(\Delta_n), \tag{2.1}$$

where $\varphi_k \in C^\infty_{pol}(\mathbb{R}^{2^n};\mathbb{R})$ *for* $k = 1, \ldots, 2^n$.

It is easy to observe that $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$, $n \in \mathbb{N}$. We define $\mathcal{P} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ the family of simple functionals, and note that $DX \in \mathcal{P}$, for $X \in \mathcal{S}$. In other words, $D : \mathcal{S} \to \mathcal{P}$. We are now in position to introduce the adjoint operator of the Malliavin derivative.

Definition 2.5 *Given a simple process* $U \in \mathcal{P}$ *of the form* (2.1), *the Skorokhod-integral* $D * U$ *of* U *is defined as [25]*

$$D * U = \sum_{k=1}^{2^n} \left(\varphi_k(\Delta_n)\Delta_n^k - \partial_{x_n^k} \varphi_k(\Delta_n) \frac{1}{2^n} \right).$$

We also write $D * U = \int_0^T U_t \diamond dW_t$.

The following two technical lemmas, proved in Pascucci [23, Chapter 16], will be central to our developments.

Lemma 2.6 *Let* $X \in \mathbb{D}^{1,2}$ *and let* U *be a second-order Skorokhod-integrable process. Then,*

$$\int_0^T XU_t \diamond dW_t = X \int_0^T U_t \diamond dW_t - \int_0^T (D_t X) U_t dt \tag{2.2}$$

and, when U_t *is adapted, the above equation can be expressed as*

$$\int_0^T XU_t \diamond dW_t = X \int_0^T U_t dW_t - \int_0^T (D_t X) U_t dt. \tag{2.3}$$

Lemma 2.7 (Stochastic integration by parts) *Let* $F \in C_b^1$, *the space of functions in* C^1 *bounded together with their derivatives, and let* $X \in \mathbb{D}^{1,2}$. *Hence, the following integration by parts*

$$\mathbb{E}[F'(X)Y] = \mathbb{E}\left[F(X) \int_0^T \frac{u_t Y}{\int_0^T u_s D_s X ds} \diamond dW_t \right] \tag{2.4}$$

holds for every random variable Y *and for every stochastic process* u *for which* (2.4) *is well defined.*

For notation clarity, we extend the notion of Malliavin derivative to a multidimensional process, by introducing the definition of partial derivative.

Definition 2.8 (Partial Malliavin derivative) *Let $W = (W^1, \dots, W^d)$ be a $d-$dimensional Brownian motion. For $s \leq t$, the partial Malliavin derivative at time s with respect to the $i-th$ component of W, denoted by D_s^i, is*

$$D_s^i W_t^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By convention, we will denote by D^V and D^S the Malliavin derivative with respect to W^V and W^S, respectively, where W^V and W^S are defined in the same way as in (1.3) and (1.2).

3. Computation of the continuation value

Consider again the GSDP generated by (1.2). For $0 \leq l \leq t$, the stochastic volatility process is

$$V_l = V_0 + \int_0^l k_V(\theta_V - V_r)dr + \int_0^l \eta V_r^\alpha dW_r^V. \tag{3.1}$$

We do not have an explicit expression of $D^V V$. We can however express it in terms of a newly introduced process, as shown in the following proposition.

Proposition 3.1 *For $0 \leq s \leq l \leq t$, $D_s^V V_l = Y_l Z_s \eta V_s^\alpha$, where Y is the solution of the stochastic differential equation*

$$Y_l = 1 - k_V \int_0^l Y_r dr + \alpha\eta \int_0^l V_r^{\alpha-1} Y_r dW_r^V, \tag{3.2}$$

and Z, is the solution of the stochastic differential equation

$$Z_l = 1 + \int_0^l \left((\alpha\eta V_r^{\alpha-1})^2 + k_V\right) Z_r dr - \int_0^l \alpha\eta V_r^{\alpha-1} Z_r dW_r^V. \tag{3.3}$$

Moreover, for all $t \geq 0$, $Y_t Z_t = 1$.

Proof From (3.2), (3.3) and the Itô formula, we have

$$\begin{aligned} d(Y_l Z_l) &= Y_l dZ_l + Z_l dY_l + d\langle Y, Z\rangle_l \\ &= Y_l Z_l \big[\left((\alpha\eta V_r^{\alpha-1})^2 + k_V\right) dl - \alpha\eta V_l^{\alpha-1} dW_l^V \\ &\quad + k_V dl + \alpha\eta V_l^{\alpha-1} dW_l^V - (\alpha\eta V_r^{\alpha-1})^2 dl \big] \\ &= 0 \end{aligned}$$

Thus, $Y_l Z_l$ is constant and since $Y_0 Z_0 = 1$, from the unicity of the representation for an Itô process, $Y_l Z_l = 1$ for any $l \geq 0$.

From (3.2), given s, we have

$$Y_l = Y_s - k_V \int_s^l Y_r dr + \alpha\eta \int_s^l V_r^{\alpha-1} Y_r dW_r^V,$$

and, since $Y_s Z_s = 1$,

$$
\begin{aligned}
Y_l Z_s \eta V_s^\alpha &= Y_s Z_s \eta V_s^\alpha - k_V \int_s^l Y_r Z_s \eta V_s^\alpha dr + \alpha\eta \int_s^l V_r^{\alpha-1} Y_r Z_s \eta V_s^\alpha dW_r^V \\
&= \eta V_s^\alpha - k_V \int_s^l Y_r Z_s \eta V_s^\alpha dr + \alpha\eta \int_s^l V_r^{\alpha-1} Y_r Z_s \eta V_s^\alpha dW_r^V . \qquad (3.4)
\end{aligned}
$$

Moreover, applying Definition 2.2 to (3.1),

$$
D_s^V V_l = \eta V_s^\alpha - k_V \int_s^l D_s^V V_r dr + \alpha\eta \int_s^l V_r^{\alpha-1} D_s^V V_r dW_r^V ,
$$

and with (3.4),

$$
D_s^V V_l = Y_l Z_s \eta V_s^\alpha .
$$

□

We now consider the expression of $\mathbb{E}[\Psi'(V_l)P(X_t, V_t)]$ for any $0 \le l \le t$ and for any function $\Psi \in C_b^{+\infty}(\mathbb{R})$, where $C_b^{+\infty}(\mathbb{R})$ is the space of bounded and infinitely differentiable functions. To alleviate the notation, we define the derivative of a stochastic process as follows.

Definition 3.2 *Let $X_t(\mu)$, $t \in \mathcal{T}$, be a continuous stochastic process, parameterized by μ, and assume that for any $t \in \mathcal{T}$, $X_t(\mu)$ is continuously differentiable with respect to μ. The derivative process $\partial_\mu X$ is the stochastic process defined as*

$$
\partial_\mu X = \{\partial_\mu X_t,\ t \in \mathcal{T}\}.
$$

We first prove the following lemma.

Lemma 3.3 *Let Λ be a continuous random variable with density $h(\cdot)$.*

a) For every f and $g \in C^1$, such that $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} g(x) = 0$, we have

$$
\mathbb{E}[f'(\Lambda)g(\Lambda)] = -\mathbb{E}[f(\Lambda)g'(\Lambda)] - \int_{-\infty}^{+\infty} f(y)g(y)h'(y)dy
$$

b) Let X be a stochastic process, function of Λ and twice continuously differentiable with respect to Λ, such that $\partial_\Lambda X \neq 0$ almost surely,

$$
\mathbb{E}[f'(X)g(X)] = -\mathbb{E}\left[f(X)\left(g'(X) - g(X)\frac{\partial_\Lambda^2 X}{(\partial_\Lambda X)^2}\right)\right] - \int_{-\infty}^{+\infty} f(y)g(y)\frac{h'(y)}{(\partial_\Lambda X)(y)}dy
$$

Proof Integrating by parts, we have

$$
\begin{aligned}
\mathbb{E}[f'(\Lambda)g(\Lambda)] &= \int_{-\infty}^{+\infty} f'(y)g(y)h(y)dy \\
&= [f(y)g(y)h(y)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(y)(g(y)h(y))'dy \\
&= -\int_{-\infty}^{+\infty} f(y)g'(y)h(y)dy - \int_{-\infty}^{+\infty} f(y)g(y)h'(y)dy.
\end{aligned}
$$

Thus, we get

$$\mathbb{E}[f'(\Lambda)g(\Lambda)] = -\mathbb{E}[f(\Lambda)g'(\Lambda)] - \int_{-\infty}^{+\infty} f(y)g(y)h'(y)dy,$$

proving a).

Since $X = X(\Lambda)$, we have

$$\frac{df(X)}{d\Lambda} = \frac{df(X)}{dX}\frac{dX}{d\Lambda} \text{ and } \frac{dg(X)}{d\Lambda} = \frac{dg(X)}{dX}\frac{dX}{d\Lambda}.$$

or, using a more compact notation,

$$\partial_\Lambda f(X) = f'(X)\partial_\Lambda X \text{ and } \partial_\Lambda g(X) = g'(X)\partial_\Lambda X.$$

Therefore,

$$\mathbb{E}[f'(X)g(X)] = \mathbb{E}\left[\partial_\Lambda f(X)\frac{g(X)}{\partial_\Lambda X}\right].$$

and using a), we can write

$$\begin{aligned}\mathbb{E}\left[f'(X)g(X)\right] &= -\mathbb{E}\left[f(X)\partial_\Lambda\left(\frac{g(X)}{\partial_\Lambda X}\right)\right] - \int_{-\infty}^{+\infty} f(y)\frac{g(y)}{(\partial_\Lambda X)(y)}h'(y)dy \\ &= -\mathbb{E}\left[f(X)\left(g'(X) - g(X)\frac{\partial^2_\Lambda X}{\partial_\Lambda X)}\right)\right] \\ &\quad - \int_{-\infty}^{+\infty} f(y)g(y)\frac{h'(y)}{(\partial_\Lambda X)(y)}dy,\end{aligned}$$

where we have used the equality

$$\partial_\Lambda\left(\frac{g(X)}{\partial_\Lambda X}\right) = g'(X) - \frac{g(X)\partial^2_\Lambda X}{(\partial_\Lambda X)^2}.$$

□

A standard, while restrictive, assumption required to apply the Malliavin calculus is the independence of the increments [9], as the distribution of the increments then depends on the time intervals uniquely. This assumption will be crucial in our developments too, when using stochastic integration by parts.

Assumption 1 (Independence of increments) Let V_t be a stochastic process. We say that V_t has independent increments if there exists a random function $M(\cdot)$ independent of V_l, such that $V_t = V_l M_{(t-l)}$ for any $l \in (0, t]$.

Assumption 1 implies that $V_t = V_0 M_t$ and $M_t = M_{(t-l)} M_l$. In line with Definition 2.1, $M_{(t-l)}$ is a function of Brownian increments $\Delta_{(t-l)} := W_t - W_l$, and, for any $t \in [0, T]$, we consider the partial derivative of the volatility process with respect to Δ, $\partial_\Delta V_t = V_0 \frac{dM_t}{d\Delta}$.

Example 3.4 *The SABR model [13] satisfies Assumption 1, as for $0 \leq l \leq t$, the stochastic process can be written as*

$$V_t = V_l e^{\eta(W_t - W_l)} = V_l e^{\eta\Delta_{(t-l)}} = V_0 e^{\eta\Delta_t}.$$

We will furthermore limit ourself to the exponential family to parametrize the stochastic process capturing the asset price or its volatility. Multiple stochastic processes used in a financial context comply with this assumption, for instance if Λ follows a normal distribution or a gamma distribution. For a random variable of the exponential family of density $h(x)$, its derivative can be written as $h'(\cdot) = r(\cdot)h(\cdot)$, a property that we will exploit to obtain the expressions of the conditional expectations.

Assumption 2 X is a stochastic process that is function of Λ, a continuous random variable from the exponential family, with density $h(\cdot)$.

Corollary 3.5 *Let X be a stochastic process satisfying Assumptions 1 and 2. Under the same conditions of Lemma 3.3, we have*

$$\mathbb{E}[f'(X)g(X)] = -\mathbb{E}\left[f(X)\left(g'(X) - g(X)\frac{\partial_\Lambda^2 X}{(\partial_\Lambda X)^2}\right)\right] - \mathbb{E}\left[f(X)g(X)\frac{r(\Lambda)}{\partial_\Lambda X}\right].$$

We are now in position to prove the following technical result.

Proposition 3.6 *If V_t is a stochastic process satisfying Assumptions 1 and 2, and for any $k > 0$, $\partial_\Delta M_k \neq 0$ almost everywhere, given $0 \leq l \leq t$, we have*

$$\begin{aligned}\mathbb{E}[\Phi'(V_l)g(V_t)] = \mathbb{E}&\left[\frac{\Phi(V_l)g(V_t)}{V_l}\left(1 - M_{(t-l)}\frac{\partial_\Delta^2 M_{(t-l)}}{(\partial_\Delta M_{(t-l)})^2} + M_l\frac{\partial_\Delta^2 M_l}{(\partial_\Delta M_l)^2}\right)\right] \\ &+ \mathbb{E}\left[\frac{\Phi(V_l)g(V_t)}{V_l}\left(M_{(t-l)}\frac{r\left(\Delta_{(t-l)}\right)}{(\partial_\Delta M_{(t-l)})} - M_l\frac{r(\Delta_l)}{\partial_\Delta M_l}\right)\right].\end{aligned}$$

Proof For $0 \leq l \leq t$, from Assumption 1 and the law of total expectation, we get

$$\mathbb{E}[\Phi'(V_l)g(V_t)] = \mathbb{E}\left[\mathbb{E}_{V_l}[\Phi'(V_l)g(mV_l)] \,|\, M_{(t-l)} = m\right]. \tag{3.5}$$

Using Corollary 3.5, the inner expectation can be written as

$$\begin{aligned}\mathbb{E}_{V_l}[\Phi'(V_l)g(mV_l)] = &- \mathbb{E}_{V_l}\left[\Phi(V_l)\left(mg'(mV_l)\right) - g(mV_l)\frac{\partial_\Delta^2 V_l}{(\partial_\Delta V_l)^2}\right] \\ &- \mathbb{E}_{V_l}\left[\Phi(V_l)g(mV_l)\frac{r(\Delta_l)}{\partial_\Delta V_l}\right].\end{aligned}$$

As $V_t = M_{(t-l)}V_l$ and $V_l = V_0 M_l$, (3.5) becomes

$$\begin{aligned}\mathbb{E}[\Phi'(V_l)g(V_t)] = -\mathbb{E}[\Phi(V_l)M_{(t-l)}g'(V_t)] &+ \mathbb{E}\left[\Phi(V_l)g(V_t)\frac{\partial_\Delta^2 V_l}{(\partial_\Delta V_l)^2}\right] \\ - \mathbb{E}&\left[\Phi(V_l)g(V_t)\frac{r(\Delta_l)}{(\partial_\Delta V_l)}\right].\end{aligned} \tag{3.6}$$

From the independence between V_l and $M_{(t-l)}$, the first term of (3.6) becomes

$$\mathbb{E}[\Phi(V_l)M_{(t-l)}g'(V_t)] = \mathbb{E}\left[\Phi(z)\mathbb{E}_M\left[M_{(t-l)}g'\left(zM_{(t-l)}\right)\right] \,\middle|\, V_l = z\right]. \tag{3.7}$$

Moreover, from Corollary 3.5,

$$\begin{aligned}\mathbb{E}_M[M_{(t-l)}g'\left(zM_{(t-l)}\right)] = &-\mathbb{E}_M\left[\frac{g\left(zM_{(t-l)}\right)}{z}\left(1 - M_{(t-l)}\frac{\partial^2_\Delta M_{(t-l)}}{\left(\partial_\Delta M_{(t-l)}\right)^2}\right)\right] \\ &-\mathbb{E}_M\left[\frac{M_{(t-l)}}{z}g\left(zM_{(t-l)}\right)\frac{r(\Delta_{(t-l)})}{\partial_\Delta M_{(t-l)}}\right].\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8). we have

$$\begin{aligned}\mathbb{E}[\Phi(V_l)M_{(t-l)}g'(V_t)] = &-\mathbb{E}\left[\frac{\Phi(V_l)g(V_t)}{V_l}\left(1 - M_{(t-l)}\frac{\partial^2_\Delta M_{(t-l)}}{\left(\partial_\Delta M_{(t-l)}\right)^2}\right)\right] \\ &-\mathbb{E}\left[\frac{M_{(t-l)}\Phi(V_l)g(V_t)}{V_l}\frac{r(\Delta_{(t-l)})}{\partial_\Delta M_{(t-l)}}\right].\end{aligned}$$

(3.6) can thus be rewritten as

$$\begin{aligned}\mathbb{E}[\Phi'(V_l)g(V_t)] = &\,\mathbb{E}\left[\frac{\Phi(V_l)g(V_t)}{V_l}\left(1 - M_{(t-l)}\frac{\partial^2_\Delta M_{(t-l)}}{(\partial_\Delta M_{(t-l)})^2}\right)\right] \\ &+\mathbb{E}\left[\frac{M_{(t-l)}\Phi(V_l)g(V_t)}{V_l}\frac{r(\Delta_{(t-l)})}{\partial_\Delta M_{(t-l)}}\right] + \mathbb{E}\left[\Phi(V_l)g(V_t)\frac{\partial^2_\Delta V_l}{(\partial_\Delta V_l)^2}\right] \\ &-\mathbb{E}\left[\Phi(V_l)g(V_t)\frac{r(\Delta_l)}{\partial_\Delta V_l}\right].\end{aligned}$$

We then obtained the announced result by noting that $V_l = V_0 M_l$. □

We can directly extend the previous proposition to asset price under varying volatility, as expressed by the following corollary, implied by Proposition 3.6.

Corollary 3.7 *Let X and V be the asset price and its volatility. Under Assumptions 1 and 2, and assuming that for any $k > 0$, $\partial_\Delta M_k \neq 0$ almost everywhere, we have, for any $0 \le l \le t$ and any $\Psi \in C_b^{+\infty}(\mathbb{R})$,*

$$\begin{aligned}\mathbb{E}[\Psi'(V_l)P(X_t,V_t)] = &\,\mathbb{E}\left[\frac{\Psi(V_l)P(X_t,V_t)}{V_l}\left(1 - M_{(t-l)}\frac{\partial^2_\Delta M_{(t-l)}}{(\partial_\Delta M_{(t-l)})^2} + M_l\frac{\partial^2_\Delta M_l}{(\partial_\Delta M_l)^2}\right)\right] \\ &+\mathbb{E}\left[\frac{\Psi(V_l)P(X_t,V_t)}{V_l}\left(M_{(t-l)}\frac{r(\Delta_{(t-l)})}{\partial_\Delta M_{(t-l)}} - M_l\frac{r(\Delta_l)}{\partial_\Delta M_l}\right)\right].\end{aligned}$$

Equipped with the previous results, we are now in position to derive the expression of the conditional expectation $\mathbb{E}[P(X_t,V_t)\,|\,V_l = \beta]$ using Malliavin calculus.

Theorem 3.8 *Let X and V be the asset price and its volatility. Under Assumptions 1 and 2, for any $\beta > 0$ and any $0 \le l \le t$, we have*

$$\mathbb{E}[P(X_t,V_t)\,|\,V_l = \beta] = \frac{\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(P(X_t,V_t))\right]}{\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(1)\right]},$$

where

$$\begin{aligned}\Upsilon_{V_l}(f(X_t,V_t)) = &\frac{f(X_t,V_t)}{V_l}\left(1 - M_{(t-l)}\frac{\partial^2_\Delta M_{(t-l)}}{(\partial_\Delta M_{(t-l)})^2} + M_l\frac{\partial^2_\Delta M_l}{(\partial_\Delta M_l)^2}\right)\\ &+\frac{f(X_t,V_t)}{V_l}\left(M_{(t-l)}\frac{r(\Delta_{(t-l)})}{\partial_\Delta M_{(t-l)}} - M_l\frac{r(\Delta_l)}{\partial_\Delta M_l}\right)\end{aligned} \tag{3.9}$$

is the Malliavin weight, H is the Heaviside function i.e. $\forall x \in \mathbb{R}$, $H(x) = 1_{x\geq 0}$, with the convention that $\mathbb{E}[P(X_t,V_t)] = 0$ when $\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(1)\right] = 0$.

Proof Referring to the work of Fournié et al. [12], we have

$$\mathbb{E}[P(X_t,V_t)\,|\,V_l=\beta] = \frac{\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(P(X_t,V_t))\right]}{\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(1)\right]},$$

where the weights $\Upsilon_{V_l}(P(X_t,V_t))$ and $\Upsilon_{V_l}(1)$ can be derived from Corollary 3.7 for $\Psi(V_l) = H(V_l-\beta) = 1_{V_l\geq\beta}$. □

Using Propositions 3.1, Corollary 3.7, and Theorem 3.8, we can compute $\mathbb{E}[P(X_t,V_t)\,|V_l=\beta]$ where V_t is generated by the GSDP (1.2). In what follows, we first develop a general technical result in Proposition 3.10, which will set the stage to formulate the main theorem of this paper, developing the computation of the continuation value $\mathbb{E}[P_t(X_t,V_t)\,|\,(X_l=\gamma,V_l=\beta)]$. Recall that from (1.4), the asset price follows the process

$$X_t = X_l A_{l,t},$$

where $A_{l,t} = exp\left(\int_l^t r ds + \int_l^t \sqrt{V_s}dW_s^S - \frac{1}{2}\int_l^t V_s ds\right)$, for any $0\leq l\leq t$. We then have the following result.

Proposition 3.9 *Let X_t be defined as in (1.4) and $G_{X_t}(\beta) := \mathbb{E}[P_t(X_t,V_t)|V_l=\beta]$. For any $\Psi \in C_b^{+\infty}(\mathbb{R})$, we have*

$$\begin{aligned}\mathbb{E}[\Psi'(X_l)G_{X_lA_{l,t}}(\beta)] =&\mathbb{E}\left[\frac{\Psi(X_l)G_{X_lA_{l,t}}(\beta)}{X_l}\left(\frac{1}{l\sqrt{1-\rho^2}}\int_0^l\frac{dW_s^S}{\sqrt{V_s}}+1\right)\right]\\ &-\mathbb{E}[A_{l,t}\Psi(X_l)G'_{X_lA_{l,t}}(\beta)].\end{aligned}$$

Proof Using the stochastic integration by parts from Lemma 2.7, we have, for any process u,

$$\mathbb{E}[\Psi'(X_l)G_{X_lA_{l,t}}(\beta)] = \mathbb{E}\left[\Psi(X_l)\int_0^l\frac{u_\tau G_{X_lA_{l,t}}(\beta)}{\int_0^l u_s D_s^S X_l ds}\diamond dW_\tau^S\right].$$

In particular, choosing $u_s = \frac{1}{\sqrt{V_s}}$ allows us to write

$$\begin{aligned}\mathbb{E}[\Psi'(X_l)G_{X_lA_{l,t}}(\beta)] &= \mathbb{E}\left[\Psi(X_l)\int_0^l\frac{G_{X_lA_{l,t}}(\beta)}{l\sqrt{1-\rho^2}\sqrt{V_\tau}X_l}\diamond dW_\tau^S\right]\\ &=\mathbb{E}\left[\frac{\Psi(X_l)}{l\sqrt{1-\rho^2}}\left(\frac{G_{X_lA_{l,t}}(\beta)}{X_l}\int_0^l\frac{dW_s^S}{\sqrt{V_s}} - \int_0^l D_s^S\left(\frac{G_{X_lA_{l,t}}(\beta)}{X_l}\right)\frac{ds}{\sqrt{V_s}}\right)\right],\end{aligned}$$

where we used the equality $D_s^S(X_l) = \sqrt{1-\rho^2}X_l\sqrt{V_s}$. The second equality follows from Lemma 2.6. By using the Malliavin derivative, the above equation becomes

$$\begin{aligned}
&\mathbb{E}[\Psi'(X_l)G_{X_lA_{l,t}}(\beta)]\\
&= \mathbb{E}\left[\frac{\Psi(X_l)}{l\sqrt{1-\rho^2}}\left(\frac{G_{X_lA_{l,t}}(\beta)}{X_l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \int_0^l \left(\frac{G'_{X_lA_{l,t}}(\beta)D_s^S(X_lA_{l,t})}{X_l} - \frac{G_{X_lA_{l,t}}(\beta)D_s^S X_l}{X_l^2}\right)\frac{ds}{\sqrt{V_s}}\right)\right]\\
&= \mathbb{E}\left[\frac{\Psi(X_l)}{l\sqrt{1-\rho^2}}\left(\frac{G_{X_lA_{l,t}}(\beta)}{X_l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \int_0^l \left(G'_{X_lA_{l,t}}(\beta)A_{l,t}\sqrt{V_s} - \frac{G_{X_lA_{l,t}}(\beta)\sqrt{V_s}}{X_l}\right)\frac{ds}{\sqrt{V_s}}\right)\right]\\
&= \mathbb{E}\left[\frac{\Psi(X_l)G_{X_lA_{l,t}}(\beta)}{X_l}\left(\frac{1}{l\sqrt{1-\rho^2}}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right] - \mathbb{E}\left[A_{l,t}\Psi(X_l)G'_{X_lA_{l,t}}(\beta)\right] \qquad (3.10)
\end{aligned}$$

In the other hand, by using the independence between X_l and $A_{l,t}$, we have

$$\begin{aligned}
\mathbb{E}\left[A_{l,t}\Psi(X_l)G'_{X_lA_{l,t}}(\beta)\right] &= \mathbb{E}\left[\mathbb{E}\left[A_{l,t}\Psi(x)G'_{xA_{l,t}}(\beta)\right]\,\Big|\, X_l = x\right]\\
&= \mathbb{E}\left[\Psi(x)\mathbb{E}\left[A_{l,t}G'_{xA_{l,t}}(\beta)\right]\,\Big|\, X_l = x\right]. \qquad (3.11)
\end{aligned}$$

By using Lemma 2.7 and choosing $u_s = \frac{1}{\sqrt{V_s}}$, we have

$$\begin{aligned}
\mathbb{E}[A_{l,t}G'_{X_lA_{l,t}}(\beta)] &= \mathbb{E}\left[G_{X_lA_{l,t}}\int_l^t \frac{u_{s'}A_{l,t}(\beta)}{\int_l^t u_s D_s^S(X_lA_{l,t})ds} \diamond dW_s^S\right]\\
&= \mathbb{E}\left[\frac{G_{X_lA_{l,t}}(\beta)}{\sqrt{1-\rho^2}X_l(t-l)}\int_l^t \frac{dW_s^S}{\sqrt{V_s}}\right],
\end{aligned}$$

allowing us to rewrite (3.11) as

$$\mathbb{E}\left[A_{l,t}\Psi(X_l)G'_{X_lA_{l,t}}(\beta)\right] = \mathbb{E}\left[\frac{\Psi(X_l)G_{X_lA_{l,t}}(\beta)}{\sqrt{1-\rho^2}(t-l)X_l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}}\right]. \qquad (3.12)$$

Combining (3.12) and (3.10), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\Psi'(X_l)G_{X_lA_{l,t}}(\beta)\right]\\
&\qquad= \mathbb{E}\left[\frac{\Psi(X_l)G_{X_lA_{l,t}}(\beta)}{X_l}\left(\frac{1}{l\sqrt{1-\rho^2}}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right] - \mathbb{E}\left[\frac{\Psi(X_l)G_{X_lA_{l,t}}(\beta)}{\sqrt{1-\rho^2}(t-l)X_l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}}\right],
\end{aligned}$$

as desired. □

Corollary 3.10 *Let* $X_t = X_l exp\left(\int_l^t rds + \int_l^t \sqrt{V_s}dW_s^S - \frac{1}{2}\int_l^t V_s ds\right)$ *for any* $0 \le l \le t$*, and for any* $\Psi \in C_b^{+\infty}(\mathbb{R})$*. We have,*

$$\mathbb{E}[\Psi'(X_l)G_{X_t}(\beta)] = \mathbb{E}\left[\Psi(X_l)\frac{G_{X_t}(\beta)}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right],$$

with $G_{X_t}(\beta) := \mathbb{E}[P_t(X_t, V_t)|V_l = \beta]$.

Proof We have

$$\mathbb{E}\left[\Psi'(X_l)G_{X_t}(\beta)\right] = \mathbb{E}\left[\frac{\Psi(X_l)G_{X_t}(\beta)}{X_l}\left(\frac{1}{l\sqrt{1-\rho^2}}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right] - \mathbb{E}\left[U_{l,t}\Psi(X_l)G'_{X_t}(\beta)\right].$$

Hence, according to Proposition 3.9, we get

$$\mathbb{E}\left[\Psi'(X_l)G_{X_t}(\beta)\right] = \mathbb{E}\left[\Psi(X_l)\frac{G_{X_t}(\beta)}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right].$$

□

The following Theorem gives the analytic expression of $\mathbb{E}\left[P_t(X_t,V_t)\,|\,(X_l=\gamma, V_l=\beta)\right]$.

Theorem 3.11 *Let* $X_t = X_l exp\left(\int_l^t r ds + \int_l^t \sqrt{V_s}dW_s^S - \frac{1}{2}\int_l^t V_s ds\right)$, *with* $0 \leq l \leq t$. *We have*

$$\mathbb{E}\left[P_t(X_t,V_t)\,|\,(X_l=\gamma, V_l=\beta)\right] = \frac{\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(G_{X_t}(\beta))\right]}{\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(1)\right]},$$

where

$$\Upsilon_{X_l}(g(X_t)) = \frac{g(X_t)}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)$$

where H *represents the Heaviside function.*

Proof From Corollary 3.10, for any $\Psi \in C_b^\infty(\mathbb{R})$, we have

$$\mathbb{E}\left[\Psi'(X_l)G_{X_t}(\beta)\right] = \mathbb{E}\left[\Psi(X_l)\Upsilon_{X_l}(G_{X_t}(\beta))\right],$$

with

$$\mathbb{E}[G_{X_t}(\beta)] = \mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(G_{X_t}(\beta))\right]$$

$$= \mathbb{E}\left[H(X_l-\gamma)\frac{G_{X_t}(\beta)}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right],$$

As in Theorem 3.8, we refer to Fournié et al. [12] to write

$$\mathbb{E}\left[P_t(X_t,V_t)\,|\,(X_l=\gamma, V_l=\beta)\right] = \frac{\mathbb{E}\left[H(X_l-\gamma)\frac{G_{X_t}(\beta)}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right]}{\mathbb{E}\left[H(X_l-\gamma)\frac{1}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right)\right]},$$

where $G_{X_t}(\beta)$ is calculated using Theorem 3.8, and the Malliavin weights $\Upsilon_{X_l}(G_{X_t}(\beta))$ and $\Upsilon_{X_l}(1)$ can be obtained from Corollary 3.10 for Ψ equal to the Heaviside function. Hence, the square integrable weight $\Upsilon_{X_l}(G_{X_t}(\beta))$ is defined as follows:

$$\Upsilon_{X_l}(G_{X_t}(\beta)) = \frac{G_{X_t}(\beta)}{X_l\sqrt{1-\rho^2}}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right).$$

The same reasoning is adopted to calculate the square integrable weight $\Upsilon_{X_l}(1)$:

$$\Upsilon_{X_l}(1) = \frac{1}{X_l\sqrt{1-\rho^2}}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right).$$

$\square$

Example 3.12 *For the SABR model where $r=0$ in (1.3) and $k_V = 0$ in (1.2). we have*

$$V_t = V_l e^{\eta(W_t - W_l)} = V_l e^{\eta\Delta_{(t-l)}} = V_0 e^{\eta\Delta_t}.$$

Using Theorem 3.8, for any $\beta > 0$ and any $0 \le l \le t$, we have

$$\mathbb{E}[P(X_t,V_t)\,|\,V_l = \beta] = \frac{\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(P(X_t,V_t))\right]}{\mathbb{E}\left[H(V_l-\beta)\Upsilon_{V_l}(1)\right]},$$

where

$$\Upsilon_{V_l}(P(X_t,V_t)) = \frac{P(X_t,V_t)}{V_l}\left(1 + \frac{r(\Delta_{(t-l)})}{\eta} - \frac{r(\Delta_l)}{\eta}\right),$$

and

$$\Upsilon_{V_l}(1) = \frac{1}{V_l}\left(1 + \frac{r(\Delta_{(t-l)})}{\eta} - \frac{r(\Delta_l)}{\eta}\right).$$

Thus,

$$\mathbb{E}[P(X_t,V_t)\,|\,V_l = \beta] = \frac{\mathbb{E}\left[H(V_l-\beta)\frac{P(X_t,V_t)}{V_l}\left(1 + \frac{r(\Delta_{(t-l)})}{\eta} - \frac{r(\Delta_l)}{\eta}\right)\right]}{\mathbb{E}\left[H(V_l-\beta)\frac{1}{V_l}\left(1 + \frac{r(\Delta_{(t-l)})}{\eta} - \frac{r(\Delta_l)}{\eta}\right)\right]}.$$

Using Theorem 3.11, this leads to

$$\mathbb{E}\left[P_t(X_t,V_t)\,|\,(X_l=\gamma, V_l=\beta)\right] = \frac{\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(G_{X_t}(\beta))\right]}{\mathbb{E}\left[H(X_l-\gamma)\Upsilon_{X_l}(1)\right]},$$

where

$$\Upsilon_{X_l}(G_{X_t}(\beta)) = \frac{1}{\sqrt{1-\rho^2}X_l}\frac{\mathbb{E}\left[H(V_l-\beta)\frac{P(X_t,V_t)}{V_l}\left(1 + \frac{r(\Delta_{(t-l)})}{\eta} - \frac{r(\Delta_l)}{\eta}\right)\right]}{\mathbb{E}\left[H(V_l-\beta)\frac{1}{V_l}\left(1 + \frac{r(\Delta_{(t-l)})}{\eta} - \frac{r(\Delta_l)}{\eta}\right)\right]} \times \left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right),$$

and

$$\Upsilon_{X_l}(1) = \frac{1}{\sqrt{1-\rho^2}X_l}\left(\frac{1}{l}\int_0^l \frac{dW_s^S}{\sqrt{V_s}} - \frac{1}{t-l}\int_l^t \frac{dW_s^S}{\sqrt{V_s}} + 1\right).$$

Example 3.13 (American option pricing) *Let $0 = t_0 < t_1 < \ldots < t_n = T$, be a discretization of the time interval $[0, T]$. The price of an American put option can be computed using the backward iterations*

$$\begin{cases} P_T = & \max\{K - S_T, 0\}, \\ P(S_{t_k}, V_{t_k}) = & \max\big\{ \max\{K - S_{t_k}, 0), \\ & e^{-r\frac{T}{n}} \mathbb{E}\left[P_{t_{k+1}}(X_{t_{k+1}}, V_{t_{k+1}}) \,|\, X_{t_k} = \gamma, V_{t_k} = \beta\right\} \big\},\ k = n-1, \ldots, 0. \end{cases}$$

The conditional expectation is then calculated using the Malliavin weights given in Theorem 3.11. Since no closed expression of the weights is available, they have to be approximated, using for instance Monte Carlo simulations.

4. Conclusion

We have explored the use of Malliavin calculus to develop expression of derivative continuation value, conditionally to the current stock price and volatility, the latter being not constant. More specifically, we were able to express such expectations in terms of Malliavin weights, allowing their use in financial derivatives as American options. The particularities of Malliavin calculus nevertheless limit its applicability, as revealed by the assumptions on the stochastic processes required by our developments. A future direction of research would be to explore to what extent we can alleviate them in order to obtain more general results, and if we can use our formulae as approximations when our assumptions do not hold. The efficiency of the method also has to be evaluated and compared to the alternative approaches. We plan to complete numerical experiments in a close future in the context of option pricing and Greeks computations, assessing the practical interest of Malliavin calculus, and if it allows to mitigate the limitations of alternative methods as regression and machine learning techniques. As the current work considers unidimensional processes uniquely, another extension would be to consider multidimensional processes, and how to take the dependencies into account in our computations.

Acknowledgements

This work has been partly supported by the Deanship of Scientific Research at Jouf University under grant No DSR-2021-03-03140, awarded to Mohamed KHARRAT, and by the Natural Sciences and Engineering Research Council Discovery Grant 2017-05798 hold by Fabian BASTIN.

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