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The radii of starlikeness and convexity of the functions including derivatives of Bessel functions

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Abstract: Let $J_\nu(z)$ denote the Bessel function of the first kind of order ν. In this paper, our aim is to determine the radii of starlikeness and convexity for three kind of normalization of the function $N_\nu(z) = az^2 J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)$ in the case where zeros are all real except for a single pair, which are conjugate purely imaginary. The key tools in the proof of our main results are the Mittag–Leffler expansion for function $N_\nu(z)$ and properties of real and complex zeros of it.

Key words: Normalized Bessel functions of the fist kind, convex functions, starlike functions, zeros of Bessel function derivatives, radius

1. Introduction

Denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\} \quad (r > 0)$ the disk of radius r and let $\mathbb{D} = \mathbb{D}_1$. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. We say that the function $f \in \mathcal{A}$ is starlike in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin. Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}_r$. For $\beta \in [0,1)$ we say that the function f is starlike of order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{D}_r$. We define by the real number

$$r^*_\beta(f) = \sup\left\{r \in (0, r_f) : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r\right\}$$

the radius of starlikeness of order β of the function f. Note that $r^*(f) = r^*_0(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^*_\beta(f)})$ is a starlike domain with respect to the origin.

The function $f \in \mathcal{A}$ is convex in the disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, the function f is convex in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}_r$. For $\beta \in [0,1)$ we say that the function f is convex of order β in $\mathbb{D}_r$ if and only if $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \mathbb{D}_r$. The radius of

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convexity of order β of the function f is defined by the real number

$$r_\beta^c(f) = \sup\left\{r \in (0, r_f) : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \text{ for all } z \in \mathbb{D}_r\right\}.$$

Note that $r^c(f) = r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c(f)})$ is a convex domain.
The Bessel function of the first kind of order ν is defined by [16, p. 217]

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \quad (z \in \mathbb{C}). \tag{1.1}$$

We know that it has all its zeros real for $\nu > -1$. In this study, we consider mainly the general function

$$N_\nu(z) = az^2 J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z)$$

where $(c = 0 \ \text{ and } \ b \neq a)$ or $(c > 0 \ \text{ and } \ b > a)$, studied by Mercer [15].
From (1.1), we have the power series representation

$$N_\nu(z) = \sum_{n=0}^{\infty} \frac{Q(2n+\nu)\,(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \quad (z \in \mathbb{C}), \tag{1.2}$$

where $Q(\nu) = a\nu^2 + (b-a)\nu + c \ (a, b, c \in \mathbb{R})$.
Note that N_ν is not belongs to $\mathcal{A}$. To prove the main results, we need normalization of the function N_ν. In this paper, we will be concerned with the following normalized forms

$$\begin{aligned} f_\nu(z) &= \left[\frac{2^\nu\Gamma(\nu+1)}{Q(\nu)} N_\nu(z)\right]^{\frac{1}{\nu}}, \\ g_\nu(z) &= \frac{2^\nu\Gamma(\nu+1)z^{1-\nu}}{Q(\nu)} N_\nu(z), \\ h_\nu(z) &= \frac{2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}}{Q(\nu)} N_\nu(\sqrt{z}). \end{aligned}$$

There are three important works on the function N_ν. First one is Mercer's paper [15], which has proved that the k^{th} positive zero of N_ν increases with ν in $\nu > 0$. Second one is the study of Ismail and Muldoon [10], who showed that under the conditions $a, b, c \in \mathbb{R}$ such that $(c = 0 \ \text{ and } \ b \neq a)$ or $(c > 0 \ \text{ and } \ b > a)$;

(i) For $\nu > 0$, the zeros of $N_\nu(z)$ are either real or purely imaginary,

(ii) For $\nu \geq \max\{0, \nu_0\}$, where ν_0 is the largest real root of the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$, the zeros of $N_\nu(z)$ are real,

(iii) If $\nu > 0$, $Q(\nu)/(b-a) > 0$ and $a/(b-a) < 0$, the zeros of $N_\nu(z)$ are all real except for a single pair, which are conjugate purely imaginary.

Lastly, Baricz, Çağlar and Deniz [4] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ν by using the results of Mercer [15], Ismail and Muldoon [10] and Shah and Trimble [17].

Recently, in [12] authors obtained radii of starlikeness and convexity of order β for the function f_ν, g_ν and h_ν under the condition (ii) related with the zeros of $N_\nu(z)$. Also, for studies on the geometric properties of Bessel functions, see [1–9, 13, 18]. In this paper, we deal with same problem for the condition (iii). The key tools in their proofs were some new Mittag–Leffler expansions for quotients of the function N_ν, special properties of the zeros of the function N_ν and their derivatives.
In the rest of this paper, the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$ will always provide on $a, b, c \in \mathbb{R}$ $(c = 0$ and $a \neq b)$ or $(c > 0$ and $a < b)$.

1.1. Zeros of hyperbolic polynomials and the Laguerre–Pólya class of entire functions

In this subsection, we recall some necessary information about polynomials and entire functions with real zeros. An algebraic polynomial is called hyperbolic if all its zeros are real. We formulate the following specific statement that we shall need, see [9] for more details.

By definition, a real entire function ψ belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2+\beta x} \prod_{k\geq 1} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}\cup\{0\}$ and $\sum x_k^{-2} < \infty$. Similarly, φ is said to be of type $\mathcal{I}$ in the Laguerre-Pólya class, written $\varphi \in \mathcal{LPI}$, if $\varphi(x)$ or $\varphi(-x)$ can be represented as

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k\geq 1} \left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N} \cup \{0\}$, $x_k > 0$ and $\sum x_k^{-1} < \infty$. The class $\mathcal{LP}$ is the complement of the space of hyperbolic polynomials in the topology induced by the uniform convergence on the compact sets of the complex plane, while $\mathcal{LPI}$ is the complement of the hyperbolic polynomials whose zeros possess a preassigned constant sign. Given an entire function φ with the Maclaurin expansion

$$\varphi(x) = \sum_{k\geq 0} \mu_k \frac{x^k}{k!},$$

its Jensen polynomials are defined by

$$P_m(\varphi; x) = P_m(x) = \sum_{k=0}^{m} \binom{m}{k} \mu_k x^k.$$

In [12], authors proved the following lemma by using above the class $\mathcal{LP}$ and Jensen polynomials.

Lemma 1.1 *[12] If $\nu \geq \max\{0, \nu_0\}$ then the functions $z \longmapsto \Psi_\nu(z) = \frac{2^\nu \Gamma(\nu+1)}{Q(\nu) z^\nu} N_\nu(z)$ has infinitely many zeros and all of them are positive. Denoting by $\lambda_{\nu,n}$ the n^{th} positive zero of $\Psi_\nu(z)$, under the same conditions the Weierstrassian decomposition*

$$\Psi_\nu(z) = \prod_{n\geq 1} \left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right)$$

is valid, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\lambda'_{\nu,n}$ *the* n^{th} *positive zero of* $\Phi'_\nu(z)$*, where* $\Phi_\nu(z) = z^\nu \Psi_\nu(z)$*, then the positive zeros of* $\Psi_\nu(z)$ *are interlaced with those of* $\Phi'_\nu(z)$*. In the other words, the zeros satisfy the chain of inequalities*

$$\lambda'_{\nu,1} < \lambda_{\nu,1} < \lambda'_{\nu,2} < \lambda_{\nu,2} < \lambda'_{\nu,3} < \lambda_{\nu,3} < \cdots .$$

Following lemmas are key tools in the proof of main results.

Lemma 1.2 *[12] The following equality holds*

$$\sum_{n\geq 1} \frac{1}{\lambda^2_{\nu,n}} = \frac{Q(\nu+2)}{4(\nu+1)Q(\nu)},$$

where $\lambda_{\nu,n}$ *is the* n^{th} *positive zero of* N_ν.

The zeros $z^{-\nu}N_\nu(z) = 0$ are taken to be $\pm\lambda_{\nu,n}$, where $n \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$. We may suppose without restricting the generality that $\lambda_{\nu,1} = i\alpha$, $\lambda'_{\nu,1} = i\alpha'$, $0 < \alpha' < \alpha$ and $\lambda'_{\nu,2} < \lambda_{\nu,2} < \lambda'_{\nu,3} < \lambda_{\nu,3} < \cdots < \lambda'_{\nu,n} < \lambda_{\nu,n} < \cdots$. It is well-known that $\alpha' < \lambda'_{\nu,2}$ and $\alpha < \lambda_{\nu,2}$.

Lemma 1.3 *[12] If* $\nu > 0,$*, then we have*

$$\begin{aligned}
\frac{zf'_\nu(z)}{f_\nu(z)} &= \frac{1}{\nu}\frac{zN'_\nu(z)}{N_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\nu,n} - z^2}, \\
\frac{zg'_\nu(z)}{g_\nu(z)} &= (1-\nu) + \frac{zN'_\nu(z)}{N_\nu(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\nu,n} - z^2}, \\
\frac{zh'_\nu(z)}{h_\nu(z)} &= (1 - \frac{\nu}{2}) + \frac{1}{2}\frac{\sqrt{z}N'_\nu(\sqrt{z})}{N_\nu(\sqrt{z})} = 1 - \sum_{n\geq 1}\frac{z}{\lambda^2_{\nu,n} - z}.
\end{aligned}$$

These series are uniformly convergent on every compact subset of $\mathbb{C} \setminus \{\pm\lambda_{\nu,n} : n \in \mathbb{N}\}$.

Lemma 1.4 *[12] If* $\nu > 0$*, then we have*

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 - \left(\frac{1}{\nu} - 1\right)\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\nu,n} - z^2} - \sum_{n\geq 1}\frac{2z^2}{\lambda'^2_{\nu,n} - z^2}, \tag{1.3}$$

$$1 + \frac{zg''_\nu(z)}{g'_\nu(z)} = 1 - \sum_{n\geq 1}\frac{2z^2}{\delta^2_{\nu,n} - z^2} \tag{1.4}$$

and

$$1 + \frac{zh''_\nu(z)}{h'_\nu(z)} = 1 - \sum_{n\geq 1}\frac{z}{\gamma^2_{\nu,n} - z}, \tag{1.5}$$

where $\lambda'_{\nu,n}, \delta_{\nu,n}$ *and* $\gamma_{\nu,n}$ *are the* n^{th} *positive zeros of the functions* N'_ν, g'_ν *and* h'_ν*, respectively.*

Lemma 1.5 *[3] If* $v \in \mathbb{C},\ \delta \in \mathbb{R}$ *and* $\delta > |v|$, *then*

$$\frac{|v|}{\delta - |v|} \geq \operatorname{Re}\left(\frac{v}{\delta - v}\right) \geq \frac{-|v|}{\delta + |v|}, \tag{1.6}$$

$$\frac{|v|}{\delta + |v|} \geq \operatorname{Re}\left(\frac{v}{\delta + v}\right) \geq \frac{-|v|}{\delta - |v|}. \tag{1.7}$$

Moreover, $v \in \mathbb{C},\ \gamma, \delta \in \mathbb{R}$ *and* $\gamma \geq \delta > r \geq |v|$, *then*

$$\frac{r^2}{(\delta - r)(\gamma + r)} \geq \operatorname{Re}\left(\frac{v^2}{(\delta + v)(\gamma - v)}\right). \tag{1.8}$$

2. Main results

2.1. Radii of starlikeness and sonvexity of the functions $f_\nu,\ g_\nu$ and h_ν

For convenience in the rest of the paper, we shall use the following notation

$$\begin{aligned} M_\nu(z) &= i^{-\nu} N_\nu(iz) = i^{-\nu}\left[az^2 J''_\nu(iz) + bz J'_\nu(iz) + cJ_\nu(iz)\right] \\ &= az^2 I''_\nu(z) + bz I'_\nu(z) + cI_\nu(z), \end{aligned}$$

where $I_\nu(z)$ denotes the modified Bessel function of the first kind and order ν. Note that $I_\nu(z) = i^{-\nu} J_\nu(iz)$ and $I_\nu(\sqrt{z}) = (-1)^{-\nu/2} J_\nu(\sqrt{-z})$.
The first principal result gives the radii of starlikeness of the functions $f_\nu,\ g_\nu$ and h_ν.

Theorem 2.1 *Let* $\beta \in [0,1),\ \nu > 0,\ Q(\nu)/(b-a) > 0,\ a/(b-a) < 0$ *and* $\frac{Q(\nu+2)}{Q(\nu)} < 0$. *The following statements hold:*

a) *The radius of starlikeness of order* β *of the function* f_ν *is the smallest positive root of the equation*

$$\frac{1}{\nu}\frac{ar^3 I'''_\nu(r) + (2a+b)r^2 I''_\nu(r) + (b+c)rI'_\nu(r)}{ar^2 I''_\nu(r) + brI'_\nu(r) + cI_\nu(r)} = \beta.$$

b) *The radius of starlikeness of order* β *of the function* g_ν *is the smallest positive root of the equation*

$$(1-\nu) + \frac{ar^3 I'''_\nu(r) + (2a+b)r^2 I''_\nu(r) + (b+c)rI'_\nu(r)}{ar^2 I''_\nu(r) + brI'_\nu(r) + cI_\nu(r)} = \beta.$$

c) *The radius of starlikeness of order* β *of the function* h_ν *is the smallest positive root of the equation*

$$(2-\nu) + \frac{ar\sqrt{r} I'''_\nu(\sqrt{r}) + (2a+b)\, r I''_\nu(\sqrt{r}) + (b+c)\sqrt{r} I'_\nu(\sqrt{r})}{ar I''_\nu(\sqrt{r}) + b\sqrt{r} I'_\nu(\sqrt{r}) + cI_\nu(\sqrt{r})} = 2\beta.$$

Proof a): In view of Lemma 1.3, we have

$$\frac{zf'_\nu(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zN'_\nu(z)}{N_\nu(z)} = 1 - \frac{1}{\nu}\sum_{n\geq 1}\frac{2z^2}{\lambda^2_{\nu,n} - z^2} \quad (\nu > 0).$$

From hypothesis of Theorem 2.1, we can write $\lambda_{\nu,1} = i\alpha \ (\alpha > 0)$. Thus, from Lemma 1.2 we obtain

$$\begin{aligned}
\frac{zf_\nu'(z)}{f_\nu(z)} &= 1+\frac{1}{\nu}\left(2\frac{z^2}{\alpha^2+z^2}-2\sum_{n\geq 2}\frac{z^2}{\lambda_{\nu,n}^2-z^2}\right) = 1+\frac{1}{\nu}\left(2\frac{\alpha^2 z^2}{\alpha^2+z^2}\frac{1}{\alpha^2}-2\sum_{n\geq 2}\frac{z^2}{\lambda_{\nu,n}^2-z^2}\right) \\
&= 1+\frac{2}{\nu}\frac{\alpha^2 z^2}{\alpha^2+z^2}\left(\sum_{n\geq 2}\frac{1}{\lambda_{\nu,n}^2}-\frac{Q(\nu+2)}{4(\nu+1)Q(\nu)}\right)-\frac{2}{\nu}\sum_{n\geq 2}\frac{z^2}{\lambda_{\nu,n}^2-z^2} \\
&= 1-\frac{1}{\nu}\left(\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2}+2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}\right).
\end{aligned}$$

On the other hand, replacing z by z^2 in (1.7) and (1.8), we get

$$\operatorname{Re}\frac{z^2}{\alpha^2+z^2} \geq \frac{-r^2}{\alpha^2-r^2} \quad \text{and} \quad \frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)} \geq \operatorname{Re}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}. \tag{2.1}$$

Therefore, since $\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)} < 0$, we have

$$\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right) \geq 1+\frac{1}{\nu}\left(\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right).$$

Consequently, the following equality holds

$$\begin{aligned}
\inf_{z\in\mathbb{U}(r)}\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right) &= 1+\frac{1}{\nu}\left(\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right) \\
&= \left.\frac{zf_\nu'(z)}{f_\nu(z)}\right|_{z=ir}.
\end{aligned} \tag{2.2}$$

Since, the mapping

$$\chi:(0,\alpha)\longrightarrow\mathbb{R}, \quad \chi(r)=\left.\frac{zf_\nu'(z)}{f_\nu(z)}\right|_{z=ir} = 1-\frac{2r^2}{\nu}\left(\frac{1}{\alpha^2-r^2}-\sum_{n\geq 2}\frac{1}{\lambda_{\nu,n}^2+r^2}\right)$$

is strictly decreasing and

$$\lim_{r\searrow 0}\chi(r)=1>\beta, \ \lim_{r\nearrow\alpha}\chi(r)=-\infty,$$

it follows that the equation $\chi(r)=\beta$ has a unique root $r_1\in(0,\alpha)$ for every $\beta\in[0,1)$. A simple calculation shows that the equation $\chi(r)=\beta$ is equivalent to

$$\frac{1}{\nu}\frac{ar^3 I_\nu'''(r)+(2a+b)r^2 I_\nu''(r)+(b+c)rI_\nu'(r)}{ar^2 I_\nu''(r)+brI_\nu'(r)+cI_\nu(r)}=\beta.$$

Finally, (2.2) implies that

$$\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)>\beta, \quad z\in\mathbb{U}(r_1),$$

where r_1 is the biggest real value having this property.

b): According to Lemma 1.3, we have

$$\frac{zg_\nu'(z)}{g_\nu(z)} = 1 - \sum_{n\geq 1} \frac{2z^2}{\lambda_{\nu,n}^2 - z^2}.$$

By replacing $\lambda_{\nu,1} = i\alpha$, we get

$$\frac{zg_\nu'(z)}{g_\nu(z)} = 1 + 2\frac{z^2}{\alpha^2+z^2} - 2\sum_{n\geq 2}\frac{z^2}{\lambda_{\nu,n}^2 - z^2} = 1 + 2\frac{\alpha^2 z^2}{\alpha^2+z^2}\frac{1}{\alpha^2} - 2\sum_{n\geq 2}\frac{z^2}{\lambda_{\nu,n}^2 - z^2}.$$

Now, using the equality in Lemma 1.2 it seems that

$$\begin{aligned}\frac{zg_\nu'(z)}{g_\nu(z)} &= 1 + 2\frac{\alpha^2 z^2}{\alpha^2+z^2}\left(\sum_{n\geq 2}\frac{1}{\lambda_{\nu,n}^2} - \frac{Q(\nu+2)}{4(\nu+1)Q(\nu)}\right) - 2\sum_{n\geq 2}\frac{z^2}{\lambda_{\nu,n}^2 - z^2} \\ &= 1 - \frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2} - 2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2 - z^2\right)}.\end{aligned}$$

Let r be a fixed number $r \in (0,\alpha)$ and $z \in \overline{\mathbb{U}(0,r)}$. The minimum principle for harmonic functions implies

$$\begin{aligned}\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) &= \operatorname{Re}\left(1 - \frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2} - 2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2 - z^2\right)}\right) \\ &\geq \min_{|z|=r}\operatorname{Re}\left(1 - \frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2} - 2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2 - z^2\right)}\right) \\ &= 1 - \frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\operatorname{Re}\frac{z^2}{\alpha^2+z^2} - 2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\operatorname{Re}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2 - z^2\right)}.\end{aligned}$$

Thus, from (2.1) and $\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)} < 0$, we have

$$\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) \geq 1 + \frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2} - 2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2 + r^2\right)}.$$

Consequently, the following equality holds

$$\begin{aligned}\inf_{z\in\mathbb{U}(r)}\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) &= 1 + \frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2} - 2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2 + r^2\right)} \\ &= \left.\frac{zg_\nu'(z)}{g_\nu(z)}\right|_{z=ir}. \end{aligned} \tag{2.3}$$

Since, the mapping

$$\varphi : (0,\alpha) \longrightarrow \mathbb{R}, \quad \varphi(r) = \left.\frac{zg_\nu'(z)}{g_\nu(z)}\right|_{z=ir} = 1 - 2r^2\left(\frac{1}{\alpha^2-r^2} - \sum_{n\geq 2}\frac{1}{\lambda_{\nu,n}^2 + r^2}\right)$$

is strictly decreasing and

$$\lim_{r\searrow 0}\varphi(r)=1>\beta,\ \lim_{r\nearrow\alpha}\varphi(r)=-\infty,$$

it follows that the equation $\varphi(r)=\beta$ has a unique root $r_2\in(0,\alpha)$ for every $\beta\in[0,1)$. A simple calculation shows that the equation $\varphi(r)=\beta$ is equivalent to

$$(1-\nu)+\frac{ar^3I_\nu'''(r)+(2a+b)r^2I_\nu''(r)+(b+c)rI_\nu'(r)}{ar^2I_\nu''(r)+brI_\nu'(r)+cI_\nu(r)}=\beta.$$

Finally, from (2.3) we have

$$\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)>\beta,\quad z\in\mathbb{U}(r_2),$$

where r_2 is the biggest real value having this property.

c): In a similar way to the proof of Theorem **a)**, we deduce from Lemmas 1.2 and 1.3

$$\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)=1-\frac{\alpha^2Q(\nu+2)}{4(\nu+1)Q(\nu)}\operatorname{Re}\frac{z}{\alpha^2+z}-\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\operatorname{Re}\frac{z^2}{(\alpha^2+z)\left(\lambda_{\nu,n}^2-z\right)},\tag{2.4}$$

where $z\in\mathbb{U}(0,r)$ and $r\in(0,\alpha^2)$. We get from (1.7), (1.8) and (2.4) the inequality

$$\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)\geq 1+\frac{\alpha^2Q(\nu+2)}{4(\nu+1)Q(\nu)}\frac{r}{\alpha^2-r}-\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^2}{(\alpha^2-r)\left(\lambda_{\nu,n}^2+r\right)},$$

holds true. Equality holds for $z=-r$. Consequently, it follows that

$$\inf_{z\in\mathbb{U}(r)}\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)=-r\frac{h_\nu'(-r)}{h_\nu(-r)}.$$

Since the function $\varphi:\left(0,\alpha^2\right)\longrightarrow\mathbb{R}$, defined by

$$\varphi(r)=-r\frac{h_\nu'(-r)}{h_\nu(-r)}=1+\frac{\alpha^2Q(\nu+2)}{4(\nu+1)Q(\nu)}\frac{r}{\alpha^2-r}-\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^2}{(\alpha^2-r)\left(\lambda_{\nu,n}^2+r\right)}$$

is strictly decreasing, and for $\frac{\alpha^2Q(\nu+2)}{2(\nu+1Q(\nu))}<0,$

$$\lim_{r\searrow 0}\varphi(r)=1>\beta\quad\text{and}\quad\lim_{r\nearrow\alpha^2}\varphi(r)=-\infty,$$

it follows that the equation $-r\frac{h_\nu'(-r)}{h_\nu(-r)}=\beta$ has a unique root $r_3\in(0,\alpha^2)$. These results imply that r_3 is the biggest value for which

$$\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)>\beta,\quad z\in\mathbb{U}(r_3).$$

A short calculation shows that $-r\frac{h_\nu'(-r)}{h_\nu(-r)}=\beta$ is equivalent to

$$(2-\nu)+\frac{ar\sqrt{r}I_\nu'''(\sqrt{r})+(2a+b)\,rI_\nu''(\sqrt{r})+(b+c)\sqrt{r}I_\nu'(\sqrt{r})}{arI_\nu''(\sqrt{r})+b\sqrt{r}I_\nu'(\sqrt{r})+cI_\nu(\sqrt{r})}=2\beta,$$

and the proof is done. □

When $\nu = 1.5$, considering the special values of $a, b, c \in \mathbb{R}$, radii of starlikeness of the functions f_ν, g_ν and h_ν are seen from the Table 1. If the values of b and c are fixed constant and the value of a is increased, radii of starlikeness of the functions f_ν, g_ν and h_ν are monotone decreasing. If the values of a and c are fixed constant and the value of b is increased or the values of a and b are fixed constant, and the value of c is increased, radii of starlikeness of the functions f_ν, g_ν and h_ν are monotone increasing. In addition, according to the increasing values of β, it is clear that radii of starlikeness of the functions f_ν, g_ν and h_ν is decreasing.

Table 1. Radii of starlikeness for f_ν, g_ν and h_ν when $\nu = 1.5$.

		$r^*(f_\nu)$		$r^*(g_\nu)$		$r^*(h_\nu)$	
		$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$
$b = -5$	$a = 3$	1.2886	1.0771	1.1666	0.9505	1.8772	1.3609
and	$a = 4$	0.9512	0.7782	0.8510	0.6766	1.0368	0.7242
$c = 0$	$a = 5$	0.7406	0.5998	0.6588	0.5181	0.6329	0.5485
$a = -2$	$b = 2$	0.9630	0.7878	0.8615	0.6848	1.0629	0.7422
and	$b = 3$	1.3384	1.1212	1.2131	0.9910	2.0225	1.4717
$c = 1$	$b = 4$	1.7306	1.4940	1.5939	1.3521	3.3226	2.5404
$a = -2$	$c = 4$	0.4920	0.3948	0.4354	0.3392	0.2812	0.1896
and	$c = 5$	0.7015	0.5663	0.6230	0.4883	0.5692	0.3881
$b = -1$	$c = 6$	0.8660	0.7034	0.7717	0.6088	0.8637	0.5955

In second principal result, we obtained the radii of convexity of the functions f_ν, g_ν and h_ν.

Theorem 2.2 *Let* $\beta \in [0, 1)$, $Q(\nu) / (b - a) > 0$, $a / (b - a) < 0$ *and* $\frac{Q(\nu+2)}{Q(\nu)} < 0$. *The following statements hold:*

a) *If* $0 < \nu \leq 1$ *then, the radius* $r^c_\beta(f_\nu)$ *is the smallest positive root of the equation*

$$1 + \frac{rM''_\nu(r)}{M'_\nu(r)} + \left(\frac{1}{\nu} - 1\right) \frac{rM'_\nu(r)}{M_\nu(r)} = \beta.$$

b) *If* $\nu > 0$ *then, the radius* $r^c_\beta(g_\nu)$ *is the smallest positive root of the equation*

$$1 + \frac{r^2 M''_\nu(r) + (2 - 2\nu) rM'_\nu(r) + \left(\nu^2 - \nu\right) M_\nu(r)}{rM'_\nu(r) + (1 - \nu) M_\nu(r)} = \beta.$$

c) *If* $\nu > 0$, *then the radius* $r^c_\beta(h_\nu)$ *is the smallest positive root of the equation*

$$1 + \frac{rM''_\nu(\sqrt{r}) + (3 - 2\nu)\sqrt{r}M'_\nu(\sqrt{r}) + \left(\nu^2 - 2\nu\right) M_\nu(\sqrt{r})}{2\sqrt{r}M'_\nu(\sqrt{r}) + 2(2 - \nu) M_\nu(\sqrt{r})} = \beta.$$

Proof **a)** Observe that

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 + \frac{zN''_\nu(z)}{N'_\nu(z)} + \left(\frac{1}{\nu} - 1\right) \frac{zN'_\nu(z)}{N_\nu(z)}.$$

Now, we consider the following infinite product representations

$$N_\nu(z) = \frac{Q(\nu)\, z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1}\left(1 - \frac{z^2}{\lambda_{\nu,n}^2}\right) \quad \text{and} \quad N'_\nu(z) = \frac{Q(\nu)\, \nu z^{\nu-1}}{2^\nu \Gamma(\nu+1)} \prod_{n\geq 1}\left(1 - \frac{z^2}{\lambda_{\nu,n}'^2}\right), \tag{2.5}$$

where $\lambda_{\nu,n}$ and $\lambda'_{\nu,n}$ are the n^{th} positive zeros of N_ν and N'_ν, respectively. The logarithmic differentiation on both sides of the above relations yields

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 - \left(\frac{1}{\nu} - 1\right) \sum_{n\geq 1} \frac{2z^2}{\lambda_{\nu,n}^2 - z^2} - \sum_{n\geq 1} \frac{2z^2}{\lambda_{\nu,n}'^2 - z^2}. \tag{2.6}$$

By using Lemma 1.1, the conditions $\nu > 0$, $Q(\nu) \big/ (b-a) > 0$ and $a \big/ (b-a) < 0$ imply $\lambda_{\nu,1} = i\alpha$, $\lambda'_{\nu,1} = i\alpha'$ and $\alpha, \alpha' > 0$, $\lambda_{\nu,n}, \lambda'_{\nu,n} > 0$ for $n \in \{2, 3, \ldots\}$. Thus, we obtain that

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 + \left(\frac{1}{\nu} - 1\right) \left[2\frac{z^2}{\alpha^2 + z^2} - \sum_{n\geq 2} \frac{2z^2}{\lambda_{\nu,n}^2 - z^2}\right] + 2\frac{z^2}{\alpha'^2 + z^2} - \sum_{n\geq 2} \frac{2z^2}{\lambda_{\nu,n}'^2 - z^2}. \tag{2.7}$$

On the other hand, the convergence of the function series in (2.6) is uniform on every compact subset of $\mathbb{C} \backslash \left(\{\lambda_{\nu,n} \mid n \in \mathbb{N}\} \cup \{\lambda'_{\nu,n} \mid n \in \mathbb{N}\}\right)$.
Additionally, from Lemma 1.2 and (2.5) we get the following equalities

$$\sum_{n\geq 1} \frac{1}{\lambda_{\nu,n}^2} = \frac{Q(\nu+2)}{4(\nu+1)Q(\nu)} \quad \text{and} \quad \sum_{n\geq 1} \frac{1}{\lambda_{\nu,n}'^2} = \frac{(\nu+2)\, Q(\nu+2)}{4\nu(\nu+1)Q(\nu)}. \tag{2.8}$$

Using above equalities, we obtain that

$$\begin{aligned} 1 + \frac{zf''_\nu(z)}{f'_\nu(z)} &= 1 + \left(\frac{1}{\nu} - 1\right) \left[2\frac{\alpha^2 z^2}{\alpha^2 + z^2} \left(\sum_{n\geq 2} \frac{1}{\lambda_{\nu,n}^2} - \frac{Q(\nu+2)}{4(\nu+1)Q(\nu)}\right) - 2\sum_{n\geq 2} \frac{z^2}{\lambda_{\nu,n}^2 - z^2}\right] \\ &+ 2\frac{\alpha'^2 z^2}{\alpha'^2 + z^2} \left(\sum_{n\geq 2} \frac{1}{\lambda_{\nu,n}'^2} - \frac{(\nu+2)\, Q(\nu+2)}{4\nu(\nu+1)Q(\nu)}\right) - 2\sum_{n\geq 2} \frac{z^2}{\lambda_{\nu,n}'^2 - z^2}. \end{aligned} \tag{2.9}$$

The equality (2.9) implies

$$\begin{aligned} 1 + \frac{zf''_\nu(z)}{f'_\nu(z)} &= 1 - \left(\frac{1}{\nu} - 1\right) \left[\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)} \frac{z^2}{\alpha^2 + z^2} + 2\sum_{n\geq 2} \frac{\alpha^2 + \lambda_{\nu,n}^2}{\lambda_{\nu,n}^2} \frac{z^4}{(\alpha^2 + z^2)\left(\lambda_{\nu,n}^2 - z^2\right)}\right] \\ &- \frac{\alpha'^2 (\nu+2)\, Q(\nu+2)}{2\nu(\nu+1)Q(\nu)} \frac{z^2}{\alpha'^2 + z^2} - 2\sum_{n\geq 2} \frac{\alpha'^2 + \lambda_{\nu,n}'^2}{\lambda_{\nu,n}'^2} \frac{z^4}{\left(\alpha'^2 + z^2\right)\left(\lambda_{\nu,n}'^2 - z^2\right)}. \end{aligned}$$

For $0<\nu\leq 1,$ by using (1.7) and (1.8), for all $z\in\mathbb{D}_{\alpha'}$ and $\frac{Q(\nu+2)}{Q(\nu)}<0$ we obtain the inequality

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right) &\geq 1+\left(\frac{1}{\nu}-1\right)\left[\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geq 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right]\\ &+ \frac{\alpha'^2\,(\nu+2)\,Q(\nu+2)}{2\nu(\nu+1)Q(\nu)}\frac{r^2}{\alpha'^2-r^2}-2\sum_{n\geq 2}\frac{\alpha'^2+\lambda_{\nu,n}'^2}{\lambda_{\nu,n}'^2}\frac{r^4}{(\alpha'^2-r^2)\left(\lambda_{\nu,n}'^2+r^2\right)}\\ &= \left.\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)\right|_{z=ir}.\end{aligned}$$

This means that

$$\begin{aligned}\inf_{z\in\mathbb{U}(r)}\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right) &= 1+ir\frac{f_\nu''(ir)}{f_\nu'(ir)}=1-\left(\frac{1}{\nu}-1\right)\left[2\frac{r^2}{\alpha^2-r^2}-\sum_{n\geq 2}\frac{2r^2}{\lambda_{\nu,n}^2+r^2}\right]\\ &- 2\frac{r^2}{\alpha'^2-r^2}+\sum_{n\geq 2}\frac{2r^2}{\lambda_{\nu,n}'^2+r^2},\end{aligned}$$

for all $r\in(0,\alpha')$. Let us consider the function $\Phi:(0,\alpha')\longrightarrow\mathbb{R},$ defined by

$$\Phi(r)=1+ir\frac{f_\nu''(ir)}{f_\nu'(ir)}=1-\left(\frac{1}{\nu}-1\right)\left[2\frac{r^2}{\alpha^2-r^2}-\sum_{n\geq 2}\frac{2r^2}{\lambda_{\nu,n}^2+r^2}\right]-2\frac{r^2}{\alpha'^2-r^2}+\sum_{n\geq 2}\frac{2r^2}{\lambda_{\nu,n}'^2+r^2}.$$

This function satisfy $\lim_{r\searrow 0}\Phi(r)=1>\beta,$ $\lim_{r\nearrow\alpha'}\Phi(r)=-\infty,$ and

$$\begin{aligned}\Phi'(r) &= -\left(\frac{1}{\nu}-1\right)\left[\frac{4r\alpha^2}{(\alpha^2-r^2)^2}-\sum_{n\geq 2}\frac{4r\lambda_{\nu,n}^2}{(\lambda_{\nu,n}^2+r^2)^2}\right]-\frac{4r\alpha'^2}{(\alpha'^2-r^2)^2}+\sum_{n\geq 2}\frac{4r\lambda_{\nu,n}'^2}{(\lambda_{\nu,n}'^2+r^2)^2}\\ &< -\left(\frac{1}{\nu}-1\right)\left[\frac{4r\alpha^2}{(\alpha^2-r^2)^2}-\sum_{n\geq 2}\frac{4r}{\lambda_{\nu,n}^2}\right]-\frac{4r\alpha'^2}{(\alpha'^2-r^2)^2}+\sum_{n\geq 2}\frac{4r}{\lambda_{\nu,n}'^2}\\ &< \left(\frac{1}{\nu}-1\right)\left[\frac{-4r\alpha^2}{(\alpha^2-r^2)^2}+\frac{4r}{\alpha^2}+\frac{Q(\nu+2)r}{(\nu+1)Q(\nu)}\right]-\frac{4r\alpha'^2}{(\alpha'^2-r^2)^2}+\frac{4r}{\alpha'^2}+\frac{(\nu+2)Q(\nu+2)r}{\nu(\nu+1)Q(\nu)}\\ &< \left(\frac{1}{\nu}-1\right)\frac{Q(\nu+2)r}{(\nu+1)Q(\nu)}+\frac{(\nu+2)Q(\nu+2)r}{\nu(\nu+1)Q(\nu)}<0.\end{aligned}$$

In other words, the function Φ maps $(0,\alpha')$ into $(-\infty,1)$ and is strictly decreasing. Thus the equation $1+ir\frac{f_\nu''(ir)}{f_\nu'(ir)}=\beta$ has exactly one root in the interval $(0,\alpha')$, and this equation is equivalent to Theorem 2.2-a. If we denote the unique root of the equation $1+ir\frac{f_\nu''(ir)}{f_\nu'(ir)}=\beta,$ by $r_4\in(0,\alpha')$. then by using the minimum principle of harmonic functions, we have that

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)>\beta\ \text{ for all }\ z\in\mathbb{U}(0,r_4),$$

$$\inf_{z\in\mathbb{U}_{r_4}} \operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)=\beta, \quad \inf_{z\in\mathbb{U}_{r}} \operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)<\beta, \qquad \text{for} \quad r>r_4.$$

b) In view of (1.4), we have

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)}=1-\sum_{n\geq 1}\frac{2z^2}{\delta_{\nu,n}^2-z^2}.$$

Let $\delta_{\nu,1}=i\theta,\ \theta>0$. Thus, we obtain

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)}=1+2\frac{z^2}{\theta^2+z^2}-2\sum_{n\geq 2}\frac{z^2}{\delta_{\nu,n}^2-z^2}=1+2\frac{\theta^2 z^2}{\theta^2+z^2}\frac{1}{\theta^2}-2\sum_{n\geq 2}\frac{z^2}{\delta_{\nu,n}^2-z^2}.$$

On the other hand, the convergence of the function series in (1.4) is uniform on every compact subset of $\mathbb{C}\ \{\delta_{\nu,n}\mid n\in\mathbb{N}\}$. Integrating both sides of the equality (1.4), it follows that

$$g_\nu'(z)=\prod_{n\geq 1}\left(1-\frac{z^2}{\delta_{\nu,n}^2}\right). \tag{2.10}$$

The convergence of this infinite product is uniform on every compact subset of $\mathbb{C}$. Comparing the coefficients of z^2 on both sides of (2.10), we get the following equality

$$\sum_{n\geq 1}\frac{1}{\delta_{\nu,n}^2}=\frac{3Q(\nu+2)}{4(\nu+1)Q(\nu)}. \tag{2.11}$$

The equality (2.11) implies

$$\frac{1}{\theta^2}=-\frac{3Q(\nu+2)}{4(\nu+1)Q(\nu)}+\sum_{n\geq 2}\frac{1}{\delta_{\nu,n}^2}$$

and using this, we obtain that

$$\begin{aligned}1+\frac{zg_\nu''(z)}{g_\nu'(z)} &= 1+2\frac{\theta^2 z^2}{\theta^2+z^2}\left(\sum_{n\geq 2}\frac{1}{\lambda_{\nu,n}^2}-\frac{3Q(\nu+2)}{4(\nu+1)Q(\nu)}\right)-2\sum_{n\geq 2}\frac{z^2}{\delta_{\nu,n}^2-z^2}\\ &= 1-\frac{3\theta^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\theta^2+z^2}-2\sum_{n\geq 2}\frac{\theta^2+\delta_{\nu,n}^2}{\delta_{\nu,n}^2}\frac{z^4}{(\theta^2+z^2)\left(\delta_{\nu,n}^2-z^2\right)}.\end{aligned}$$

On the other hand, we have $\frac{3\theta^2 Q(\nu+2)}{4(\nu+1)Q(\nu)}<0$, and taking $v=z^2$ in the inequality (1.7), we get

$$\operatorname{Re}\frac{z^2}{\theta^2+z^2}\geq\frac{-r^2}{\theta^2-r^2}$$

for all $|z|\leq r<\theta$. Moreover, taking $r=z^2$ in the inequality (1.8) it follows that

$$\frac{r^4}{(\theta^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\geq\operatorname{Re}\frac{z^4}{(\theta^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}$$

for all $|z| \leq r < \theta$ and $n \in \{2,3,\ldots\}$. Summarizing, provided that $|z| \leq r < \theta$, the following inequality holds

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right) \geq 1+\frac{3\theta^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\theta^2-r^2}-2\sum_{n\geq 2}\frac{\theta^2+\gamma_{\nu,n}^2}{\gamma_{\nu,n}^2}\frac{r^4}{(\theta^2-r^2)\left(\gamma_{\nu,n}^2+r^2\right)}.$$

This means that

$$\inf_{z\in\mathbb{U}_r}\left\{\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)\right\}=1+ir\frac{g_\nu''(ir)}{g_\nu'(ir)}=1-2\frac{r^2}{\theta^2-r^2}+2\sum_{n\geq 2}\frac{r^2}{\delta_{\nu,n}^2+r^2},$$

for all $r\in(0,\theta)$. Now, let us consider the function $\Theta:(0,\theta)\to\mathbb{R}$, defined by

$$\Theta(r)=1-2\frac{r^2}{\theta^2-r^2}+2\sum_{n\geq 2}\frac{r^2}{\delta_{\nu,n}^2+r^2}=1+ir\frac{g_\nu''(ir)}{g_\nu'(ir)}.$$

For $\nu>0$, $Q(\nu)\diagup(b-a)>0$ and $a\diagup(b-a)<0$, this function satisfy $\lim_{r\searrow 0}\Theta(r)=1>\beta$, $\lim_{r\nearrow\theta}\Theta(r)=-\infty$ and

$$\begin{aligned}\Theta'(r) &= -\frac{4r\theta^2}{(\theta^2-r^2)^2}+\sum_{n\geq 2}\frac{4r\delta_{\nu,n}^2}{(\delta_{\nu,n}^2+r^2)^2}<-\frac{4r\theta^2}{(\theta^2-r^2)^2}+\sum_{n\geq 2}\frac{4r}{\delta_{\nu,n}^2}\\ &= -\frac{4r\theta^2}{(\theta^2-r^2)^2}+\frac{4r}{\theta^2}+\frac{3rQ(\nu+2)}{(\nu+1)Q(\nu)}<\frac{3rQ(\nu+2)}{(\nu+1)Q(\nu)}.\end{aligned}$$

In other words, the function Θ maps $(0,\theta)$ into $(-\infty,1)$ and is strictly decreasing. Thus, the equation $1+ir\frac{g_\nu''(ir)}{g_\nu'(ir)}=\beta$ has exactly one root in the interval $(0,\theta)$, and this equation is equivalent to Theorem 2.2-b. If we denote the unique root of the equation $1+ir\frac{g_\nu''(ir)}{g_\nu'(ir)}=\beta$ by $r_5\in(0,\theta)$, then by using the minimum principle of harmonic functions we have that

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)>\beta \quad \text{for all} \quad z\in\mathbb{U}(0,r_5),$$

$$\inf_{z\in\mathbb{U}_{r_5}}\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)=\beta,\qquad \inf_{z\in\mathbb{U}_r}\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)<\beta,\qquad \text{for}\quad r>r_5,$$

and the proof is done.

c) Lemma 1.5 implies that

$$1+\frac{zh_\nu''(z)}{h_\nu'(z)}=1-\sum_{n\geq 1}\frac{z}{\gamma_{\nu,n}^2-z}.$$

Let $\gamma_{\nu,1}=i\kappa,\ \kappa>0$. Then, we have

$$1+\frac{zh_\nu''(z)}{h_\nu'(z)}=1+\frac{z}{\kappa^2+z}-\sum_{n\geq 2}\frac{z}{\gamma_{\nu,n}^2-z}=1+\frac{\kappa^2 z}{\kappa^2+z}\frac{1}{\kappa^2}-\sum_{n\geq 2}\frac{z}{\gamma_{\nu,n}^2-z}.$$

On the other hand, similarly to Theorem **a)**, we obtain the following equality

$$\sum_{n\geq 1}\frac{1}{\gamma_{\nu,n}^2}=\frac{Q(\nu+2)}{2(\nu+1)Q(\nu)},$$

which in turn implies that

$$-\frac{1}{\kappa^2}+\sum_{n\geq 2}\frac{1}{\gamma_{\nu,n}^2}=\frac{Q(\nu+2)}{2(\nu+1)Q(\nu)}.$$

Thus, we get

$$\begin{aligned} 1+\frac{zh_\nu''(z)}{h_\nu'(z)} &= 1+\frac{\kappa^2 z}{\kappa^2+z}\left(\sum_{n\geq 2}\frac{1}{\gamma_{\nu,n}^2}-\frac{Q(\nu+2)}{2(\nu+1)Q(\nu)}\right)-\sum_{n\geq 2}\frac{z}{\gamma_{\nu,n}^2-z} \\ &= 1-\frac{\kappa^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z}{\kappa^2+z}-\sum_{n\geq 2}\frac{\kappa^2+\gamma_{\nu,n}^2}{\gamma_{\nu,n}^2}\frac{z^2}{(\kappa^2+z)\left(\gamma_{\nu,n}^2-z\right)}. \end{aligned} \tag{2.12}$$

Since $|z|\leq r<\kappa^2<\gamma_{\nu,n}^2$, then the inequalities (1.7) and (1.8),implies that

$$\operatorname{Re}\left[-\frac{\kappa^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z}{\kappa^2+z}\right]\geq\frac{\kappa^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r}{\kappa^2-r}.$$

and

$$\operatorname{Re}\left[-\frac{\kappa^2+\gamma_{\nu,n}^2}{\gamma_{\nu,n}^2}\frac{z^2}{(\kappa^2+z)\left(\gamma_{\nu,n}^2-z\right)}\right]\geq-\frac{\kappa^2+\gamma_{\nu,n}^2}{\gamma_{\nu,n}^2}\frac{r^2}{(\kappa^2-r)\left(\gamma_{\nu,n}^2+r\right)}.$$

Then, from above last two inequalities and (2.12) we have

$$\operatorname{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)\geq 1+\frac{\kappa^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r}{\kappa^2-r}-\sum_{n\geq 2}\frac{\kappa^2+\gamma_{\nu,n}^2}{\gamma_{\nu,n}^2}\frac{r^2}{(\kappa^2-r)\left(\gamma_{\nu,n}^2+r\right)}=1-r\frac{h_\nu''(-r)}{h_\nu'(-r)}.$$

Thus, we get

$$\inf_{z\in\mathbb{U}_r}\left\{\operatorname{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)\right\}=1-r\frac{h_\nu''(-r)}{h_\nu'(-r)}\quad\text{for all}\quad r\in(0,\kappa^2).$$

Now, consider the function $\psi:(0,\kappa^2)\to\mathbb{R}$, defined by

$$\psi(r)=1-r\frac{h_\nu''(-r)}{h_\nu'(-r)}=1+\frac{\kappa^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r}{\kappa^2-r}-\sum_{n\geq 2}\frac{\kappa^2+\gamma_{\nu,n}^2}{\gamma_{\nu,n}^2}\frac{r^2}{(\kappa^2-r)\left(\gamma_{\nu,n}^2+r\right)}.$$

Since for $\nu>0,\ Q(\nu)/(b-a)>0$ and $a/(b-a)<0$,

$$\lim_{r\searrow 0}\psi(r)=1>\beta,\quad \lim_{r\nearrow\kappa^2}\psi(r)=-\infty,$$

it follows that equation $1-r\frac{h_\nu''(-r)}{h_\nu'(-r)}=\beta$ has at last one real root in the interval $(0,\kappa^2)$. Let r_6 denote the smallest positive real root of the equation $1-r\frac{h_\nu''(-r)}{h_\nu'(-r)}=\beta$. we have

$$\operatorname{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)>\beta,\quad z\in\mathbb{U}(0,r_6),$$

and if $r > r_6$ then $\inf_{z\in\mathbb{U}_r} \operatorname{Re}\left(1+\frac{zh''_\nu(z)}{h'_\nu(z)}\right) < \beta$.

In order to finish the proving, we remark that $1 - r\frac{h''_\nu(-r)}{h'_\nu(-r)} = \beta$ is equivalent to

$$1+\frac{rM''_\nu(\sqrt{r})+(3-2\nu)\sqrt{r}M'_\nu(\sqrt{r})+\left(\nu^2-2\nu\right)M_\nu(\sqrt{r})}{2\sqrt{r}M'_\nu(\sqrt{r})+2\,(2-\nu)\,M_\nu(\sqrt{r})}=\beta.$$

□

In the Table 2, the monotonicity properties of convexity radii according to special values of $a, b, c \in \mathbb{R}$ are similar to radii of starlikeness in the Table 1.

Moreover, looking at the values $\nu = 0.5$, $a = -2$, $b = 2$, $c = 1$ and $\beta = 0$ in Table 2, it is seen that $r_c(h_{1/2}) = 0.9350$. In the Figure, the images of function $h_{1/2}(z)$ for $r_0^c(h_{1/2}) = 0.9350$ and $r_0^c(h_{1/2}) = 1.1$ are convex and not convex, respectively

Table 2. Radii of convexity for f_ν, g_ν and h_ν when $\nu = 0.5$.

		$r^c(f_\nu)$		$r^c(g_\nu)$		$r^c(h_\nu)$	
		$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$	$\beta = 0$	$\beta = 0.5$
$b = -5$	$a = 4$	0.5429	0.4428	0.6711	0.5591	0.8051	0.6045
and	$a = 5$	0.4199	0.3311	0.5391	0.4360	0.5611	0.4059
$c = 0$	$a = 6$	0.3577	0.2786	0.4668	0.3726	0.4375	0.3103
$a = -2$	$b = 0$	0.2821	0.2170	0.3752	0.2949	0.2979	0.2055
and	$b = 1$	0.3972	0.3100	0.5167	0.4134	0.5316	0.3786
$c = 1$	$b = 2$	0.5873	0.4798	0.7247	0.6046	0.9350	0.7035
$a = -2$	$c = 4$	0.4560	0.3566	0.5917	0.4745	0.6924	0.4949
and	$c = 5$	0.5354	0.4236	0.6842	0.5553	0.8928	0.6499
$b = -1$	$c = 6$	0.6164	0.4947	0.7744	0.6371	1.1031	0.8173

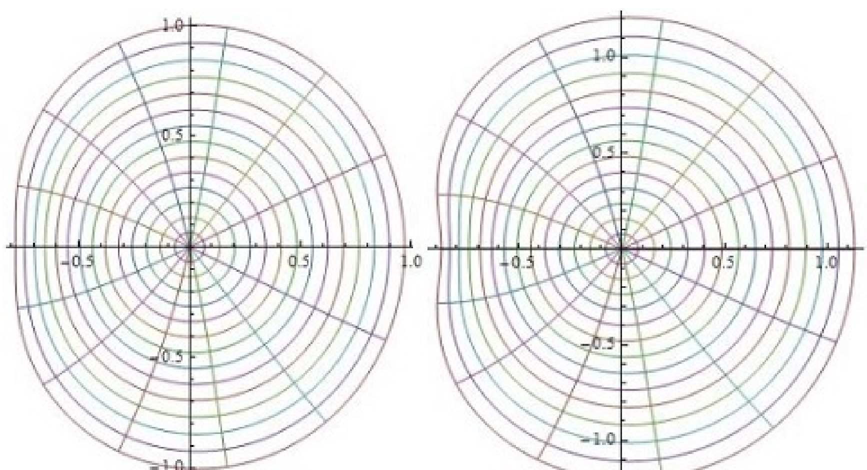

Figure. Images of function $h_{1/2}(z)$ for $r_0^c(h_{1/2}) = 0.9350$ and $r_0^c(h_{1/2}) = 1.1$, respectively.

3. Conclusion

In this section, some results and examples for special values of ν in main theorems will be given.
Indeed, we can write the function $N_\nu(z)$ in terms of elementary trigonometric functions for the value $\nu = 3/2$ as follows:

$$N_{3/2}(z) = \frac{\left[a\left(4z^2-15\right)+6b-4c\right]z\cos z+\left[4\left(b-2a\right)z^2+15a-6b+4c\right]\sin z}{2\sqrt{2\pi}z\sqrt{z}}.$$

Thus, we have

$$f_{3/2}(z) = 3^{2/3}\left(\frac{\left[a\left(4z^2-15\right)+6b-4c\right]z\cos z+\left[4\left(b-2a\right)z^2+15a-6b+4c\right]\sin z}{\left(3a+6b+4c\right)z\sqrt{z}}\right)^{2/3},$$

$$g_{3/2}(z) = \frac{3\left[a\left(4z^2-15\right)+6b-4c\right]z\cos z+3\left[4\left(b-2a\right)z^2+15a-6b+4c\right]\sin z}{\left(3a+6b+4c\right)z^2}$$

and

$$h_{3/2}(z) = \frac{3\left[a\left(4z-15\right)+6b-4c\right]\sqrt{z}\cos\sqrt{z}+3\left[4\left(b-2a\right)z+15a-6b+4c\right]\sin\sqrt{z}}{\left(3a+6b+4c\right)\sqrt{z}}.$$

Now, we state the following results for the functions $f_{3/2}$, $g_{3/2}$ and $h_{3/2}$ in Theorem 2.1.

Corollary 3.1 *Let* $\beta \in [0,1)$, $(3a+6b+4c)\diagup 4(b-a) > 0$, $a\diagup(b-a) < 0$ *and* $\frac{35a+14b+4c}{3a+6b+4c} < 0$. *The following statements are true.*

- ***a)*** *The radius* $r_\beta^*\left(f_{3/2}\right)$ *is the smallest positive root of the equation*

$$\frac{\left[4\left(2b-a\right)r^2-45a+18b-12c\right]r\cosh r+\left[a\left(8r^4+22r^2+45\right)+8\left(c-b\right)r^2-18b+12c\right]\sinh r}{3\left[a\left(4r^2+15\right)-6b+4c\right]r\cosh r+3\left[4\left(b-2a\right)r^2-15a+6b-4c\right]\sinh r} = \beta.$$

- ***b)*** *The radius* $r_\beta^*\left(g_{3/2}\right)$ *is the smallest positive root of the equation*

$$\frac{2-\left[2\left(a-b\right)r^2+15a-6b+4c\right]r\cosh r+\left[a\left(4r^4+15r^2+30\right)-2\left(3b-2c\right)\left(r^2+2\right)\right]\sinh r}{\left[a\left(4r^2+15\right)-6b+4c\right]r\cosh r+\left[4\left(b-2a\right)r^2-15a+6b-4c\right]\sinh r} = \beta.$$

- ***c)*** *The radius* $r_\beta^*\left(h_{3/2}\right)$ *is the smallest positive root of the equation*

$$\frac{\left[b\left(4r+6\right)-15a-4c\right]\sqrt{r}\cosh\sqrt{r}+\left[a\left(4r^2+7r+15\right)-2b\left(r+3\right)+4c\left(r+1\right)\right]\sinh\sqrt{r}}{2\left[a\left(4r+15\right)-6b+4c\right]\sqrt{r}\cosh\sqrt{r}-2\left[a\left(8r+15\right)-4br-6b+4c\right]\sinh\sqrt{r}} = \beta.$$

Example 3.2 *Taking* $a = 3$, $b = -5$, $c = 0$ *and* $\beta = 0$ *in Corollary 3.1, we have the radii of starlikeness of the functions:*

- $r_0^*\left(f(z)\right) = 1.2886$, *where* $f(z) = \left(\frac{3\left(-4z^2+25\right)z\cos z+\left(44z^2-75\right)\sin z}{7z^{3/2}}\right)^{2/3}$
- $r_0^*\left(g(z)\right) = 1.1666$, *where* $g(z) = \frac{3\left(-4z^2+25\right)z\cos z+\left(44z^2-75\right)\sin z}{7z^2}$

- $r_0^*\left(h(z)\right) = 1.8772,$ *where* $h(z) = \frac{(-12z+75)\sqrt{z}\cos\sqrt{z}+(44z-75)\sin\sqrt{z}}{7\sqrt{z}}.$

Similarly, for $\nu = 1/2$ we get

$$N_{1/2}(z) = \frac{4\,(b-a)\,z\cos z + \left[a\left(3-4z^2\right) - 2b + 4c\right]\sin z}{2\sqrt{2\pi}\sqrt{z}}.$$

Thus, we have

$$f_{1/2}(z) = \frac{\left[4\,(a-b)\,z\cos z + \left(4az^2 - 3a + 2b - 4c\right)\sin z\right]^2}{\left(a-2b-4c\right)^2 z},$$

$$g_{1/2}(z) = \frac{4\,(a-b)\,z\cos z + \left(4az^2 - 3a + 2b - 4c\right)\sin z}{a-2b-4c}$$

and

$$h_{1/2}(z) = \frac{4\,(a-b)\,z\cos\sqrt{z} + (4az - 3a + 2b - 4c)\sqrt{z}\sin\sqrt{z}}{a-2b-4c}.$$

Taking $\nu = 1/2$ in Theorem 2.2, we have following results.

Corollary 3.3 *Let* $\beta \in [0,1),$ $(-a+2b+4c)\big/4(b-a) > 0,$ $a\big/(b-a) < 0$ *and* $\frac{15a+10b+4c}{-a+2b+4c} < 0.$ *The following statements are true.*

a) *The radius of convexity of order* β *of the function* $f_{1/2}$ *is the smallest positive root of the equation*

$$\begin{aligned} 1 \quad &+ \quad \frac{2\left(4ar^2 + a + 4c\right) r\cosh r + \left(4ar^2 + 8br^2 - A\right)\sinh r}{8\,(b-a)\,r\cosh r + 2\left(4ar^2 + A\right)\sinh r} \\ &+ \quad \frac{4\left(4\,(2a+b)\,r^2 - 2a + b - 4c\right) r\cosh r + \left(16ar^4 + 8\,(a+b+2c)\,r^2 + 3A\right)\sinh r}{4\left(4ar^2 + a + 4c\right) r\cosh r + 2\left(4ar^2 + 8br^2 - A\right)\sinh r} = \beta \end{aligned}$$

where $A = 3a - 2b + 4c$.

b) *The radius of convexity of order* β *of the function* $g_{1/2}$ *is the smallest positive root of the equation*

$$\frac{\left[4\,(4a+b)\,r^2 - a + 2b + 4c\right]\cosh r + \left[4ar^2 + 7a + 10b + 4c\right] r\sinh r}{\left[\left(4ar^2 - a + 2b + 4c\right)\cosh r + 4\,(a+b)\,r\sinh r\right] r} = \beta.$$

c) *The radius of convexity of order* β *of the function* $h_{1/2}$ *is the smallest positive root of the equation*

$$\frac{\left[4\,(6a+b) - 7a + 10b + 12c\right]\sqrt{r}\cosh\sqrt{r} + \left[a\left(4r^2 + 19r + 3\right) + 2b\,(9x-1) + 4c\,(x+1)\right]\sinh\sqrt{r}}{2\,(4ar - 5a + 6b + 4c)\sqrt{r}\cosh\sqrt{r} + 2\,(8ar + 4br + 3a - 2b + 4c)\sinh\sqrt{r}} = \beta.$$

Example 3.4 *Taking* $a = -2,$ $b = 2,$ $c = 1$ *and* $\beta = 0$ *in Corollary 3.3, we have the radii of convexity of the functions:*

- $r_0^c\left(f(z)\right) = 0.5873,$ *where* $f(z) = \frac{\left(8z\cos z + 2\left(4z^2-3\right)\sin z\right)^2}{100z}$
- $r_0^c\left(g(z)\right) = 0.7247,$ *where* $g(z) = \frac{8z\cos z + \left(4z^2-3\right)\sin z}{5}$
- $r_0^c\left(h(z)\right) = 0.9350,$ *where* $h(z) = \frac{8z\cos\sqrt{z} + (4z-3)\sqrt{z}\sin\sqrt{z}}{5}.$

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