The Arens-Michael envelopes of Laurent Ore extensions

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Abstract: For an Arens-Michael algebra A we consider a class of A-$\hat{\otimes}$-bimodules which are invertible with respect to the projective bimodule tensor product. We call such bimodules topologically invertible over A. Given a Fréchet-Arens-Michael algebra A and a topologically invertible Fréchet A-$\hat{\otimes}$-bimodule M, we construct an Arens-Michael algebra $\widehat{L}_A(M)$ which serves as a topological version of the Laurent tensor algebra $L_A(M)$. Also, for a fixed algebra B we provide a condition on an invertible B-bimodule N which allows us to explicitly describe the Arens-Michael envelope of $L_B(N)$ as a topological Laurent tensor algebra. In particular, we provide an explicit description of the Arens-Michael envelope of an invertible Ore extension $A[x, x^{-1}; \alpha]$ for a metrizable algebra A.

Key words: Arens-Michael envelopes, topological bimodules, locally convex algebras, Ore extensions

1. Introduction

We would like to begin the paper by demonstrating a connection between Arens-Michael envelopes and noncommutative geometry.

Noncommutative geometry is a branch of mathematics which, in particular, arose from such fundamental results as the first Gelfand-Naimark theorem or the Nullstellensatz, or, more precisely, their categorical interpretations:

Theorem 1.1 (the first Gelfand-Naimark theorem) *Denote the category of commutative unital C^*-algebras by $\mathbf{CUC}^*$ and the category of compact Hausdorff topological spaces by $\mathbf{Comp}$. Then the pair of functors $\mathcal{F} : \mathbf{Comp} \to \mathbf{CUC}^*$ and $\mathcal{G} : \mathbf{CUC}^* \to \mathbf{Comp}$, where $\mathcal{F}(X) = C(X)$ and $\mathcal{G}(A) = Spec_m(A)$, is an antiequivalence of categories.*

Theorem 1.2 (Nullstellensatz) *Let $\mathbb{K}$ be an algebraically closed field. Denote the category of affine algebraic $\mathbb{K}$-varieties by $\mathbf{Aff}$ and the category of commutative finitely generated reduced unital $\mathbb{K}$-algebras by $\mathbf{Alg}$. Then a pair of functors $\mathcal{F} : \mathbf{Aff} \to \mathbf{Alg}$ and $\mathcal{G} : \mathbf{Alg} \to \mathbf{Aff}$, where $\mathcal{F}(X) = \mathbb{K}[X]$ and $\mathcal{G}(A) = Spec_m(A)$, is an antiequivalence of categories.*

As the reader can see, these theorems state that some categories of geometrical objects are antiequivalent to the category of algebras of functions on them. This observation, for example, serves as a motivation to think of noncommutative C^*-algebras as the function spaces on "noncommutative topological spaces".

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The notion of an Arens-Michael envelope was discovered by J. Taylor in [11] due to the problem of a multioperator functional calculus existence. It is worth noting that the terminology the author used was different from that we use nowadays: Taylor defined the Arens-Michael envelopes as "completed locally multiplicative convex envelopes". The current terminology is due to A. Helemskii, see [4].

The following theorem stands as a reason to study Arens-Michael envelopes in the context of noncommutative geometry.

Theorem 1.3 *The Arens-Michael envelope of* $\mathbb{C}[t_1, \ldots, t_n]$ *is topologically isomorphic to the algebra of holomorphic functions* $\mathcal{O}(\mathbb{C}^n)$ *endowed with the compact-open topology.*

This fact, discovered by J. Taylor, can be formulated as follows: the Arens-Michael envelope of the algebra of regular functions on $\mathbb{C}^n$ is isomorphic to the algebra of holomorphic functions on $\mathbb{C}^n$. In fact, the same holds true for an arbitrary affine complex algebraic variety, see [8, Example 3.6]. Therefore, it makes sense to define the algebra of "holomorphic functions" on a noncommutative affine algebraic variety as the Arens-Michael envelope of the algebra of "regular functions" on it. In other words, the notion of Arens-Michael envelope serves as a "bridge" between algebra and functional analysis.

In this paper we are concerned with "computing" the Arens-Michael envelopes for some interesting noncommutative associative finitely generated algebras over $\mathbb{C}$. By this, we mean that, for an algebra A, we aim to explicitly construct the Arens-Michael algebra B which turns out to be isomorphic to the Arens-Michael envelope $\widehat{A}$. In most (nondegenerate) cases, such algebra B is constructed as a power series algebra. In other words, the underlying locally convex space of B turns out to be a Köthe space.

Suppose that A is an algebra with an endomorphism $\alpha \in \mathrm{End}(A)$ and an α-derivation $\delta : A \to A$. Then, under some reasonable conditions on α and δ, the Arens-Michael envelope of its Ore extension $A[t; \alpha, \delta]$ admits a description in terms of the Arens-Michael envelope $\widehat{A}$.

A lot of naturally occurring noncommutative algebras can be represented as iterated Ore extensions, for example, q-deformations of classical algebras, such as $\mathrm{Mat}_q(2)$ or $U_q(\mathfrak{g})$.

Let A be a unital associative complex algebra, $\alpha \in \mathrm{End}(A)$ and $\delta : A \to A$ be an α-derivation. Consider the Ore extension $A[t; \alpha, \delta]$. Then there are several cases:

1. Suppose that the pair (α, δ) is "nice enough" in the sense that their extensions to $\widehat{A}$ behave well enough with respect to the topology on $\widehat{A}$ (m-localizable families of morphisms). Then the Arens-Michael envelope of $A[t; \alpha, \delta]$ admits a relatively simple description, see [8, Proposition 4.5] and [8, Theorem 5.17].

   Now suppose that $\delta = 0$ for simplicity.

2. Consider now the case of general $\alpha \in \mathrm{End}(A)$. Then $A[t; \alpha]$ still admits an explicit description of the Arens-Michael envelope, which utilizes an analytic version of the notion of tensor algebra associated with a bimodule. However, the seminorms which were used to describe the topology on $A\{t; \alpha\} \simeq \widehat{A[t; \alpha]}$ are difficult to compute in the general case, see [8, Proposition 4.9], [8, Corollary 5.6] and [8, Example 4.3].

3. Now suppose that α is invertible. Then one can define the Laurent Ore extension $A[t, t^{-1}; \alpha]$. If the pair (α, α^{-1}) is "nice enough", then the Arens-Michael envelope admits a description similar to the one in Case 1, see [8, Proposition 4.15] and [8, Theorem 5.21].

4. In this paper we treat the case of arbitrary $\alpha \in \text{Aut}(A)$, in other words, we do not assume that the pair (α, α^{-1}) is (m)-localizable. It is worth mentioning that the approach is inspired by methods used in [8]. In particular, we introduce an analytic version of the Laurent tensor algebra associated with a topologically invertible bimodule.

5. The most general case, $A[t; \alpha, \delta]$, is still out of reach, unfortunately.

Let us present the main results of this paper. For a $\hat{\otimes}$-algebra A and an A-$\hat{\otimes}$-module M let us denote their Arens-Michael envelopes by $\widehat{A}$ and $\widehat{M}$, respectively (see Definition 2.5 and Definition 2.9).

(1) First of all, we define the notion of topologically invertible $\hat{\otimes}$-bimodules, and we utilize several ideas from [9] in order to understand the following problem: if M and M^{-1} are invertible A-modules, is it true that $\widehat{M}$ and $\widehat{M^{-1}}$ are topologically invertible over $\widehat{A}$? We prove that this is true in a particular case (see Theorem 3.11), but the general case remains open.

(2) Given a pair of topologically inverse $\hat{\otimes}$-bimodules $\tilde{M}, \tilde{M}^{-1}$ over a $\hat{\otimes}$-algebra $\tilde{A}$, we define the topological version of the Laurent tensor algebra via universal property, similar to the algebraic case. We manage to prove that it exists (see Theorem 3.14) and it is unique in the metrizable setting. Moreover, we show that if A is an associative algebra, and M, M^{-1} are inverse bimodules, then, under reasonable conditions, the Arens-Michael envelope of the algebraic Laurent algebra $L_A(M)$ is the topological Laurent algebra corresponding to the Arens-Michael envelopes $\widehat{A}, \widehat{M}, \widehat{M^{-1}}$ (see Proposition 3.15).

(3) If $\alpha : A \to A$ is an automorphism, we can define the A-bimodule A_α as follows: it coincides with A as a left A-module and for any $x \in A_\alpha$ and $a \in A$ we have $x \cdot a = x\alpha(a)$, where $\cdot$ denotes the action of A on A_α via right multiplication. In our paper, we prove that the Arens-Michael envelope of $A[t, t^{-1}; \alpha]$ is the topological Laurent algebra of A_α over A (see Corollary 3.16).

(4) Finally, if A is a Frechét-Arens-Michael algebra and α is an arbitrary continuous automorphism of A, we provide an explicit power series representation for the topoligcal Laurent algebra of A_α over A (see Corollary 4.7).

The structure of this paper is as follows. In Section 2, we give the definitions of different types of topological algebras and their Arens-Michael envelopes. Subsections 3.1 and 3.2 are devoted to defining several algebraic constructions, in particular, the Laurent tensor algebra $L_A(M)$ of an invertible bimodule. Throughout the next subsections, we introduce the analytic analogue of $L_A(M)$, formulate and prove one of the main results of the paper.

In Section 4, we tackle the special case $M = A_\alpha$ and describe the topological Laurent tensor algebra $\widehat{L}_A(A_\alpha)$ as explicitly as possible for any Fréchet-Arens-Michael algebra A and any continuous automorphism $\alpha : A \to A$. In Section 5, we state some open problems related to Arens-Michael envelopes. We also provide some examples in Appendix A.

2. Definitions
2.1. Basic notions
All algebras are considered over $\mathbb{C}$, and assumed to be unital, associative.

Definition 2.1 *Let A be a locally convex space with a multiplication $\mu : A \times A \to A$, such that (A, μ) is an algebra.*

(1) If μ is separately continuous then A is called a locally convex algebra.

(2) If A is a Fréchet space and μ is separately continuous then we call A a Fréchet algebra.

(3) If A is a complete locally convex space, and μ is jointly continuous then A is called a $\hat{\otimes}$-algebra.

Definition 2.2 *A locally convex algebra A is called m-convex if the topology on it can be defined by a family of submultiplicative seminorms.*

Definition 2.3 *A complete locally m-convex algebra is called an Arens-Michael algebra.*

For us, it will be important to keep in mind the following examples of Arens-Michael algebras:

1. Any Banach algebra is an Arens-Michael algebra.

2. For any $n \in \mathbb{N}$ the algebra $\mathcal{O}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$, endowed with the compact-open topology, is an Arens-Michael algebra.

3. For any locally compact space X, the algebra of continuous functions $C(X)$, endowed with the compact-open topology, is an Arens-Michael algebra.

   Also, keep in mind that every Arens-Michael algebra is a $\hat{\otimes}$-algebra.

Definition 2.4 *Let A be a $\hat{\otimes}$-algebra and let M be a complete locally convex space equipped with an A-bimodule structure such that the natural maps $A \times M \to M$ and $M \times A \to M$ are jointly continuous. Then M is called a A-$\hat{\otimes}$-bimodule.*

For a detailed introduction to the theory of locally convex spaces and algebras the reader can see [12], [6], [7] or [5].

2.2. Arens-Michael envelopes

Definition 2.5 ([4]) *Let A be an algebra. An Arens-Michael envelope of A is a pair $(\widehat{A}, i_A)$, where $\widehat{A}$ is an Arens-Michael algebra and $i_A : A \to \widehat{A}$ is an algebra homomorphism, satisfying the following universal property: for any Arens-Michael algebra B and algebra homomorphism $\varphi : A \to B$ there exists a unique continuous algebra homomorphism $\hat{\varphi} : \widehat{A} \to B$ extending φ, i.e. $\varphi = \hat{\varphi} \circ i_A$:*

$$
\begin{array}{ccc}
\widehat{A} & \overset{\hat{\varphi}}{\dashrightarrow} & B \\
{\scriptstyle i_A} \uparrow & \nearrow {\scriptstyle \varphi} & \\
A & &
\end{array}
$$

The Arens-Michael envelope of an algebra always exists and is unique up to a topological isomorphism, it is isomorphic to the completion of A with respect to the family of all submultiplicative seminorms on A.

We have already mentioned Theorem 1.3, which serves as a fundamental example of a computation of the Arens-Michael envelope. Here are some other important examples, which we borrow from [11] and [8]:

Example 2.6 *Denote the free algebra with generators $\xi_1, \ldots, \xi_n$ over $\mathbb{C}$ by F_n. Then its Arens-Michael envelope is a locally convex algebra, looks as follows:*

$$\mathcal{F}_n := \left\{ a = \sum_{w \in W_n} a_w \xi^w : \|a\|_\rho = \sum_{w \in W_n} |a_w| \rho^{|w|} < \infty \; \forall \, 0 < \rho < \infty \right\}.$$

In particular, $\mathcal{F}_n$ is a nuclear Fréchet algebra.

Example 2.7 *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. The Arens-Michael envelope of $U(\mathfrak{g})$ is isomorphic to the direct product $\prod_{V \in \hat{\mathfrak{g}}} Mat(V)$, where $\hat{\mathfrak{g}}$ is the set of the equivalence classes of finite-dimensional irreducible representations of $\mathfrak{g}$.*

Sometimes the Arens-Michael envelope of an algebra is isomorphic to the zero algebra:

Example 2.8 *Suppose that A is an algebra generated by x and y with the single relation $xy - yx = 1$. Then $\widehat{A} = 0$, because an arbitrary nonzero Banach algebra B cannot contain elements $x, y \in B$ such that $[x, y] = 1$.*

The definition of Arens-Michael envelopes can be given in case of bimodules, too.

Definition 2.9 ([8]) *Let A be an algebra and suppose that M is an A-bimodule. Then an Arens-Michael envelope of M is a pair $(\widehat{M}, i_M)$, where $\widehat{M}$ is a $\widehat{A}$-$\hat{\otimes}$-bimodule and $i_M : M \to \widehat{M}$ is a A-bimodule homomorphism, which satisfies the following universal property: for any $\widehat{A}$-$\hat{\otimes}$-bimodule N and A-bimodule homomorphism $f : M \to N$ there exists a unique continuous $\widehat{A}$-$\hat{\otimes}$-bimodule homomorphism $\hat{f} : \widehat{M} \to N$ which extends f:*

$$
\begin{array}{ccc}
\widehat{M} & \xdashrightarrow{\hat{f}} & N \\
{\scriptstyle i_A} \big\uparrow & \nearrow{\scriptstyle f} & \\
M & &
\end{array}
$$

In this paper, we will use [10, Proposition 6.1], which, basically, states that the Arens-Michael functor commutes with quotients:

Theorem 2.10 *Suppose that A is an algebra and $I \subset A$ is a two-sided ideal. Denote by J the closure of $i_A(I)$ in $\widehat{A}$. Then J is a closed two-sided ideal in $\widehat{A}$ and the induced homomorphism $A/I \to \widehat{A}/J$ extends to a topological algebra isomorphism*

$$\widehat{(A/I)} \simeq (\widehat{A}/J)^{\sim},$$

where $\widetilde{A}$ denotes the completion of A as a locally convex space.

Moreover, if $\widehat{A}$ is a Fréchet algebra, then we do not need to complete the quotient, so we have

$$\widehat{(A/I)} \simeq \widehat{A}/J.$$

Remark. As a corollary from this theorem and Example 2.6, we have that the Arens-Michael algebra of any finitely generated algebra over $\mathbb{C}$ is a Fréchet algebra.

3. Topological analogues of invertible bimodules and their Laurent tensor algebras

3.1. Some algebraic constructions

Firstly, let us recall the definitions of some crucial algebraic constructions, which we will use throughout this paper:

Definition 3.1 *Let A be an algebra and consider an endomorphism α of A. Then we define A_α as a A-bimodule which coincides with A as a left A-module and $x \circ a = x\alpha(a)$ for $x \in A_\alpha$, $a \in A$. Similarly, one defines an A-bimodule $_\alpha A$.*

Definition 3.2 *Let A be an algebra, $\alpha \in End(A)$ and $\delta \in Der(A, _\alpha A)$, or, equivalently,*

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \quad \forall a, b \in A.$$

Then the Ore extension of A with respect to α and δ is the vector space

$$A[t; \alpha, \delta] = \left\{ \sum_{i=0}^{n} a_i t^i : a_i \in A \right\}$$

with the multiplication, which is uniquely defined by the following conditions:

1. *The relation $ta = \alpha(a)t + \delta(a)$ holds for any $a \in A$*

2. *The natural inclusions $A \hookrightarrow A[t; \alpha, \delta]$ and $\mathbb{C}[t] \hookrightarrow A[t; \alpha, \delta]$ are algebra homomorphisms.*

Also, if $\delta = 0$ and α is invertible, then one can define the Laurent Ore extension

$$A[t, t^{-1}; \alpha] = \left\{ \sum_{i=-n}^{n} a_i t^i : a_i \in A \right\}$$

with the multiplication defined in a similar way.

3.2. Invertible bimodules and the Laurent tensor algebra

Remark. We were not able to find any published references related to Laurent tensor algebras of invertible modules in the literature, so we decided to provide a basic but completely self-contained exposition in this section, inspired by the unpublished notes of R.C. Cannings and M.P. Holland, "Tensor Algebras of Invertible Bimodules".

Definition 3.3 *Let an A be an algebra and consider an A-bimodule M. Then M is called an invertible A-bimodule if there exist an A-bimodule M^{-1} together with A-bimodule isomorphisms $i_1 : M \otimes_A M^{-1} \simeq A$ and $i_2 : M^{-1} \otimes_A M \simeq A$ (which we shall call convolutions) such that the following diagrams commute:*

$$
\begin{array}{ccc}
M \otimes_A M^{-1} \otimes_A M \xrightarrow{Id_M \otimes i_2} M \otimes_A A & \quad & M^{-1} \otimes_A M \otimes_A M^{-1} \xrightarrow{Id_M \otimes i_1} M^{-1} \otimes_A A \\
\downarrow{\scriptstyle i_1 \otimes Id_M} \qquad \downarrow{\scriptstyle m \otimes a \to ma} & & \downarrow{\scriptstyle i_2 \otimes Id_M} \qquad \downarrow{\scriptstyle n \otimes a \to na} \\
A \otimes_A M \xrightarrow{a \otimes m \to am} M & & A \otimes_A M^{-1} \xrightarrow{a \otimes n \to an} M^{-1}
\end{array}
\tag{3.1}
$$

With any A-bimodule M, one associates the tensor algebra $T_A(M)$:

$$T_A(M) := A \oplus \bigoplus_{n \in \mathbb{N}} M^{\otimes n},$$

where $M^{\otimes n} := \underbrace{M \otimes_A \cdots \otimes_A M}_{n \text{ times}}$. In turn, for every invertible A-bimodule, we can define a complex vector space which will be denoted by $L_A(M)$:

$$L_A(M) := \bigoplus_{n \in \mathbb{Z}} M^{\otimes n}, \tag{3.2}$$

where $M^{\otimes -n} := (M^{-1})^{\otimes n}$ and $M^{\otimes 0} := A$.

The elements belonging to $M^{\otimes n}$ for some $n \in \mathbb{Z}$ will be called homogeneous of degree n. The following proposition states that $L_A(M)$ admits a natural algebra structure:

Proposition 3.4 *Suppose that A is an algebra and M is an invertible A-bimodule. Then $L_A(M)$ admits a unique multiplication μ such that $(L_A(M), \mu)$ becomes an associative algebra and μ satisfies the following conditions:*

(1) the natural inclusions $j_M : T_A(M) \to L_A(M)$ and $j_{M^{-1}} : T_A(M^{-1}) \to L_A(M)$ are algebra homomorphisms.

(2) for any $m \in M$ and $n \in M^{-1}$ we have $m \cdot n = i_1(m \otimes n)$ and $n \cdot m = i_2(n \otimes m)$.

Proof It suffices to define the multiplication on the homogeneous elements of $L_A(M)$. Fix $m_1 \otimes \cdots \otimes m_k \in M \otimes \cdots \otimes M$ and $n_1 \otimes \cdots \otimes n_l \in M^{-1} \otimes \cdots \otimes M^{-1}$. Then define

$$(m_1 \otimes \ldots m_k) \cdot (n_1 \otimes \cdots \otimes n_l) = (m_1 \otimes \ldots m_{k-1} i_1(m_k \otimes n_1)) \cdot (n_2 \otimes \cdots \otimes n_l)$$

and

$$(n_1 \otimes \cdots \otimes n_l) \cdot (m_1 \otimes \ldots m_k) = (n_1 \otimes \ldots n_{l-1} i_2(n_l \otimes m_1)) \cdot (m_2 \otimes m_k),$$

then we repeat the process until we get a homogeneous element of $L_A(M)$. The associativity of the resulting algebra is a straightforward corollary from the commutativity of (3.1) in the Definition 3.3. $\square$

Let us call $L_A(M)$ the Laurent tensor algebra of an invertible bimodule M. The following proposition immediately follows from the constructions of $T_A(M)$ and $L_A(M)$.

Proposition 3.5 *Suppose that A is an algebra and α is an automorphism of A.*

(1) A_α and $A_{\alpha^{-1}}$ are inverse A-bimodules with respect to the maps

$$i_1(a \otimes b) = a\alpha(b), \quad i_2(b \otimes a) = b\alpha^{-1}(a).$$

(2) Moreover, $T(A_\alpha) \simeq A[t; \alpha]$, $L(A_\alpha) \simeq A[t, t^{-1}; \alpha]$.

The algebra $L_A(M)$ satisfies the following universal property:

Definition 3.6 *Let A be an algebra and consider an A-algebra B with respect to a homomorphism $\theta : A \to B$ together with A-bimodule homomorphisms $\alpha : M \to B$, $\beta : M^{-1} \to B$. Then we will call the triple of morphisms $(\theta, \alpha, \beta, B)$ compatible if and only if the following diagram is commutative:*

$$
\begin{array}{ccccc}
M \otimes_A M^{-1} & \xrightarrow{\ i_1\ } & A & \xleftarrow{\ i_2\ } & M^{-1} \otimes_A M \\
\downarrow{\scriptstyle \alpha \otimes \beta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \beta \otimes \alpha} \\
B \otimes_A B & \xrightarrow{\ m\ } & B & \xleftarrow{\ m\ } & B \otimes_A B
\end{array}
\tag{3.3}
$$

Proposition 3.7 *The triple of morphisms $(i_A, i_M, i_{M^{-1}}, L_A(M))$, where all morphisms are tautological inclusions into $L_A(M)$, is a universal compatible triple, i.e. for any other algebra B and any compatible triple of morphisms $(\theta, \alpha, \beta, B)$ there exists a unique A-algebra homomorphism $f : L_A(M) \to B$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
L_A(M) \xrightarrow{\ f\ } B & \qquad L_A(M) \xrightarrow{\ f\ } B & \qquad L_A(M) \xrightarrow{\ f\ } B \\
{\scriptstyle i_A}\uparrow \ \ \nearrow{\scriptstyle \theta} & {\scriptstyle i_M}\uparrow \ \ \nearrow{\scriptstyle \alpha} & {\scriptstyle i_{M^{-1}}}\uparrow \ \ \nearrow{\scriptstyle \beta} \\
A & M & M^{-1}
\end{array}
\tag{3.4}
$$

Proof It suffices to check the existence, as the uniqueness will follow as a standard category-theoretic argument.

And the existence is straightforward: for every $m_1 \otimes \cdots \otimes m_k \in M^{\otimes k}$ define

$$
f(m_1 \otimes \cdots \otimes m_k) = \alpha(m_1) \ldots \alpha(m_k),
$$

and for $n_1 \otimes \cdots \otimes n_l \in M^{\otimes -l}$ we define

$$
f(n_1 \otimes \cdots \otimes n_l) = \beta(n_1) \ldots \beta(n_l),
$$

and $f(a) = \theta(a)$ for every $a \in A$.

The commutativity of (3.3) ensures that f is a well-defined homomorphism of A-algebras. And the diagrams (3.4) commute due to the construction of f. $\qquad\square$

3.3. Topologically invertible bimodules

Now, using the language of locally convex vector spaces, we will construct topological versions of the notions we described above.

Definition 3.8 *Let A be a $\hat{\otimes}$-algebra and M be an A-$\hat{\otimes}$-bimodule. Then we will call M a topologically invertible A-$\hat{\otimes}$-bimodule if there exist an A-$\hat{\otimes}$-bimodule M^{-1} and two A-$\hat{\otimes}$-bimodule topological isomorphisms $i_1 : M \hat{\otimes}_A M^{-1} \simeq A$ and $i_2 : M^{-1} \hat{\otimes}_A M \simeq A$, such that the following diagrams commute:*

$$
\begin{array}{ccc}
M\hat{\otimes}_A M^{-1}\hat{\otimes}_A M \xrightarrow{Id_M \hat{\otimes} i_2} M\hat{\otimes}_A A & \qquad M^{-1}\hat{\otimes}_A M\hat{\otimes}_A M^{-1} \xrightarrow{Id_M \hat{\otimes} i_1} M^{-1}\hat{\otimes}_A A \\
\downarrow{\scriptstyle i_1 \hat{\otimes} Id_M} \qquad \downarrow{\scriptstyle m\otimes r \to mr} & \qquad \downarrow{\scriptstyle i_2 \hat{\otimes} Id_M} \qquad \downarrow{\scriptstyle n\otimes r \to nr} \\
A\hat{\otimes}_A M \xrightarrow{r\otimes m \to rm} M & \qquad A\hat{\otimes}_A M^{-1} \xrightarrow{r\otimes n \to rn} M^{-1}
\end{array}
\tag{3.5}
$$

The following proposition is the topological version of the Proposition 3.5.

Proposition 3.9 *Let A be a $\hat{\otimes}$-algebra and suppose that α is an automorphism of A. Then A_α and $A_{\alpha^{-1}}$ are topologically inverse A-$\hat{\otimes}$-bimodule with respect to the maps*

$$i_1(a \otimes b) = a\alpha(b), \quad i_2(b \otimes a) = b\alpha^{-1}(a).$$

More information on topologically invertible bimodules can be found in [9].

There is a natural question related to Arens-Michael envelopes: is it true that the Arens-Michael envelope of an invertible bimodule is topologically invertible? At the moment, we can state a conjecture:

Conjecture 3.10 *Suppose that A is an algebra and M is an invertible A-bimodule. Then there exist topological A-$\hat{\otimes}$-bimodule isomorphisms $\hat{i}_1 : \widehat{M}\hat{\otimes}_{\widehat{A}}\widehat{M^{-1}} \to \widehat{A}$ and $\hat{i}_2 : \widehat{M^{-1}}\hat{\otimes}_{\widehat{A}}\widehat{M} \to \widehat{A}$, satisfying the following conditions:*

(1) $\widehat{M}$ is a topologically invertible $\widehat{A}$-$\hat{\otimes}$-bimodule w.r.t. i_1 and i_2.

(2) The following diagram is commutative:

$$
\begin{array}{ccccc}
M \otimes_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1} \otimes_A M \\
\downarrow{\scriptstyle i_M \otimes i_{M^{-1}}} & & \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_{M^{-1}} \otimes i_M} \\
\widehat{M}\hat{\otimes}_{\widehat{A}}\widehat{M^{-1}} & \xrightarrow{\hat{i}_1} & \widehat{A} & \xleftarrow{\hat{i}_2} & \widehat{M^{-1}}\hat{\otimes}_{\widehat{A}}\widehat{M}
\end{array}
\tag{3.6}
$$

where the left arrow maps $a \otimes b$ to $i_M(a) \otimes i_{M^{-1}}(b)$, and the right arrow maps $b \otimes a$ to $i_{M^{-1}}(b) \otimes i_M(a)$.

It turns out that there is a particular case in which, at least, the first statement of the above conjecture holds.

Remark. Here we consider the Arens-Michael envelopes of $\hat{\otimes}$-algebras and $\hat{\otimes}$-bimodules, see [8, Section 3] for the details.

Theorem 3.11 *Consider a $\hat{\otimes}$-algebra A and a pair of topologically invertible A-$\hat{\otimes}$-bimodules M, M^{-1}. Suppose that the following condition is satisfied:*

$$M\hat{\otimes}_A\widehat{A} \simeq \widehat{A}\hat{\otimes}_A M, M^{-1}\hat{\otimes}_A\widehat{A} \simeq \widehat{A}\hat{\otimes}_A M^{-1} \text{ as } A\text{-}\hat{\otimes}\text{-bimodules.}$$

Then $\widehat{M}$ and $\widehat{M^{-1}}$ are topologically invertible $\widehat{A}$-bimodules.

Proof This is an immediate corollary of the fact that $\widehat{M} \simeq \widehat{A}\hat{\otimes}_A M\hat{\otimes}_A\widehat{A}$, which is the statement of [8, Remark 3.8], and [9, Proposition 10.4]. The idea is to write the following chain of $\widehat{A}$-$\hat{\otimes}$-bimodule isomorphisms:

$$
\begin{aligned}
\widehat{M}\hat{\otimes}_{\widehat{A}}\widehat{M^{-1}} &\overset{R3.8}{\simeq} \widehat{A}\hat{\otimes}_A M\hat{\otimes}_A\widehat{A}\hat{\otimes}_{\widehat{A}}\widehat{A}\hat{\otimes}_A M^{-1}\hat{\otimes}_A\widehat{A} \simeq \widehat{A}\hat{\otimes}_A M\hat{\otimes}_A\widehat{A}\hat{\otimes}_A M^{-1}\hat{\otimes}_A\widehat{A} \overset{\text{P10.4 (ii)}}{\simeq} \\
&\simeq \widehat{A}\hat{\otimes}_A M\hat{\otimes}_A M^{-1}\hat{\otimes}_A\widehat{A}\hat{\otimes}_A\widehat{A} \overset{(1)}{\simeq} \widehat{A}\hat{\otimes}_A M\hat{\otimes}_A M^{-1}\hat{\otimes}_A\widehat{A} \xrightarrow{i_1} \widehat{A}\hat{\otimes}_A\widehat{A}\hat{\otimes}_A\widehat{A} \overset{R3.8}{\simeq} \widehat{A}.
\end{aligned}
\tag{3.7}
$$

In a similar fashion, we can show that the associativity diagrams commute. $\qquad\square$

As a corollary, consider an algebra A and a pair of invertible bimodules M, M^{-1} of at most countable dimension. Then [8, Proposition 2.3] implies the following statements:

(1) A_s is a $\hat{\otimes}$-algebra, M_s, $(M^{-1})_s$ are A_s-$\hat{\otimes}$-bimodules.

(2) These bimodules are topologically invertible as A_s-bimodules.

Suppose that A_s, M_s and $(M^{-1})_s$ satisfy the conditions of the Theorem 3.11. Then $\widehat{M}$ and $\widehat{M^{-1}}$ are topologically invertible $\widehat{A}$-bimodules.

Proposition 3.12 *Conjecture 3.10 holds in the case of $M = A_\alpha$, where A is an arbitrary associative algebra and $\alpha \in Aut(A)$.*

Proof We refer to [8, Corollary 5.6] which states that $\widehat{A_\alpha} \simeq (\widehat{A})_{\hat{\alpha}}$. Taking the necessary isomorphisms from the Proposition 3.9, we get (1), and the following computation proves the commutativity of the left side of (3.6):

$$\hat{i}_1(i_A(a) \otimes i_A(b)) = i_A(a\alpha(b)) = i_A \circ i_1(a \otimes b) \quad (a \in M, b \in M^{-1})$$

A similar argument also shows that the right quadrant of the diagram (3.6) is commutative too. □

3.4. Topological Laurent tensor algebras

Fix an Arens-Michael algebra A and a pair of topologically inverse A-$\hat{\otimes}$-bimodules M and M^{-1}.

Definition 3.13 *Let B be an Arens-Michael algebra, which is an A-algebra with respect to a continuous homomorphism $\theta : A \to B$, also let $\alpha : M \to B$, $\beta : M^{-1} \to B$ be continuous A-$\hat{\otimes}$-bimodule homomorphisms. Then we will call the triple $(\theta, \alpha, \beta, B)$ topologically compatible if and only if the following diagram is commutative:*

$$
\begin{array}{ccccc}
M\hat{\otimes}_A M^{-1} & \xrightarrow{i_1} & A & \xleftarrow{i_2} & M^{-1}\hat{\otimes}_A M \\
\downarrow{\scriptstyle \alpha\hat{\otimes}\beta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \beta\hat{\otimes}\alpha} \\
B\hat{\otimes}_A B & \xrightarrow{m} & B & \xleftarrow{m} & B\hat{\otimes}_A B
\end{array}
\tag{3.8}
$$

Now we will formulate and prove one of the main theorems of the paper:

Theorem 3.14 *Let A be a Fréchet-Arens-Michael algebra and consider topologically inverse Fréchet A-$\hat{\otimes}$-bimodules M, M^{-1}. Then there exist an Arens-Michael algebra $\widehat{L}_A(M)$ and a topologically compatible triple of morphisms $(\theta, \alpha, \beta, \widehat{L}_A(M))$ that satisfies the following universal property: for every Arens-Michael algebra B and a topologically compatible triple of morphisms $(\theta', \alpha', \beta', B)$ there exists a unique continuous A-algebra homomorphism $f : \widehat{L}_A(M) \to B$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
\widehat{L}_A(M) \xrightarrow{f} B & \qquad \widehat{L}_A(M) \xrightarrow{f} B & \qquad \widehat{L}_A(M) \xrightarrow{f} B \\
\theta \uparrow \quad \nearrow{\scriptstyle \theta'} & \alpha \uparrow \quad \nearrow{\scriptstyle \alpha'} & \beta \uparrow \quad \nearrow{\scriptstyle \beta'} \\
A & M & M^{-1}
\end{array}
\tag{3.9}
$$

We will call it the topological(or analytic) Laurent tensor algebra of the A-$\hat{\otimes}$-bimodule M.

The proof of the existence of the universal object will be given in the next subsection. What we want to do now is to establish the connection between analytic Laurent tensor algebras and Arens-Michael envelopes.

Proposition 3.15 *Now suppose that A is an algebra and M, M^{-1} is a pair of (algebraically) inverse A-bimodules. Suppose that the following condition holds for $\widehat{A}$, $\widehat{M}$ and $\widehat{M^{-1}}$:*

(1) The underlying LCS of $\widehat{A}$, $\widehat{M}$ and $\widehat{M^{-1}}$ are Fréchet spaces.

(2) $\widehat{M}$ and $\widehat{M^{-1}}$ are topologically inverse as $\widehat{A}$-$\hat{\otimes}$-bimodules which satisfy Conjecture 3.10.

Then, if $(\theta, \alpha, \beta, \widehat{L}_{\widehat{A}}(\widehat{M}))$ is the resulting topologically compatible triple in Theorem 3.14, then $\widehat{L_A(M)} \simeq \widehat{L}_{\widehat{A}}(\widehat{M})$.

Proof   Firstly, we need to construct an algebra homomorphism $i : L_A(M) \to \widehat{L}_{\widehat{A}}(\widehat{M})$. Consider the following morphisms: $\theta i_A : A \to \widehat{A} \to \widehat{L}_{\widehat{A}}(\widehat{M})$, $\alpha i_M : M \to \widehat{M} \to \widehat{L}_{\widehat{A}}(\widehat{M})$ and $\beta i_{M^{-1}} : M^{-1} \to \widehat{M^{-1}} \to \widehat{L}_{\widehat{A}}(\widehat{M})$. It turns out that this triple of morphisms is (algebraically) compatible; however, this statement is not as obvious as one might think: look at the diagram, commutativity of which we aim to prove:

$$\begin{array}{ccccc} M \otimes_A M^{-1} & \xrightarrow{\ i_1\ } & A & \xleftarrow{\ i_2\ } & M^{-1} \otimes_A M \\ {\scriptstyle \alpha i_M \otimes \beta i_{M^{-1}}}\downarrow & & {\scriptstyle \theta i_A}\downarrow & & \downarrow{\scriptstyle \beta i_{M^{-1}} \otimes \alpha i_M} \\ \widehat{L}_{\widehat{A}}(\widehat{M}) \otimes_A \widehat{L}_{\widehat{A}}(\widehat{M}) & \xrightarrow{\ m\ } & \widehat{L}_{\widehat{A}}(\widehat{M}) & \xleftarrow{\ m\ } & \widehat{L}_{\widehat{A}}(\widehat{M}) \otimes_A \widehat{L}_{\widehat{A}}(\widehat{M}) \end{array} \qquad (3.10)$$

Notice that we deal with the algebraic tensor product of $\widehat{L}_{\widehat{A}}(\widehat{M})$, not with completed projective tensor product. However, we can write

$$\theta i_A \circ i_1(x \otimes y) \stackrel{3.6}{=} \theta \circ \hat{i}_1(i_M(x) \otimes i_{M^{-1}}(y)) \stackrel{3.8}{=} m \circ (\alpha \hat{\otimes} \beta)(i_M(x) \otimes i_{M^{-1}}(y)) =$$
$$= m \circ \varphi \circ (\alpha i_M \otimes i_{M^{-1}}\beta)(x \otimes y) = m(\alpha i_M \otimes i_{M^{-1}}\beta)(x \otimes y),$$

where

$$\varphi : \widehat{L}_{\widehat{A}}(\widehat{M}) \otimes_A \widehat{L}_{\widehat{A}}(\widehat{M}) \to \widehat{L}_{\widehat{A}}(\widehat{M}) \hat{\otimes}_{\widehat{A}} \widehat{L}_{\widehat{A}}(\widehat{M}), \quad \varphi(b_1 \otimes b_2) = b_1 \otimes b_2.$$

If we denote the algebra $\widehat{L}_{\widehat{A}}(\widehat{M})$ by B, then the argument can be illustrated by the following three-dimensional diagram:

$$ (3.11) $$

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And if the triple $(\theta i_A, \alpha i_M, \beta i_{M^{-1}}, \widehat{L}_{\widehat{A}}(\widehat{M}))$ is compatible, we get the morphism $i : L_A(M) \to \widehat{L}_{\widehat{A}}(\widehat{M})$.

Secondly, we need to prove that the pair $(\widehat{L}_{\widehat{A}}(\widehat{M}), i)$ satisfies the universal property. Let us consider the following continuous morphisms: $\widehat{\varphi|_A} : \widehat{A} \to X$, $\widehat{\varphi|_M} : \widehat{M} \to X$ and $\widehat{\varphi|_{M^{-1}}} : \widehat{M^{-1}} \to X$, which come from the respective universal properties. The first map is an algebra homomorphism, and the latter are $\widehat{A}$-bimodule morphisms. The resulting triple is topologically compatible, and the argument is basically the same as the one we gave in the first step of the proof, we only need to keep in mind that elementary tensors span a dense subspace in a completed projective tensor products of locally convex spaces. From that we get a unique $\widehat{A}$-algebra morphism $\hat{\varphi} : \widehat{L}_{\widehat{A}}(\widehat{M}) \to X$. The last thing that is left is to show that it really extends φ. However, if we restrict φ on A, M or M^{-1}, the statement holds, so it is true for $L_A(M)$. □

The following is a corollary of Propositions 3.12 and 3.15.

Corollary 3.16 *Suppose that A is an associative algebra with the Arens-Michael envelope which is a Fréchet algebra, and let $\alpha \in Aut(A)$ be an arbitrary algebra automorphism. Then the following isomorphism takes place:*

$$(A[t, t^{-1}; \alpha])\widehat{} = \widehat{L_A(A_\alpha)} \cong \widehat{L}_{\widehat{A}}(\widehat{A}_{\widehat{\alpha}}).$$

3.5. Constructing the universal object

To prove Theorem 3.14, we need to utilize the construction of the analytic tensor algebra, described in [8]. This approach is natural because the resulting topological Laurent tensor algebra is expected to contain an appropriate version of the topological tensor algebra, and this object is already defined in [8].

Suppose that A is an Arens-Michael algebra and M is an A-$\hat{\otimes}$-bimodule. Fix a directed generating family of seminorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ on M. Consider the locally convex space

$$\widehat{T}_A(M)^+ = \left\{ (x_n) \in \prod_{i=1}^{\infty} M^{\hat{\otimes}n} : \|(x_n)\|_{\nu,\rho} := \sum_{n=1}^{\infty} \|x_n\|_\nu^{\hat{\otimes}n} \rho^n < \infty, \nu \in \Lambda, 0 < \rho < \infty \right\}, \qquad (3.12)$$

where $M^{\hat{\otimes}n} := \underbrace{M \hat{\otimes}_A \dots \hat{\otimes}_A M}_{n \text{ times}}$. By definition, the seminorms $\|\cdot\|_{\nu,\rho}$ generate the topology on $\widehat{T}_A(M)^+$.

Definition 3.17 *The topological (analytic) tensor algebra of M is a locally convex space*

$$\widehat{T}_A(M) := A \oplus \widehat{T}_A(M)^+.$$

In [8], it is proven that $\widehat{T}_A(M)$ admits a multiplication which makes a natural inclusion $f : T_A(M) \to \widehat{T}_A(M)$ into an algebra homomorphism and turns $\widehat{T}_A(M)$ into an Arens-Michael algebra. We also will use [8, Proposition 4.8], which states that $\widehat{T}_A(M)$ is defined by its universal property.

Fix a Fréchet-Arens-Michael algebra A, a pair of topologically inverse Fréchet A-$\hat{\otimes}$-bimodules M and M^{-1} with respect to the topological A-bimodule isomorphisms $i_1 : M \hat{\otimes}_A M^{-1} \to A$ and $i_2 : M^{-1} \hat{\otimes}_A M \to A$.

Now, for any $x \in M$ and $y \in M^{-1}$, consider the elements

$$(0, 0, (x, 0) \otimes (0, y), 0, \dots) - (i_1(x \otimes y), 0, 0, \dots) \in \widehat{T}_A(M) \qquad (3.13)$$

and

$$(0, 0, (0, y) \otimes (x, 0), 0, \dots) - (i_2(y \otimes x), 0, 0, \dots) \in \widehat{T}_A(M). \tag{3.14}$$

It would be reasonable to assume that these elements are equal to zero in $\widehat{L}_A(M)$. This idea serves as a motivation for the following definition:

Definition 3.18 *Let $\widehat{L}_A(M)' := \widehat{T}_A(M \oplus M^{-1})/I$, where I is the closure of the two-sided ideal generated by elements of form* (3.13) *and* (3.14) *for any $x \in M$, $y \in M^{-1}$.*

Remark. Actually, this is the only place where we use the Fréchet assumption. If $\widehat{T}_A(M \oplus M^{-1})$ is not Fréchet, the quotient might not be complete, we would have to complete the resulting algebra and the following proof, in fact, will still work; however, this assumption makes everything easier.

Let us also denote some morphisms associated with this object:

$$j_0 : A \hookrightarrow \widehat{T}_A(M \oplus M^{-1})$$

$$j_M : M \hookrightarrow M \oplus M^{-1} \to \widehat{T}_A(M \oplus M^{-1})$$

$$j_{M^{-1}} : M^{-1} \hookrightarrow M \oplus M^{-1} \to \widehat{T}_A(M \oplus M^{-1})$$

$$\pi : \widehat{T}_A(M \oplus M^{-1}) \to \widehat{L}_A(M)'$$

.

If A is a Fréchet-Arens-Michael algebra, so are $\widehat{T}_A(M \oplus M^{-1})$ and $\widehat{L}_A(M)'$. Consider the triple of morphisms $i_A = \pi \circ j_0$, $i_M = \pi \circ j_M$, $i_{M^{-1}} = \pi \circ j_{M^{-1}}$.

Lemma 3.19 *The triple $(i_A, i_M, i_{M^{-1}}, \hat{L}_A(M)')$ is topologically compatible.*

Proof We need to prove the commutativity of the following diagram:

$$\begin{array}{ccccc}
M \hat{\otimes}_A M^{-1} & \xrightarrow{\ i_1\ } & A & \xleftarrow{\ i_2\ } & M^{-1} \hat{\otimes}_A M \\
\downarrow{\scriptstyle i_M \otimes i_{M^{-1}}} & & \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_{M^{-1}} \otimes i_M} \\
\widehat{L}_A(M)' \hat{\otimes}_A \widehat{L}_A(M)' & \xrightarrow{\ m\ } & \widehat{L}_A(M)' & \xleftarrow{\ m\ } & \widehat{L}_A(M)' \hat{\otimes}_A \widehat{L}_A(M)'
\end{array} \tag{3.15}$$

For every $x \in M$ and $y \in M^{-1}$, we can consider an elementary tensor $x \otimes y \in M \hat{\otimes}_A M^{-1}$.

$$m \circ (i_M \otimes i_{M^{-1}})(x \otimes y) = m((0, (x, 0), 0, 0, \dots) \cdot (0, (0, y), 0, 0, \dots)) = (0, 0, (x, 0) \otimes (0, y), 0, \dots)$$

$$= (i_1(x \otimes y), 0, 0, \dots) = i_A \circ i_1(x \otimes y).$$

We finish the proof by using the fact that elementary tensors span a dense subspace in $M \hat{\otimes}_A M^{-1}$. $\qquad\square$

Proposition 3.20 $\widehat{L}_A(M)' \simeq \widehat{L}_A(M)$.

Proof We must check that the triple $(i_A, i_M, i_{M^{-1}}, \hat{L}_A(M)')$ satisfies the universal property. Suppose that B is a Banach algebra and that $(\theta, \gamma, \delta, B)$ is an topologically compatible triple. Consider the direct sum of γ and δ: $\gamma \oplus \delta : M \oplus M^{-1} \to B$. It is a continuous A-bimodule morphism which, by [8, Proposition 4.8], can be uniquely extended to a continuous A-algebra morphism $\varphi : \widehat{T}_A(M \oplus M^{-1}) \to B$. From the fact that $(\theta, \gamma, \delta, B)$ is topologically compatible, it easily follows that $\varphi(I) = 0$, so, in fact, we obtain a unique continuous A-algebra homomorphism $\tilde{\varphi} : \hat{L}_A(M)' \to B$. Due to the construction, it extends θ, γ and δ. $\qquad \square$

This concludes the proof of the Theorem 3.14.

4. The case of $M = A_\alpha$

4.1. Localizable linear maps between locally convex spaces

Definition 4.1 *Let A be an Arens-Michael algebra and let $\mathcal{F} \subset \mathcal{L}(A)$ be a family of continuous linear maps $A \to A$.*

Then F is called an m-localizable family (see [8]) if the topology on A can be defined by a family of submultiplicative seminorms $\{\|\cdot\|_\lambda\}_{\lambda \in \Lambda}$, satisfying the following property: for every $T \in \mathcal{F}$ there exists a constant $C_T > 0$, such that

$$\|Ta\|_\lambda \leq C_T \|a\|_\lambda \quad \text{for every } a \in A.$$

An operator $T \in \mathcal{L}(E)$ is called m-localizable $\iff \{T\}$ is a m-localizable family.

Suppose now that A is an Arens-Michael algebra, α is a continuous automorphism of A, such that $\{\alpha, \alpha^{-1}\}$ is a m-localizable family. Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$, then we can define the following vector space:

$$\mathcal{O}(\mathbb{C}^\times, A) := \left\{ f = \sum_{i=-\infty}^{\infty} a_i t^i : \|f\|_{\lambda, \rho} := \sum_{-\infty}^{\infty} \|a_i\|_\lambda \rho^i < \infty \ \forall \lambda \in \Lambda, 0 < \rho < \infty \right\}. \tag{4.1}$$

This vector space with topology, generated by $\|\cdot\|_{\lambda, \rho}$ becomes a complete locally convex space. Moreover, [8, Lemma 4.12] and [8, Proposition 4.15] state that in our case $\mathcal{O}(\mathbb{C}^\times, A)$ admits a unique multiplication, which is compatible with α (i.e. $ta = \alpha(a)t, t^{-1}a = \alpha^{-1}(a)t^{-1}$ for every $a \in A$) and makes $\mathcal{O}(\mathbb{C}^\times, A)$ into an Arens-Michael algebra, which is denoted by $\mathcal{O}(\mathbb{C}^\times, A; \alpha)$.

Proposition 4.2 *Under assumptions made above, $\widehat{L}_A(A_\alpha) \simeq \mathcal{O}(\mathbb{C}^\times, A; \alpha)$.*

Proof Firstly, we must consider natural morphisms

$$i_A : A \hookrightarrow \mathcal{O}(\mathbb{C}^\times, A; \alpha),$$

$$i_{A_\alpha} : A_\alpha \to \mathcal{O}(\mathbb{C}^\times, A; \alpha), i_{A_\alpha}(1) = t$$

$$i_{A_{\alpha^{-1}}} : A_{\alpha^{-1}} \to \mathcal{O}(\mathbb{C}^\times, A; \alpha), i_{A_{\alpha^{-1}}}(1) = t^{-1}.$$

We aim to prove that the triple of morphisms $(i_A, i_{A_\alpha}, i_{A_{\alpha^{-1}}}, \mathcal{O}(\mathbb{C}^\times, A; \alpha))$ is a topologically compatible triple, which satisfies the universal property. The first part is obviously true due to the construction of $\mathcal{O}(\mathbb{C}^\times, A; \alpha)$.

Suppose that $(\theta, \alpha, \beta, B)$ is another topologically compatible triple. Notice that

$$\alpha(1)\beta(1) = \beta(1)\alpha(1) = 1,$$

so $\alpha(1) \in B$ is an invertible element. Then, due to [8, Proposition 4.14], there exists a unique continuous algebra homomorphism $f : \mathcal{O}(\mathbb{C}^{\times}, A; \alpha) \to B$, $f(t) = \alpha(1)$. It is easily seen that $f i_A = \theta$, $f i_{A_\alpha} = \alpha$, $f i_{A_{\alpha^{-1}}} = \beta$. $\quad\square$

4.2. The general case

In this section, A is an Arens-Michael algebra and α is an automorphism of A. We aim to obtain a description of $\widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$, similar to the description of $\widehat{T}_A(A_\alpha)$, obtained in [8, Proposition 4.9], without putting any localizability assumptions on the pair (α, α^{-1}).

For every tuple $w \in W_2$ we denote the k-th symbol of w by $w(k)$. Also consider the functions $c_1 : W_2 \to \mathbb{Z}_{\geq 0}$ and $c_2 : W_2 \to \mathbb{Z}_{\geq 0}$ which count the number of instances of 1 and 2 in a tuple, respectively. Also denote $c(w) = c_1(w) - c_2(w)$. For every element in W_2 define an A-$\hat{\otimes}$-bimodule as follows:

(1) $A_\emptyset := A$

(2) $A_{(1)} := A_\alpha$, $A_{(2)} := A_{\alpha^{-1}}$

(3) for every $w_1, w_2 \in W_2$ we have $A_{w_1 w_2} := A_{w_1} \hat{\otimes}_A A_{w_2}$

Let $w \in W_2$ be a nonempty element and let $1 \leq k \leq |w|$. Replace all numbers 2 in w with -1 and denote the new tuple by w'. Let us define a function $p(w, k)$ as follows:

$$p(w, k) = \sum_{i=1}^{k} w'(j) = \sum_{i=1}^{k} 3 - 2w(j). \tag{4.2}$$

Proposition 4.3 *For every $w \in W_2$ consider a mapping*

$$i_w : \prod_{i=1}^{|w|} A_{\alpha^{w'(i)}} \to A_{\alpha^n}, \quad i_w(x_1, \ldots x_{|w|}) := x_1 \prod_{i=2}^{|w|} \alpha^{p(w, i-1)}(x_i),$$

where $n = c(w)$.

Then i_w is a continuous A-balanced map which induces a A-$\hat{\otimes}$-bimodule isomorphism $i_w : A_w \simeq A_{\alpha^n}$.

Proof First of all, let us prove that i_w is an A-balanced map:

$$i_w(x_1, \ldots, x_i \circ r, x_{i+1}, \ldots x_{|w|}) =$$

$$= x_1 \alpha^{p(w,1)}(x_2) \ldots \alpha^{p(w,i-1)}(x_i \alpha^{w'(i)}(r)) \alpha^{p(w,i)}(x_{i+1}) \ldots \alpha^{p(w,|w|-1)}(x_{|w|}) =$$

$$= x_1 \alpha^{p(w,1)}(x_2) \ldots \alpha^{p(w,i-1)}(x_i) \alpha^{w'(i)+p(w,i-1)}(r) \alpha^{p(w,i)}(x_{i+1}) \ldots \alpha^{p(w,|w|-1)}(x_{|w|}).$$

However, by definition, $w'(i) + p(w, i-1) = p(w, i)$, so we get

$$x_1 \alpha^{p(w,1)}(x_2) \ldots \alpha^{p(w,i-1)}(x_i) \alpha^{w'(i)+p(w,i-1)}(r) \alpha^{p(w,i)}(x_{i+1}) \ldots \alpha^{p(w,|w|-1)}(x_{|w|}) =$$

$$= x_1 \alpha^{p(w_1)}(x_2) \ldots \alpha^{p(w,i-1)}(x_i) \alpha^{p(w,i)}(r) \alpha^{p(w,i)}(x_{i+1}) \ldots \alpha^{p(w,|w|-1)}(x_{|w|}) =$$

$$= i_w(x_1, \ldots, x_i, r \circ x_{i+1}, \ldots, x_{|w|}).$$

Therefore, i_w is balanced.

Now suppose that $f : \prod_{i=1}^{|w|} A_{\alpha^{w'(i)}} \to M$ is a continuous A-balanced A-$\hat{\otimes}$-bimodule homomorphism. Then we define

$$\tilde{f} : A_{\alpha^n} \to M, \quad \tilde{f}(a) = f(a, 1, \ldots, 1).$$

This map is a well-defined homomorphism of A-$\hat{\otimes}$-bimodules: for any $b \in A$ we have

$$\tilde{f}(ba) = f(ba, 1, \ldots, 1) = bf(a, 1, \ldots, 1) = b\tilde{f}(a),$$

$$\tilde{f}(a)b = f(a, 1, \ldots, 1) \circ b = f(a, 1, \ldots, \alpha^{p(w,|w|)}(b)) =$$

$$= f(a, 1, \ldots, 1, \alpha^{p(w,|w|-1)+p(w,|w|)}(r), 1) = \cdots = f(a\alpha^n(b), 1, \ldots, 1).$$

We prove that $f = \tilde{f} \circ i_w$ by using a similar argument, which we will omit here. $\square$

Lemma 4.4 *The following diagram is commutative:*

$$
\begin{array}{ccc}
A_{w_1} \hat{\otimes}_A A_{w_2} & =\!=\!= & A_{w_1 w_2} \\
\downarrow{\scriptstyle i_{w_1} \otimes i_{w_2}} & & \downarrow{\scriptstyle i_{w_1 w_2}} \\
A_{\alpha^{k_1}} \hat{\otimes}_A A_{\alpha^{k_2}} & \xrightarrow{\varphi} & A_{\alpha^{k_1+k_2}}
\end{array}
\quad , \tag{4.3}
$$

where $\varphi(a \otimes b) = a\alpha^{k_1}(b)$.

Proof Again, it suffices to look at elementary tensors. Let $x = x_1 \otimes \cdots \otimes x_{|w_1|} \in A_{w_1}$ and $y = y_1 \otimes \cdots \otimes y_{|w_2|} \in A_{w_2}$. Then we have

$$\varphi \circ (i_{w_1} \otimes i_{w_2})(x \otimes y) = \varphi \left(\prod_{i=1}^{|w_1|} \alpha^{p(w_1,i-1)}(x_i) \otimes \prod_{i=1}^{|w_2|} \alpha^{p(w_2,i-1)}(y_i) \right) =$$

$$= \prod_{i=1}^{|w_1|} \alpha^{p(w_1,i-1)}(x_i) \cdot \prod_{i=1}^{|w_2|} \alpha^{p(w_2,i-1)+k_1}(y_i).$$

Notice that $k_1 = p(w_1, |w_1|)$; therefore,

$$\prod_{i=1}^{|w_1|} \alpha^{p(w_1,i-1)}(x_i) \cdot \prod_{i=1}^{|w_2|} \alpha^{p(w_2,i-1)+k_1}(y_i) = \prod_{i=1}^{|w_1|} \alpha^{p(w_1 w_2,i-1)}(x_i) \cdot \prod_{i=|w_1|+1}^{|w_2|} \alpha^{p(w_1 w_2,i-1)}(y_i) = i_{w_1 w_2}(x \otimes y).$$

$\square$

Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on A.

Definition 4.5 *Define the following locally convex space:*

$$A\{x_1, x_2; \alpha\} = \left\{ a = \sum_{w \in W_2} a_w x^w : \|a\|_{\lambda, \rho} = \sum_{w \in W_2} \|a_w\|_\lambda^{(w)} \rho^{|w|} < \infty \; \forall \lambda \in \Lambda, 0 < \rho < \infty \right\}, \qquad (4.4)$$

where $\|r\|_\lambda^{(w)}$ are seminorms on A, induced by the seminorm on A_{α_n} (for $n = c(w)$)

$$\|r\|_\lambda^{(w)} = \inf_{r = \sum_{j=1}^k i_w(r_{1,j} \otimes \cdots \otimes r_{|w|,j})} \sum_{j=1}^k \|r_{1,j}\|_\lambda \cdots \|r_{|w|,j}\|_\lambda,$$

identifying A_{α_n} with A, as their underlying LCS are the same. Finally, we define $\|\cdot\|_\lambda^{(\emptyset)} = \|\cdot\|_\lambda$.

Remark. Actually, $\|\cdot\|_\lambda^{(1)} = \|\cdot\|_\lambda^{(2)} = \|\cdot\|_\lambda$ due to the definition of i_w.

The space $A\{x_1, x_2; \alpha\}$ with the topology, generated by $\|\cdot\|_{\lambda, \rho}$, is a complete locally convex space.

Theorem 4.6 *The space $A\{x_1, x_2; \alpha\}$ admits a unique multiplication which satisfies the following conditions:*

(1) the natural inclusions $A[t; \alpha] \hookrightarrow A\{x_1, x_2; \alpha\}$ and $A[s; \alpha^{-1}] \hookrightarrow A\{x_1, x_2, \alpha\}$, where
$\sum a_n t^n \to \sum a_n x_1^n$ and $\sum a_n s^n \to \sum a_n x_2^n$, are algebra homomorphisms.

(2) there exists a canonical topological A-algebra isomorphism $\psi : A\{x_1, x_2; \alpha\} \to \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$.

As a corollary, $A\{x_1, x_2; \alpha\}$ becomes an Arens-Michael algebra.

Proof Fix a generating directed family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on A. For every $k > 0$ we identify $(A_\alpha \oplus A_{\alpha^{-1}})^{\hat{\otimes} k}$ with $\bigoplus_{|w|=k} A_w$. If we denote the projective tensor product of k copies of $\|\cdot\|_\lambda + \|\cdot\|_\lambda$ by $\|\cdot\|_{\lambda, \lambda}^{\hat{\otimes} n}$, we can rewrite the definition of $\widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$ as follows:

$$\widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}}) = \left\{ (x_w) \in \prod_{w \in W_2} A_w : \|(x_w)\|_{\lambda, \rho} = \sum_{n \geq 0} \|(x_w)_{|w|=n}\|_{\lambda, \lambda}^{\hat{\otimes} n} \rho^n < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}.$$

Moreover, notice that for every $x_w \in A_w$, $\lambda \in \Lambda$ we have

$$\left\|(x_w)_{|w|=n}\right\|_{\lambda, \lambda}^{\hat{\otimes} n} = \sum_{|w|=n} \|i_w(x_w)\|_\lambda^{(w)}$$

by the definition of $\|\cdot\|_\lambda^{(w)}$.

For any element $a \in A\{x_1, x_2; \alpha\}$ we define ψ as follows:

$$(\psi(a))_w = a_w \otimes 1 \cdots \otimes 1.$$

Therefore, for any $0 < \rho < \infty$ and $\lambda \in \Lambda$ we have

$$\|\psi(a)\|_{\lambda,\rho} = \sum_{n=0}^{\infty} \left(\left\| (a_w \otimes 1 \cdots \otimes 1)_{|w|=n} \right\|_{\lambda}^{\hat{\otimes} n} \right) \rho^n = \sum_{n=0}^{\infty} \left(\sum_{|w|=n} \|i_w(a_w \otimes 1 \cdots \otimes 1)\|_{\lambda}^{(w)} \right) \rho^n =$$

$$= \sum_{n=0}^{\infty} \left(\sum_{|w|=n} \|a_w\|_{\lambda}^{(w)} \right) \rho^n = \|a\|_{\lambda,\rho}.$$

Therefore, we have proven that ψ is a topological isomorphism of locally convex spaces, and Lemma 4.4 ensures that ψ is an algebra homomorphism, and the existence of natural inclusions

$$T_A(A_\alpha) \hookrightarrow \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}}), \quad T_A(A_{\alpha^{-1}}) \hookrightarrow \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}})$$

implies (1). □

Corollary 4.7 *Suppose that A is a Fréchet-Arens-Michael algebra and α is an automorphism of A. Then*

$$\widehat{L}_A(A_\alpha) \simeq A\{x_1, x_2; \alpha\}/\overline{(x_1 x_2 - 1, x_2 x_1 - 1)}.$$

We also provide some examples of explicit computations of $A\{x, y; \alpha\}$ in Appendix A.

5. Open questions

1. How can one characterize the Arens-Michael envelopes among all Arens-Michael algebras? In particular, is every Arens-Michael algebra isomorphic to the Arens-Michael envelope of an associative algebra?

   Remark. This question is nontrivial because, in general, the Arens-Michael envelope of an Arens-Michael algebra A is not isomorphic to A itself, even when A is a Banach algebra, as follows from [3, Theorems 5.1.17, 5.1.18]. The basic idea is that there might be multiple non-equivalent norms on an algebra which turn it into a Banach algebra.

2. Consider an element $f \in F_2 = \mathbb{C}\langle x, y \rangle$. Is there a way to determine whether the Arens-Michael envelope of $\widehat{F_2/(f)}$ is isomorphic to the zero algebra? This question is equivalent to finding a nonzero Banach algebra B and a nontrivial pair $(x, y) \in B^2$ satisfying $f(x, y) = 0$. In particular, is it a decidable problem?

   A slightly different case, featuring the relations of form $f(x, y) = [x, y] - h(y)$, where h is a holomorphic function, is considered in a recent preprint [2] of O. Aristov (in Russian).

3. Does Conjecture 3.10 hold for every algebra A and an invertible R-bimodule M?

4. There are a lot of interesting algebras for which the Arens-Michael envelopes are yet to be explicitly described. For example, consider the quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$.

   Definition 5.1 *The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is the associative unital algebra generated by E, F, K, K^{-1} with the following relations:*

   $$KE = q^2 EK, \ KF = q^{-2}FK, \ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

This algebra can be represented as an iterated Ore extension:

$$U_q(\mathfrak{sl}_2) \simeq \mathbb{C}[K, K^{-1}][F; \alpha_0][E; \alpha_1, \delta],$$

and we have the following result due to D. Pedchenko [*]:

Theorem 5.2 *Consider* $|q| = 1, q \neq -1, 1$. *Then*

$$\widehat{U_q(\mathfrak{sl}_2)} \simeq \left\{ f = \sum_{i,j \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}} c_{ijk} K^i F^j E^k : \|f\|_\rho := \sum_{i,j,k} |c_{ijk}| \rho^{i+j+k} < \infty \; \forall \rho > 0 \right\},$$

with multiplication, uniquely defined by the relations in Definition 5.1.

When $|q| \neq 1$ this representation becomes useless to us, because the morphisms cease to be m-localizable.

In fact, this problem was what motivated us to tackle the description of the Arens-Michael envelope of Laurent Ore extensions in the general case. Consider the following isomorphism:

$$U_q(\mathfrak{sl}_2) \simeq \frac{\mathbb{C}\langle E, F \rangle [K, K^{-1}; \alpha]}{([E, F] - \frac{K - K^{-1}}{q - q^{-1}})},$$

where $\alpha(E) = q^2 E$ and $\alpha(F) = q^{-2} F$. Then we use the main result:

$$(\mathbb{C}\langle E, F \rangle [K, K^{-1}; \alpha])\hat{} \simeq \widehat{L}_{\mathcal{F}_2}((\mathcal{F}_2)_\alpha) \simeq \mathcal{F}_2\{x_1, x_2; \alpha\}/\overline{(x_1 x_2 - 1, x_2 x_1 - 1)}. \tag{5.1}$$

Unfortunately, the algebra $\mathcal{F}_2\{x, y; \alpha\}$ itself turned out be too difficult to describe explicitly. As the readers can see, Example A.5 demonstrates how computationally difficult this approach is.

However, there is some progress made by O. Aristov[†], where he provides a description of the Arens-Michael envelope of $U_q(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$ and $|q| \neq 1$ in terms of the equivalence classes of irreducible finite-dimensional representations of $U_q(\mathfrak{g})$, in the spirit of J. Taylor's result. Unfortunately, this approach does not provide a power-series representation of $\widehat{U_q(\mathfrak{sl}_2)}$ for $|q| \neq 1$. So, for now, (5.1) is the closest thing to a power series representation of the Arens-Michael envelope of $U_q(\mathfrak{g})$ that is available to us.

A. Several examples of explicit computations of $\widehat{T}_A(A_\alpha)$ and $\widehat{L}_A(A_\alpha)$

Here we will provide several examples, which illustrate the complexity of "extensions" $\widehat{T}_A(A_\alpha) = A\{x; \alpha\}, \widehat{T}_A(A_\alpha \oplus A_{\alpha^{-1}}) = A\{x, y; \alpha\}$ (and $\widehat{L}_A(A_\alpha)$ as a corollary) even for the simplest and most natural cases.

We want to consider the case of non-m-localizable pairs $\{\alpha, \alpha^{-1}\}$, because the m-localizable case has been already treated in Section 3.1.

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[*]D. Pedchenko (2020). The Arens-Michael envelopes of the Jordanian Plane and $U_q(sl(2))$ [online]. Website https://arxiv.org/abs/2009.06477 [accessed 17 February 2022].

[†]O. Aristov (2020). Banach space representations of Drinfeld-Jimbo algebras and their complex-analytic forms [online]. Website https://arxiv.org/abs/2012.12565 [accessed 17 February 2022].

Lemma A.1 *Consider an Arens-Michael algebra A with topology, generated by a family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ and $\alpha \in Aut(A)$. Denote*

$$w_n = (\underbrace{1,1,\ldots,1,1}_{n \ times}, \underbrace{2,2,\ldots,2,2}_{n \ times}), \quad w'_n = (\underbrace{2,2,\ldots,2,2}_{n \ times}, \underbrace{1,1,\ldots,1,1}_{n \ times}).$$

Suppose that there is an element $r \in A$ such that

$$\lim_{n\to\infty} (\|r\|_\lambda^{(w_n)} \rho^{2n}) = 0 \tag{A.1}$$

for every $\lambda \in \Lambda$ and $\rho > 0$ (in other words, the sequence $(\|r\|_\lambda^{(w_n)})$ is rapidly decaying), or

$$\lim_{n\to\infty} (\|r\|_\lambda^{(w'_n)} \rho^{2n}) = 0. \tag{A.2}$$

Then $r \in I \subset A\{x_1, x_2; \alpha\}$, where I is the smallest closed two-sided ideal, which contains $x_1 x_2 - 1$ and $x_2 x_1 - 1$. In particular, if there exists an invertible element $r \in A$, which satisfies (A.1) or (A.2), then $\widehat{L}_A(A_\alpha) = 0$.

Proof   Notice that $x_1^k x_2^k - 1 \in I$ for any $k > 0$; therefore, $r - r x_1^k x_2^k \in I$, but

$$\|(r - r x_1^k x_2^k) - r\|_{\lambda,\rho} = \|r\|_\lambda^{(w_k)} \rho^{2k} \xrightarrow[k\to\infty]{} 0.$$

As we can see, the sequence $r - r x_1^k x_2^k$ converges to r in the topology of $A\{x_1, x_2; \alpha\}$ due to the assumptions in our Lemma; therefore, $r \in \overline{I}$. $\square$

Example A.2 *Consider $A = C(\mathbb{R})$ and $\alpha(f)(x) = f(x-1)$ for $f \in C(\mathbb{R}), x \in \mathbb{R}$. Recall that the topology on A is generated by the family $\|f\|_K := \sup_{x \in K} |f(x)|$, where $K \subset \mathbb{R}$ is a compact subset. Notice that instead of all K we could take all the intervals $[-x, x]$ for $x > 0$ or even $[-a_n, a_n]$, where (a_n) is an arbitrary increasing unbounded sequence.*

Let $|w| > 1$. Then we can write down a lower estimate for $\|\cdot\|_{[-n,n]}^{(w)}$ as follows:

$$\|f\|_{[-n,n]}^{(w)} = \inf_{f=\sum_{j=1}^k i_w(f_{1,j}\otimes\cdots\otimes f_{|w|,j})} \sum_{j=1}^k \|f_{1,j}\|_{[-n,n]} \cdots \|f_{|w|,j}\|_{[-n,n]} =$$

$$= \inf_{f=\sum_{j=1}^k i_w(f_{1,j}\otimes\cdots\otimes f_{|w|,j})} \sum_{j=1}^k \|f_{1,j}\|_{[-n,n]} \cdots \left\|\alpha^{p(w,|w|-1)}(f_{|w|,j})\right\|_{[-n+p(w,|w|-1),n+p(w,|w|-1)]} \geq$$

$$\geq \inf_{f=\sum_{j=1}^k i_w(f_{1,j}\otimes\cdots\otimes f_{|w|,j})} \sum_{j=1}^k \|f_{1,j}\|_{I_n^{(w)}} \cdots \left\|\alpha^{p(w,|w|-1)}(f_{|w|,j})\right\|_{I_n^{(w)}} \geq \|f\|_{I_n^{(w)}},$$

where

$$I_n^{(w)} = \bigcap_{i=1}^{|w|-1} [-n + p(w,i), n + p(w,i)]$$

for $|w| > 1$, and $I_n^{(w)} = [-n, n]$ for $|w| \leq 1$.

Notice that if the intersection is empty, then we say that the respective seminorm is identically zero. If we denote

$$k_{min}(w) = \begin{cases} \min\limits_{1 \le i \le |w|-1} p(w,i) & , |w| > 1, \\ 0 & , |w| \le 1, \end{cases} \quad k_{max}(w) = \begin{cases} \max\limits_{1 \le i \le |w|-1} p(w,i) & , |w| > 1, \\ 0 & , |w| \le 1, \end{cases}$$

then

$$I_n^{(w)} = [-n + k_{max}(w), n + k_{min}(w)].$$

Now we aim to prove that $\|f\|_{[-n,n]}^{(w)} = \|f\|_{I_n^{(w)}}$. Consider the representation

$$f = \alpha^{k_{max}(w)}(g) + \alpha^{k_{min}(w)}(h) + (f - \alpha^{k_{max}(w)}(g) - \alpha^{k_{min}(w)}(h)) \text{ for any } g, h \in C(\mathbb{R}).$$

Thus, we get an upper estimate:

$$\|f\|_{[-n,n]}^{(w)} \le \inf_{g,h \in C(\mathbb{R})} \|g\|_{[-n,n]} + \|h\|_{[-n,n]} + \left\| f - \alpha^{k_{max}(w)}(g) - \alpha^{k_{min}(w)}(h) \right\|_{[-n,n]} \tag{A.3}$$

Denote the function $\tilde{f} = f - \alpha^{k_{max}(w)}(g) - \alpha^{k_{min}(w)}(h)$. Suppose that $-n + k_{max} \le n + k_{min}$. Then consider the following g_m and h_m:

$$g_m(x) = \begin{cases} f(x + k_{max}), & x < -n - 1/m \\ (-m(x + n))f(x + k_{max}), & x \in [-n - 1/m, -n], \\ 0, & x > -n \end{cases} \tag{A.4}$$

$$h_m(x) = \begin{cases} 0, & x < n \\ (m(x - n))f(x + k_{min}), & x \in [n, n + 1/m] \\ f(x + k_{min}), & x > n + 1/m \end{cases} \tag{A.5}$$

Then we have to look at the right hand side of (A.3):

$$\|g_m\|_{[-n,n]} = \|h_m\|_{[-n,n]} = 0,$$

$$\alpha^{k_{max}(w)}(g_m)(x) = \begin{cases} f(x), & x < -n - 1/m + k_{max}(w) \\ (-m(x - k_{max}(w) + n))f(x), & x \in [-n - 1/m + k_{max}(w), -n + k_{max}(w)], \\ 0, & x > -n + k_{max}(w) \end{cases} \tag{A.6}$$

$$\alpha^{k_{min}(w)}(h_m)(x) = \begin{cases} 0, & x < n + k_{min}(w) \\ (m(x - k_{min}(w) - n))f(x), & x \in [n + k_{min}(w), n + k_{min}(w) + 1/m] \\ f(x), & x > n + 1/m + k_{min}(w) \end{cases} \tag{A.7}$$

$$\tilde{f}(x) = \begin{cases} 0 & x < -n - 1/m + k_{max}(w) \\ (1 + m(x - k_{max}(w) + n))f(x) & x \in [-n - 1/m + k_{max}(w), -n + k_{max}(w)] \\ f(x) & x \in [-n + k_{max}(w), n + k_{min}(w)] = I_n^{(w)} \\ (1 - m(x - k_{min}(w) - n))f(x) & x \in [n + k_{min}(w), n + 1/m + k_{min}(w)] \\ 0 & x > n + 1/m + k_{min}(w), \end{cases} \tag{A.8}$$

so

$$\|f\|_{[-n,n]}^{(w)} = \left\|\tilde{f}\right\|_{[-n,n]} \le \|f\|_{[-n+k_{max}(w)-1/m,n+1/m+k_{min}(w)]}.$$

By taking $m \to 0$, we get the desired equality.

If $-n + k_{max}(w) > n + k_{min}(w)$, then we can look at $g_3(x)$ and $h_3(x)$. Notice that

$$-n + k_{max}(w) - 1/3 > n + k_{min}(w) + 1/3,$$

so the computations above show us that the supports of $\alpha^{k_{max}(w)}(g_3)$ and $\alpha^{k_{min}(w)}(h_3)$ have empty intersection, so $\tilde{f} \equiv 0$ for g_3 and h_3; therefore,

$$\|f\|_{[-n,n]}^{(w)} = \left\|\tilde{f}\right\|_{[-n,n]} = 0.$$

This argument worked for $|w| > 1$, but we know that

$$\|\cdot\|_{[-n,n]}^{(w)} = \|\cdot\|_{[-n,n]}$$

when $|w| \le 1$.

To sum everything up, we have deduced that the algebras $C(\mathbb{R})\{x; \alpha\}$ and $C(\mathbb{R})\{x_1, x_2; \alpha\}$ look as follows:

$$C(\mathbb{R})\{x; \alpha\} = \left\{ a = \sum_{k \ge 0} a_k x^k : \|a\|_{n,\rho} := \|a_0\|_{[-n,n]} + \sum_{k \ge 1} \|a_k\|_{[-n+k-1,n-k+1]} \rho^k < \infty, \ \forall n > 0, 0 < \rho < \infty \right\},$$

$$C(\mathbb{R})\{x_1, x_2; \alpha\} = \left\{ a = \sum_{w \in W_2} a_w x^w : \|a\|_{n,\rho} := \sum_{w \in W_2} \|a_w\|_{I_n^{(w)}} \rho^{|w|} < \infty, \ \forall w \in W_2, 0 < \rho < \infty \right\}.$$

It is easily seen that the isomorphism $C(\mathbb{R})\{x; \alpha\} \simeq C(\mathbb{R})[[x]]$ takes place, because $\|\cdot\|_{[-n,n]}^{(k)} = 0$ for $k > 2n$. Therefore, Lemma A.1 implies that $\widehat{L}_A(A_\alpha) = 0$, because the $(\|1\|_{[-n,n]}^{(w_k)})_{k \in \mathbb{N}} \in c_{00}$.

Remark. As the readers can see, computing the topological tensor algebras using just the definition requires a lot of effort. In the examples that will follow next, we are going to use several tricks which simplify the computations.

Example A.3 *What happens if we consider the shift automorphism on the algebra of holomorphic functions $\mathcal{O}(\mathbb{C})$ instead of $C(\mathbb{R})$? We get a result, which is similar to what we got in the Example A.2, as stated in [8, Example 4.3]:*

Proposition A.4 *Let $R = \mathcal{O}(\mathbb{C})$. Consider an automorphism $\alpha(f)(z) = f(z-1)$. Then $R\{x; \alpha\} \cong R[[x]]$ as locally convex spaces, where the topology on $R[[x]]$ is generated by $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$, where $\left\|\sum_k^\infty a_k x^k\right\|_n = \|a_n\|$.*

The proof cleverly utilizes Mergelyan's approximation theorem. In particular, A. Yu. Pirkovskii proves that $\|\cdot\|_\rho^{(n+1)} = 0$ *for* $n > \lfloor 2\rho \rfloor + 1$; *therefore,* $\|\cdot\|_\rho^{(w_{n+1})} \le \|\cdot\|_\rho^{(n+1)} = 0$, *so* $\widehat{L}_R(R_\alpha) = 0$, *as well. Equivalently, we have*

$$(\mathbb{C}[x][y, y^{-1}; \alpha])\widehat{} = 0,$$

where $\alpha(f)(x) = f(x - 1)$.

Example A.5 *Let* $A = \mathcal{O}(\mathbb{C})$ *and consider the automorphism* $\beta_q : A \to A$, $\beta_q(f)(z) = f(qz)$, *where* $|q| \ne 1$. *Fix a generating family of seminorms* $\{\|\cdot\|_\rho : 0 < \rho < \infty\}$ *on* A, *where*

$$\|f\|_\rho = \left\|\sum_k f^{(k)} z^k\right\|_\rho := \sum_k |f^{(k)}| \rho^k.$$

Let us try to describe $\widehat{L}_A(A_{\beta_q})$. *To do this, we will need the following lemma:*

Lemma A.6 *Let* α *be an automorphism of an associative algebra* A. *Suppose that for any Banach algebra* B *and any algebra homomorphism* $\varphi : A[x; \alpha] \to B$ *the element* $\varphi(x)$ *is nilpotent in* B. *Then*

$$(A[x; \alpha])\widehat{} \simeq \widehat{A}[[x]], \quad (A[x, x^{-1}; \alpha])\widehat{} \simeq 0$$

as a locally convex space.

Proof Our argument follows the idea given in [8, Example 5.1]. If $||\cdot||_\nu$ is a submultiplicative seminorm on A, then for every $n > 0$ we can define the following submultiplicative seminorm $||\cdot||_{\nu,n}$ on $A[x; \alpha]$ as follows:

$$\left\|\sum_{i=0}^m a_i x^i\right\|_{\nu,n} = \sum_{i=0}^n ||a_i||_\nu.$$

Now, suppose that $||\cdot||$ is a submultiplicative seminorm on $A[x; \alpha]$. As $||x^N|| = 0$ for some $N > 0$, we get that

$$\left\|\sum_{i=0}^m a_i x^i\right\| = \left\|\sum_{i=0}^{N-1} a_i x^i\right\| \le \sum_{i=0}^{N-1} ||x^i|| ||a_i|| \le (\max_i ||x^i||) \sum_{i=0}^{N-1} ||a_i||.$$

However, $(\max_i ||x^i||)$ is just a constant, and we have proven that $||\cdot||$ has to be dominated by a seminorm of form $||\cdot||_{\nu', N-1}$. Therefore, the Arens-Michael envelope of $(A[x; \alpha])\widehat{}$ is the completion of $A[x; \alpha]$ with respect to $||\cdot||_{\nu,n}$ for all ν and $n > 0$, but this completion is, indeed, isomorphic to $\widehat{A}[[x]]$.

For the second isomorphism, we just need to notice that there are no nontrivial Banach algebras with invertible nilpotents, so there are no nontrivial homomorphisms from $A[x, x^{-1}; \alpha]$ to Banach algebras. Therefore, the Arens-Michael envelope is trivial. □

Now all that remains is to identify $\mathcal{O}(\mathbb{C})$ *with the Arens-Michael envelope of* $\mathbb{C}[z]$, *then we make use of several isomorphisms:*

$$\widehat{L}_A(A_{\beta_q}) \simeq (\mathbb{C}[z][x, x^{-1}; \beta_q])\widehat{} \simeq (\mathbb{C}[x, x^{-1}][z; \beta_q^{-1}])\widehat{}.$$

*A well-known fact about Banach algebras states that if x, y are (nonzero) elements of a Banach algebra B, where x is invertible and $xy = qyx$ holds for some $|q| \neq 1$ and $q \neq 0$, then y is nilpotent (see [**?** , Lemma 1.2] for proof). This allows us to use Lemma A.6, which immediately yields*

$$\widehat{L}_A(A_{\beta_q}) \simeq (\mathbb{C}[x, x^{-1}][z; \beta_q^{-1}])\hat{} \simeq \mathcal{O}(\mathbb{C}^\times)[[z]]$$

as a locally convex space. This computation agrees with the result obtained in [1, Proposition 8.13]. Keep in mind that the multiplication on the Arens-Michael envelope is uniquely defined by the relations in $\mathbb{C}[z][x, x^{-1}; \beta_q]$

Remark. The same idea about considering nilpotent elements could be used in Example A.2, because the relation $xt = t(x - 1)$ also forces all homomorphic images of t to be nilpotent. The only difference is that $C(\mathbb{R})$ is already a Banach algebra itself, and it is not immediately clear how to express it as an Arens-Michael envelope. However, the argument can be modified considering the fact that both topological tensor algebras still satisfy their respective universal properties.

Remark. The previous version of this preprint relied on a massive computation which used the definition of the topological Laurent tensor product directly. Finally, it is worth noting that this trick does not immediately work for $A = \mathcal{F}_2$, as we cannot interchange the extra Ore extension variable with the generators of $\mathcal{F}_2$ due to noncommutativity issues.

Acknowledgments

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