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On extended interpolative single and multivalued F-contractions

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Abstract: The main objective of this paper is to study an extended interpolative single and multivalued Hardy-Rogers type F-contractions in complete metric spaces. We prove some fixed point theorems for such mappings. Further, we give an application to integral equations to verify our main results. The results presented in this paper improve the recent works of Karapinar et al. [12] and Mohammadi et al. [16].

Key words: Interpolation, F-contractions, fixed point, Hardy-Rogers type contractions

1. Introduction

A new concept of contractions obtained from the definition of the Kannan contraction under the aspect of interpolation was introduced by Karapinar [11] as follows:

Definition 1.1 *[11] Let (Q,ρ) be a metric space. The mapping $S:Q\to Q$ is called an interpolative Kannan type contraction, if there are constants $\alpha\in[0,1)$ and $c_1\in(0,1)$ such that*

$$\rho(Sp,Sq)\leq\alpha\left[\rho(p,Sp)\right]^{c_1}\left[\rho(q,Sq)\right]^{1-c_1},$$

for all $p,q\in Q\setminus Fix(S)$ where $Fix(S)=\{p\in Q:Sp=p\}$.

Karapinar et al. [12] showed that an interpolative Kannan type contraction has a fixed point in complete metric spaces.

After, Karapinar et al. [13] introduced the concept of interpolative Reich–Rus–Ciri´c type contractions as follows. They also proved that these contractions have a fixed point in partial metric spaces.

Definition 1.2 *[13] Let (Q,ρ) be a metric space. A mapping $S:Q\to Q$ is said to be an interpolative Reich–Rus–Ciri´c type contraction, if there are constants $\alpha\in[0,1)$ and $c_1,c_2\in(0,1)$ such that*

$$\rho(Sp,Sq)\leq\alpha\left[\rho(p,q)\right]^{c_1}\left[\rho(p,Sp)\right]^{c_2}\left[\rho(q,Sq)\right]^{1-c_1-c_2},$$

for all $p,q\in Q\setminus Fix(S)$.

Inspired by the definition above, the concept of interpolative Hardy-Rogers type contractions was given by Karapinar et al. [12] as follows and they showed that these contractions have a fixed point in complete metric spaces.

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Definition 1.3 *Let (Q,ρ) be a metric space. We say that the self-mapping $S:Q\to Q$ is an interpolative Hardy-Rogers type contraction if there exists $\alpha\in[0,1)$ and $c_1,c_2,c_3\in(0,1)$ with $c_1+c_2+c_3<1$, such that*

$$\rho(Sp,Sq)\le\alpha\left[\rho(p,q)\right]^{c_1}\left[\rho(p,Sp)\right]^{c_2}\left[\rho(q,Sq)\right]^{c_3}\left[\frac{\rho(p,Sq)+\rho(q,Sp)}{2}\right]^{1-c_1-c_2-c_3} \tag{1.1}$$

for all $p,q\in Q\setminus Fix(S)$.

For other studies related to interpolative type contraction, see [5–9, 15, 18]. On the other hand in 2012, Wardowski [20] gave the concept of F-contraction. Later, many authors have studied fixed point theorems for the class of F-contraction [1–4, 14, 17, 19].

Definition 1.4 *[10] Assume that $\mathcal{F}$ is the family of all functions. The mapping $F:\mathbb{R}^+\to\mathbb{R}$ with the following properties:*

- $(F1)$ *F is strictly increasing;*
- $(F2)$ *for each sequence $\{p_n\}\subset\mathbb{R}^+$ of positive numbers $\lim_{n\to\infty}p_n=0\Leftrightarrow\lim_{n\to\infty}F(p_n)=-\infty$;*
- $(F3)$ *there exists $m\in(0,1)$ such that $\lim_{p\to0^+}p^mF(p)=0$.*

Example 1.5 *[10] The following functions $F_i:\mathbb{R}^+\to\mathbb{R}, i=1,2,3$ belong to $\mathcal{F}$*

- *(i)* $F_1(p)=\ln\left(p^2+p\right)$, $p>0$;
- *(ii)* $F_2(p)=-\frac{1}{\sqrt[2]{p}}$, $p>0$;
- *(iii)* $F_3(p)=\ln p$, $p>0$.

Recently, Mohammadi et al. [16] gave the notation of extended interpolative Ćirić–Reich–Rus type F-contraction as follows and they obtained some fixed point results for such mappings.

Definition 1.6 *Let (Q,ρ) be a metric space. A mapping $S:Q\to Q$ is called an extended interpolative Ćirić–Reich–Rus type F-contraction if there exists $c_1,c_2\in[0,1)$ with $c_1+c_2<1$, $\tau>0$ and $F\in\mathcal{F}$ such that*

$$\tau+F(\rho(Sp,Sq))\le c_1F(\rho(p,q))+c_2F(\rho(p,Sp))+(1-c_1-c_2)F(\rho(q,Sq)) \tag{1.2}$$

for all $p,q\in Q\setminus Fix(S)$ where $Fix(S)=\{p\in Q:Sp=p\}$ with $\rho(Sp,Sq)>0$.

Inspired by the above studies and the approach of Wardowski [20], we consider extended interpolative single and multivalued Hardy-Rogers type F-contractions in complete metric space. We prove some fixed point results for such mappings. Further, we provide a nontrivial application to integral equations to illustrate the impact of our main results.

2. Main results

In this section, firstly we give the notation of extended interpolative Hardy-Rogers type F-contractions as follows.

Definition 2.1 *Let (Q,ρ) be a metric space. A mapping $S:Q\to Q$ is called an extended interpolative Hardy-Rogers type F-contraction if there exists $c_1,c_2,c_3\in[0,1)$ with $c_1+c_2+c_3<1$, $\tau>0$ and $F\in\mathcal{F}$ such that*

$$\begin{aligned}\tau+F\left(\rho\left(Sp,Sq\right)\right) \quad&\leq\quad c_1F\left(\rho\left(p,q\right)\right)+c_2F\left(\rho\left(p,Sp\right)\right)+c_3F\left(\rho\left(q,Sq\right)\right)\\&\quad\quad+\left(1-c_1-c_2-c_3\right)F\left(\frac{\rho\left(p,Sq\right)+\rho\left(q,Sp\right)}{2}\right)\end{aligned}\tag{2.1}$$

for all $p,q\in Q\setminus Fix(S)$ where $Fix(S)=\{p\in Q:Sp=p\}$ with $\rho\left(Sp,Sq\right)>0$.

Theorem 2.2 *Let (Q,ρ) be a complete metric space and $S:Q\to Q$ be an extended interpolative Hardy-Rogers type F-contraction. Then S has a fixed point in Q.*

Proof Let $\{p_n\}$ be a Picard sequence given as $p_n=Sp_{n-1}=Sp_0$ for starting point $p_0\in Q$. If we take $p_n=p_{n-1}$ for some $n\in\mathbb{N}$, we get that p_{n-1} is a fixed point of S. This implies that the existence of a fixed point of S is clear. Now we assume that $p_n\neq p_{n-1}$ for all $n\in\mathbb{N}$. Using (2.1) with $p=p_n$ and $q=p_{n-1}$ for all $n\in\mathbb{N}$, we have that

$$\begin{aligned}\tau+F\left(\rho(p_{n+1},p_n)\right) \quad&=\quad \tau+F\left(\rho(Sp_n,Sp_{n-1})\right)\leq c_1F\left(\rho\left(p_n,p_{n-1}\right)\right)+c_2F\left(\rho\left(p_n,Sp_n\right)\right)\\&\quad\quad+c_3F\left(\rho\left(p_{n-1},Sp_{n-1}\right)\right)+\left(1-c_1-c_2-c_3\right)F\left(\frac{\rho\left(p_n,Sp_{n-1}\right)+\rho\left(p_{n-1},Sp_n\right)}{2}\right)\\&=\quad c_1F\left(\rho\left(p_n,p_{n-1}\right)\right)+c_2F\left(\rho\left(p_n,p_{n+1}\right)\right)+c_3F\left(\rho\left(p_{n-1},p_n\right)\right)\\&\quad\quad+\left(1-c_1-c_2-c_3\right)F\left(\frac{\rho\left(p_{n-1},p_{n+1}\right)}{2}\right)\\&\leq\quad \left(c_1+c_3\right)F\left(\rho\left(p_n,p_{n-1}\right)\right)+c_2F\left(\rho\left(p_n,p_{n+1}\right)\right)\\&\quad\quad+\left(1-c_1-c_2-c_3\right)F\left(\frac{\rho\left(p_{n-1},p_n\right)+\rho\left(p_n,p_{n+1}\right)}{2}\right).\end{aligned}\tag{2.2}$$

Suppose that $\rho\left(p_{n-1},p_n\right)<\rho(p_n,p_{n+1})$ for some $n\geq1$. From (2.2), we have

$$\begin{aligned}\tau+F\left(\rho(p_{n+1},p_n)\right) \quad&\leq\quad \left(c_1+c_3\right)F\left(\rho\left(p_n,p_{n+1}\right)\right)+c_2F\left(\rho\left(p_n,p_{n+1}\right)\right)\\&\quad\quad+\left(1-c_1-c_2-c_3\right)F\left(\rho\left(p_n,p_{n+1}\right)\right)\\&=\quad F\left(\rho(p_{n+1},p_n)\right)\end{aligned}\tag{2.3}$$

which is a contradiction. Then, we have $\rho(p_n,p_{n+1})\leq\rho\left(p_{n-1},p_n\right)$ for all $n\geq1$. Again from (2.2), we obtain

$$\tau+F\left(\rho(p_{n+1},p_n)\right)\leq F\left(\rho(p_n,p_{n-1})\right).$$

Thus,

$$F\left(\rho(p_{n+1},p_n)\right)\leq F\left(\rho(p_n,p_{n-1})\right)-\tau\leq\cdots\leq F\left(\rho(p_1,p_0)\right)-n\tau\tag{2.4}$$

for all $n\geq1$. Taking limit as $n\to\infty$ on both sides of the inequality (2.4), we obtain that

$$\lim_{n\to\infty}F\left(\rho(p_{n+1},p_n)\right)=-\infty.\tag{2.5}$$

Using (2.5) and $(F2)$, we also get

$$\lim_{n\to\infty} \rho(p_{n+1}, p_n) = 0. \tag{2.6}$$

Denote $\theta_n = \rho(p_{n+1}, p_n)$ for all $n \in \mathbb{N}$. Then, from (2.6) we know that $\lim_{n\to\infty} \theta_n = 0$. From $(F3)$, there exists $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} \theta_n^\lambda F(\theta_n) = 0.$$

It follows from (2.4) that

$$\theta_n^\lambda F(\theta_n) - \theta_n^\lambda F(\theta_0) \le -\theta_n^\lambda n\tau \le 0. \tag{2.7}$$

Taking limit as $n \to \infty$ on above inequality, we get

$$\lim_{n\to\infty} n\theta_n^\lambda = 0. \tag{2.8}$$

From (2.8), there exists $n_1 \in \mathbb{N}$ such that $n\theta_n^\lambda \le 1$ for all $n \ge n_1$. So, we have

$$\theta_n \le \frac{1}{n^{\frac{1}{\lambda}}}, \tag{2.9}$$

for all $n \ge 1$. Now, we will show that $\{p_n\}$ is a Cauchy sequence. For this, we consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. Using (2.9) and the triangular inequality, we get

$$\begin{aligned} \rho(p_n, p_m) &\le \rho(p_n, p_{n+1}) + \rho(p_{n+1}, p_{n+2}) + \cdots + \rho(p_{m-1}, p_m) \\ &= \theta_n + \theta_{n+1} + \cdots + \theta_{m-1} \\ &= \sum_{i=n}^{m-1} \theta_i \\ &\le \sum_{i=n}^{\infty} \theta_i \\ &\le \sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{\lambda}}}. \end{aligned} \tag{2.10}$$

If we take limit as $n \to \infty$ in the inequality (2.10), we obtain that $\lim_{m,n\to\infty} \rho(p_n, p_m) = 0$. This shows that $\{p_n\}$ is a Cauchy sequence in Q. Since (Q, ρ) is a complete metric space, there exists $p^* \in Q$ such that $p_n \to p^* \in Q$ as $n \to \infty$. Now, we will show that p^* is a fixed point of S. Suppose to the contrary $p^* \ne Sp^*$. We will consider the following two cases:

Case 1: There is a subsequence $\{p_{n_t}\}$ of the sequence $\{p_n\}$ such that $Sp_{n_t} = Sp^*$ for all $n \ge n_1$. Then,

$$\rho(p^*, Sp^*) = \lim_{t\to\infty} \rho(p_{n_t+1}, Sp^*) = \lim_{t\to\infty} \rho(Sp_{n_t}, Sp^*) = 0.$$

Case 2: There is a natural number n_1 such that $Sp_n \ne Sp^*$ for all $n \ge n_1$. Applying (2.1), for $p = p_n$

and $q = p^*$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned}\tau + F\left(\rho\left(p_{n+1}, Sp^*\right)\right) &= \tau + F\left(\rho\left(Sp_n, Sp^*\right)\right) \leq c_1 F\left(\rho\left(p_n, p^*\right)\right) + c_2 F\left(\rho\left(p_n, Sp_n\right)\right) \\ &\quad + c_3 F\left(\rho\left(p^*, Sp^*\right)\right) + \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_n, Sp^*\right) + \rho\left(p^*, Sp_n\right)}{2}\right) \qquad (2.11) \\ &= c_1 F\left(\rho\left(p_n, p^*\right)\right) + c_2 F\left(\rho\left(p_n, p_{n+1}\right)\right) \\ &\quad + \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_n, p^*\right) + \rho\left(p^*, p_{n+1}\right)}{2}\right).\end{aligned}$$

Taking limit as $n \to \infty$ in the inequality (2.11), we find that $\lim_{n\to\infty} F\left(\rho\left(p_{n+1}, Sp^*\right)\right) = -\infty$ and so $\lim_{n\to\infty} \rho\left(p_{n+1}, Sp^*\right) = 0$. Therefore,

$$\rho\left(p^*, Sp^*\right) = \lim_{n\to\infty} \rho\left(p_{n+1}, Sp^*\right) = 0.$$

This implies that $\rho\left(p^*, Sp^*\right) = 0$ and so $Sp^* = p^*$. □

Remark 2.3 *(i) If we take $c_3 = 0$ in Theorem 2.2, we obtain that Theorem 2.2 in [16].*

(ii) If we take $F(p) = \ln p$ (for $p > 0$) in Theorem 2.2, the inequality (2.1) reduces to the following inequality;

$$\rho\left(Sp, Sq\right) \leq e^{-\tau} \left[\rho\left(p, q\right)\right]^{c_1} \left[\rho\left(p, Sp\right)\right]^{c_2} \left[\rho\left(q, Sq\right)\right]^{c_3} \left[\frac{\rho\left(p, Sq\right) + \rho\left(q, Sp\right)}{2}\right]^{1-c_1-c_2-c_3} \qquad (2.12)$$

for all $p, q \in Q \backslash Fix(S)$. Then, the contraction (2.12) implies the contraction (1.1). So, Theorem 2.2 generalizes Theorem 4 in [12] to extended interpolative F-contraction type mappings.

Next, we will consider Theorem 2.2 for the multivalued mappings. For this, we recall the following concepts about multivalued mappings.

Let (Q, ρ) be a metric space and let $CB(Q)$ be the collection of all nonempty bounded and closed subsets of Q. Let H be a Hausdorff metric induced by the metric ρ of Q, that is

$$H(Q_1, Q_2) = max\left\{\sup_{p\in Q_1} \rho(p, Q_2), \sup_{q\in Q_2} \rho(q, Q_1)\right\},$$

for every $Q_1, Q_2 \in CB(Q)$ with

$$\rho(p, Q_2) = inf\left\{\rho(p, q) : q \in Q_2\right\}.$$

Let S be a mapping from Q to $CB(Q)$. If $p \in Sp$, then $p \in Q$ is called a fixed point of multivalued mapping S.

Now, we will give the definition of multivalued version of extended interpolative Hardy-Rogers type F-contraction as follows:

Definition 2.4 *Let (Q, ρ) be a metric space. A mapping $S : Q \to CB(Q)$ is called an extended interpolative multivalued Hardy-Rogers type F-contraction if there exists $c_1, c_2, c_3 \in [0, 1)$ with $c_1 + c_2 + c_3 < 1$, $\tau > 0$ and*

$F \in \mathcal{F}$ such that

$$\begin{aligned}\tau + F\left(H\left(Sp, Sq\right)\right) \quad &\leq \quad c_1 F\left(\rho\left(p, q\right)\right) + c_2 F\left(\rho\left(p, Sp\right)\right) + c_3 F\left(\rho\left(q, Sq\right)\right) \\ &+ \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p, Sq\right) + \rho\left(q, Sp\right)}{2}\right)\end{aligned} \tag{2.13}$$

for all $p, q \in Q \setminus Fix(S)$ with $H\left(Sp, Sq\right) > 0$.

Theorem 2.5 *Let (Q, ρ) be a complete metric space and $S : Q \to Q$ be an extended interpolative multivalued Hardy-Rogers type F-contraction. Suppose that the following condition is fulfilled;*

$$(W): \ F\left(\inf U\right) = \inf F\left(U\right).$$

Then S has a fixed point in Q.

Proof Let $p_0 \in Q$ be a starting point, $p_1 \in Sp_0$. If we take $p_1 \in Sp_1$ and $p_0 \in Sp_0$, we obtain that p_0 and p_1 are fixed points of S. We suppose that $p_1 \notin Sp_1$ and $p_0 \notin Sp_0$. This implies that $Sp_0 \neq Sp_1$. Taking $p = p_0$ and $q = p_1$ in the inequality (2.13), we have that

$$\begin{aligned}\frac{\tau}{3} + F\left(\rho(p_1, Sp_1)\right) \quad &< \quad \tau + F\left(H(Sp_0, Sp_1)\right) \leq c_1 F\left(\rho\left(p_0, p_1\right)\right) + c_2 F\left(\rho\left(p_0, Sp_0\right)\right) \\ &+ c_3 F\left(\rho\left(p_1, Sp_1\right)\right) + \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_0, Sp_1\right) + \rho\left(p_1, Sp_0\right)}{2}\right) \\ &\leq \quad c_1 F\left(\rho\left(p_0, p_1\right)\right) + c_2 F\left(\rho\left(p_0, p_1\right)\right) + c_3 F\left(\rho\left(p_1, Sp_1\right)\right) \\ &+ \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_0, p_1\right) + \rho\left(p_1, Sp_1\right)}{2}\right) \\ &\leq \quad c_1 F\left(\rho\left(p_0, p_1\right)\right) + c_2 F\left(\rho\left(p_0, p_1\right)\right) + c_3 F\left(\rho\left(p_1, Sp_1\right)\right) \\ &+ \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_0, p_1\right) + \rho\left(p_1, Sp_1\right)}{2}\right).\end{aligned} \tag{2.14}$$

If we take $\rho\left(p_0, p_1\right) < \rho(p_1, Sp_1)$, from (2.14), we get

$$\begin{aligned}\frac{\tau}{3} + F\left(\rho(p_1, Sp_1)\right) \quad &\leq \quad c_1 F\left(\rho(p_1, Sp_1)\right) + c_2 F\left(\rho(p_1, Sp_1)\right) + c_3 F\left(\rho\left(p_1, Sp_1\right)\right) \\ &+ \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho(p_1, Sp_1) + \rho\left(p_1, Sp_1\right)}{2}\right) \\ &= \quad F\left(\rho(p_1, Sp_1)\right),\end{aligned}$$

which is a contradiction. So, we have that $\rho(p_1, Sp_1) \leq \rho\left(p_0, p_1\right)$. Again from (2.14), we obtain

$$\frac{\tau}{3} + F\left(\rho(p_1, Sp_1)\right) \leq F\left(\rho\left(p_0, p_1\right)\right).$$

From the condition (W) and above inequality, we obtain that $p_2 \in Sp_1$ such that

$$\frac{\tau}{3} + F\left(\rho(p_1, p_2)\right) < \frac{\tau}{3} + F\left(\rho(p_1, Sp_1)\right) \leq F\left(\rho\left(p_0, p_1\right)\right).$$

If we continue similar process, we can find that the sequence $\{p_n\}$ in W so that $p_{n+1} \in Sp_n, p_n \notin Sp_n$ and

$$\frac{\tau}{3} + F\left(\rho(p_n, p_{n+1})\right) \leq F\left(\rho\left(p_{n-1}, p_n\right)\right).$$

for all $n \geq 1$. Taking n_0 such that $p_{n_0} = p_{n_0+1}$, we obtain that p_{n_0} is a fixed point of S. Assume that $p_{n_0} \neq p_{n_0+1}$. Then

$$F\left(\rho(p_n, p_{n+1})\right) \leq F\left(\rho\left(p_{n-1}, p_n\right)\right) - \frac{\tau}{3} \leq \cdots \leq F\left(\rho\left(p_0, p_1\right)\right) - n\frac{\tau}{3}.$$

Using similar method in proof of Theorem 2.2, we get that $\{p_n\}$ is a Cauchy sequence. Assume that $p_n \to p^*$ and $p^* \notin Sp^*$. Now, we consider two cases:

Case 1: There is a subsequence $\{p_{n_t}\}$ of the sequence $\{p_n\}$ such that $Sp_{n_t} = Sp^*$ for all $n \geq n_1$. Then,

$$\rho\left(p^*, Sp^*\right) = \lim_{t\to\infty} \rho\left(p_{n_t+1}, Sp^*\right) = \lim_{t\to\infty} H\left(Sp_{n_t}, Sp^*\right) = 0.$$

Case 2: There is a natural number n_1 such that $Sp_n \neq Sp^*$ for all $n \geq n_1$. Applying (2.1), for $p = p_n$ and $q = p^*$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned}
\tau + F\left(\rho\left(p_{n+1}, Sp^*\right)\right) &\leq \tau + F\left(H\left(Sp_n, Sp^*\right)\right) \leq c_1 F\left(\rho\left(p_n, p^*\right)\right) + c_2 F\left(\rho\left(p_n, Sp_n\right)\right) \\
&\quad + c_3 F\left(\rho\left(p^*, Sp^*\right)\right) + \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_n, Sp^*\right) + \rho\left(p^*, Sp_n\right)}{2}\right) \qquad (2.15) \\
&= c_1 F\left(\rho\left(p_n, p^*\right)\right) + c_2 F\left(\rho\left(p_n, p_{n+1}\right)\right) \\
&\quad + \left(1 - c_1 - c_2 - c_3\right) F\left(\frac{\rho\left(p_n, p^*\right) + \rho\left(p^*, p_{n+1}\right)}{2}\right).
\end{aligned}$$

Taking limit as $n \to \infty$ in the inequality (2.15), we find that $\lim_{n\to\infty} F\left(\rho\left(p_{n+1}, Sp^*\right)\right) = -\infty$ and so $\lim_{n\to\infty} \rho\left(p_{n+1}, Sp^*\right) = 0$. Therefore,

$$\rho\left(p^*, Sp^*\right) = \lim_{n\to\infty} \rho\left(p_{n+1}, Sp^*\right) \leq \lim_{n\to\infty} F\left(H\left(Sp_n, Sp^*\right)\right) = 0.$$

This implies that $\rho\left(p^*, Sp^*\right) = 0$ and so $p^* \in Sp^*$. That is, S has a fixed point in Q. □

2.1. Application

Problems in physics, engineering, and many other fields can be modeled with linear differential equations and integral equations. Fixed point theorems can be used while showing the existence and uniqueness of the solutions of such equations. Now we provide an application of our results to the solution of integral equations:

$$p(t) = r(t) + \int_0^s W(t,u) h\left(u, p(u)\right) du \quad, t \in [0, s] \qquad (2.16)$$

where

(i) $r : J \to \mathbb{R}$ and $h : J \times \mathbb{R} \to \mathbb{R}$ are continuous for all $t \in J = [0, s]$,

(ii) $W : J \times J \to \mathbb{R}$ is continuous and measurable at $u \in I$ for all $t \in J$,

(iii) $W(t,u) \geq 0$ for all $t,u \in J$ and $\int_0^s W(t,u)du \leq 1$ for all $t \in J$.

Let $Q = C(J,\mathbb{R})$ be the set of all real valued continuous functions and

$$\rho(p,q) = \sup\{|p(t)-q(t)| : t \in J\} = \|p-q\|.$$

Theorem 2.6 *Assume that the conditions (i)-(iii) hold. Suppose that there are $\tau > 0$ and $c_1, c_2, c_3 \in (0,1)$ with $c_1 + c_2 + c_3 < 1$ such that*

$$\begin{aligned} &|h(t,p(t)) - h(t,q(t))| \\ \leq\ & \frac{|p(t)-q(t)|}{\left[\begin{array}{c} \tau\|p-q\| + c_1 + c_2 \sqrt[2]{\frac{\|p-q\|}{\left\|p-\int_0^s W(t,u)h(u,p(u))du\right\|}} + c_3 \sqrt[2]{\frac{\|p-q\|}{\left\|q-\int_0^s W(t,u)h(u,q(u))du\right\|}} \\ +(1-c_1-c_2-c_3)\sqrt[2]{\frac{2\|p-q\|}{\left\|p-\int_0^s W(t,u)h(u,q(u))du\right\|+\left\|q-\int_0^s W(t,u)h(u,p(u))du\right\|}} \end{array}\right]^2} \end{aligned} \tag{2.17}$$

for each $t \in J$ and for all $p,q \in C(J,\mathbb{R})$ such that

$$\begin{aligned} p(t) &\neq \int_0^s W(t,u)h(u,p(u))\,du, \\ q(t) &\neq \int_0^s W(t,u)h(u,q(u))\,du, \end{aligned}$$

and

$$\int_0^s W(t,u)h(u,p(u))\,du \neq \int_0^s W(t,u)h(u,q(u))\,du.$$

Then the integral equation (2.16) has a solution in $C(J,\mathbb{R})$.

Proof Define the mapping $S : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as

$$Sp(t) = r(t) + \int_0^s W(t,u)h(u,p(u))\,du \tag{2.18}$$

for $t \in [0,s]$. From (2.17) and (2.18), we get

$$\begin{aligned}
|Sp(t)-Sq(t)| &= \left|\int_0^s W(t,u)\left(h\left(u,p(u)\right)-h\left(u,q(u)\right)\right)du\right| \\
&\leq \int_0^s W(t,u)\left|h\left(u,p(u)\right)-h\left(u,q(u)\right)\right|du \\
&\leq \int_0^s \frac{W(t,u)\left|p(t)-q(t)\right|}{\left[\begin{array}{c}\tau\left\|p-q\right\|+c_1+c_2\sqrt[2]{\frac{\|p-q\|}{\|p-Sp\|}}+c_3\sqrt[2]{\frac{\|p-q\|}{\|q-Sq\|}} \\ +\left(1-c_1-c_2-c_3\right)\sqrt[2]{\frac{2\|p-q\|}{\|p-Sq\|+\|q-Sp\|}}\end{array}\right]^2}du \\
&\leq \left[\begin{array}{c}\tau\left\|p-q\right\|+c_1+c_2\sqrt[2]{\frac{\|p-q\|}{\|p-Sp\|}}+c_3\sqrt[2]{\frac{\|p-q\|}{\|q-Sq\|}} \\ +\left(1-c_1-c_2-c_3\right)\sqrt[2]{\frac{2\|p-q\|}{\|p-Sq\|+\|q-Sp\|}}\end{array}\right]^2\int_0^s W(t,u)du \\
&\leq \left[\begin{array}{c}\tau\left\|p-q\right\|+c_1+c_2\sqrt[2]{\frac{\|p-q\|}{\|p-Sp\|}}+c_3\sqrt[2]{\frac{\|p-q\|}{\|q-Sq\|}} \\ +\left(1-c_1-c_2-c_3\right)\sqrt[2]{\frac{2\|p-q\|}{\|p-Sq\|+\|q-Sp\|}}\end{array}\right]^2.
\end{aligned}$$

Taking supremum in above inequality, we get

$$\begin{aligned}
\rho\left(Sp,Sq\right) &= \left\|Sp-Sq\right\| \leq \left[\begin{array}{c}\tau\left\|p-q\right\|+c_1+c_2\sqrt[2]{\frac{\|p-q\|}{\|p-Sp\|}}+c_3\sqrt[2]{\frac{\|p-q\|}{\|q-Sq\|}} \\ +\left(1-c_1-c_2-c_3\right)\sqrt[2]{\frac{2\|p-q\|}{\|p-Sq\|+\|q-Sp\|}}\end{array}\right] \\
&= \frac{\rho\left(p,q\right)}{\left[\begin{array}{c}\tau\rho\left(p,q\right)+c_1+c_2\sqrt[2]{\frac{\rho(p,q)}{\rho(p,Sp)}}+c_3\sqrt[2]{\frac{\rho(p,q)}{\rho(q,Sq)}} \\ +\left(1-c_1-c_2-c_3\right)\sqrt[2]{\frac{2\rho(p,q)}{\rho(p,Sq)+\rho(q,Sp)}}\end{array}\right]^2}.
\end{aligned} \tag{2.19}$$

Using (2.19), we write

$$\begin{aligned}
\frac{1}{\sqrt[2]{\rho\left(Sp,Sq\right)}} &\geq \tau+c_1\left(\frac{1}{\sqrt[2]{\rho\left(p,q\right)}}\right)+c_2\left(\frac{1}{\sqrt[2]{\rho\left(p,Sp\right)}}\right)+c_3\left(\frac{1}{\sqrt[2]{\rho\left(q,Sq\right)}}\right) \\
&\quad +\left(1-c_1-c_2-c_3\right)\left(\frac{\sqrt[2]{2}}{\sqrt[2]{\rho\left(p,Sq\right)+\sqrt[2]{\rho\left(q,Sp\right)}}}\right)
\end{aligned}$$

which implies that

$$\begin{aligned}
\tau+\left(\frac{-1}{\sqrt[2]{\rho\left(Sp,Sq\right)}}\right) &\leq c_1\left(\frac{-1}{\sqrt[2]{\rho\left(p,q\right)}}\right)+c_2\left(\frac{-1}{\sqrt[2]{\rho\left(p,Sp\right)}}\right)+c_3\left(\frac{-1}{\sqrt[2]{\rho\left(q,Sq\right)}}\right) \\
&\quad +\left(1-c_1-c_2-c_3\right)\left(\frac{-\sqrt[2]{2}}{\sqrt[2]{\rho\left(p,Sq\right)+\sqrt[2]{\rho\left(q,Sp\right)}}}\right).
\end{aligned}$$

Then,

$$\begin{aligned}\tau+\left(\frac{-1}{\sqrt[2]{\rho\left(Sp,Sq\right)}}\right) &\leq c_1\left(\frac{-1}{\sqrt[2]{\rho\left(p,q\right)}}\right)+c_2\left(\frac{-1}{\sqrt[2]{\rho\left(p,Sp\right)}}\right)+c_3\left(\frac{-1}{\sqrt[2]{\rho\left(q,Sq\right)}}\right)\\ &\quad+\left(1-c_1-c_2-c_3\right)\left(\frac{-\sqrt[2]{2}}{\sqrt[2]{\rho\left(p,Sq\right)+\sqrt[2]{\rho\left(q,Sp\right)}}}\right).\end{aligned}$$

If we take $F(t)=-\frac{1}{\sqrt[2]{t}}$, we obtain

$$\begin{aligned}\tau+F\left(\rho\left(Sp,Sq\right)\right) &\leq c_1F\left(\rho\left(p,q\right)\right)+c_2F\left(\rho\left(p,Sp\right)\right)+c_3F\left(\rho\left(q,Sq\right)\right)\\ &\quad+\left(1-c_1-c_2-c_3\right)F\left(\frac{\rho\left(p,Sq\right)+\rho\left(q,Sp\right)}{2}\right)\end{aligned}$$

for all $p,q\in Q\setminus Fix(S)$ with $\rho\left(Sp,Sq\right)>0$. That is, the conditions of Theorem 2.2 are held. So, the mapping S has a fixed point. Thus, the integral equation (2.16) has a solution. □

3. Conclusion

The purpose of this paper was to introduce extended interpolative single and multivalued Hardy-Rogers type F-contractions in complete metric spaces. We gave some fixed point results for classes of such mappings. We also provided an application to illustrate the main result in this paper.

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