

Properties of Abelian-by-cyclic shared by soluble finitely generated groups

Fares GHERBI* , Nadir TRABELSI 

Department of Mathematics, Faculty of Sciences, University Ferhat Abbas Setif 1, Setif, Algeria

Received: 08.12.2021

Accepted/Published Online: 31.01.2022

Final Version: 11.03.2022

Abstract: Our main result states that if G is a finitely generated soluble group having a normal Abelian subgroup A , such that G/A and $\langle x, a \rangle$ are nilpotent (respectively, finite-by-nilpotent, periodic-by-nilpotent, nilpotent-by-finite, finite-by-supersoluble, supersoluble-by-finite) for all $(x, a) \in G \times A$, then so is G . We deduce that if \mathfrak{X} is a subgroup and quotient closed class of groups and if all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are nilpotent (respectively, finite-by-nilpotent, periodic-by-nilpotent, nilpotent-by-finite, finite-by-supersoluble, supersoluble-by-finite), then so are all finitely generated soluble groups of \mathfrak{X} . We give examples that show that our main result is not true for other classes of groups, like the classes of Abelian, supersoluble, and FC -groups.

Key words: Coherent, polycyclic, nilpotent, supersoluble, soluble, Abelian-by-cyclic

1. Introduction and results

In [6, Lemma] (respectively, [4, Proposition 8 and Theorem]), it is proved that a finitely generated soluble group G having a normal Abelian subgroup A such that G/A and $\langle x, a \rangle$ are polycyclic (respectively, coherent) for all $(x, a) \in G \times A$, is itself polycyclic (respectively, coherent). Recall that a group G is said to be coherent if every finitely generated subgroup is finitely presented. Here we will generalize this result to other classes, more precisely we will prove the following result.

Theorem 1.1 *If G is a finitely generated soluble group having a normal Abelian subgroup A , such that G/A and $\langle x, a \rangle$ are nilpotent (respectively, finite-by-nilpotent, periodic-by-nilpotent, nilpotent-by-finite, finite-by-supersoluble, supersoluble-by-finite) for all $(x, a) \in G \times A$, then so is G .*

Recall that G is said to be supersoluble if it has a finite normal series with cyclic factors.

In [3, Theorem 1.1], it is proved that if all metabelian groups of a subgroup and quotient closed class \mathfrak{X} of groups are periodic-by-nilpotent, then so are all soluble groups of \mathfrak{X} . As consequences of Theorem 1.1, we have results in the same spirit.

Corollary 1.2 *Let \mathfrak{X} be a subgroup and quotient closed class of groups. If all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are nilpotent (respectively, finite-by-nilpotent, periodic-by-nilpotent, nilpotent-by-finite, finite-by-supersoluble, supersoluble-by-finite), then so are all finitely generated soluble groups of \mathfrak{X} .*

*Correspondence: fares.gherbi@univ-setif.dz

2010 AMS Mathematics Subject Classification: 20F16, 20F99.

We deduce, by Corollary 1.2, by [4, Proposition 8 and Theorem] and by the fact that coherent groups form a local class, the following immediate consequence.

Corollary 1.3 *Let \mathfrak{X} be a subgroup and quotient closed class of groups.*

1. *If all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are nilpotent, then all locally soluble groups of \mathfrak{X} are locally nilpotent.*
2. *If all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are periodic-by-nilpotent, then all locally soluble groups of \mathfrak{X} are periodic-by-(locally nilpotent).*
3. *If all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are coherent, then all locally soluble groups of \mathfrak{X} are coherent.*

The consideration of the group $U(3, \mathbb{Z})$ of all 3×3 unitriangular matrices over \mathbb{Z} , which is finitely generated, torsion-free and nilpotent of class 2 [8, p. 123-124], shows that Theorem 1.1 is not true for the classes of Abelian, finite-by-Abelian, Abelian-by-finite and FC -groups. We give in subsection 2.5 below, two examples which show that Theorem 1.1 is not true also for supersoluble groups.

2. Proof of the results

We prove first Theorem 1.1 starting with the finite-by-nilpotent case.

2.1. The finite-by-nilpotent case

Proposition 2.1 *Let G be a finitely generated soluble group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are finite-by-nilpotent for all $(x, a) \in G \times A$. Then G is finite-by-nilpotent.*

Proof Let G and A be as stated. Clearly G/A and $\langle x, a \rangle$ are polycyclic for all x in G and a in A , so by [6, Lemma], G is polycyclic and thus it satisfies the maximal condition on subgroups. Now, as A is Abelian and finitely generated, its torsion subgroup T is finite and normal in G , but the hypotheses on G are inherited by quotients, so we may therefore assume that A is torsion-free. Let $x \in G$ and $a \in A$. Since $\langle x, a \rangle$ is finite-by-nilpotent, there exists a positive integer $c = c(x, a)$ such that $\gamma_{c+1}(\langle x, a \rangle)$ is finite. The normality of A in G gives that $\gamma_{c+1}(\langle x, a \rangle) \leq \gamma_{c+1}(A \langle x \rangle) \leq A$, so that $\gamma_{c+1}(\langle x, a \rangle) = 1$ because A is torsion-free, and thus $[a, c x] = 1$. This means that $A \leq R(G)$ the set of right Engel elements of G . It follows, by [9, Theorem 7.21], that $A \leq Z_n(G)$ for some positive integer n . Finally, since $G/Z_n(G) \simeq (G/A)/(Z_n(G)/A)$ is finite-by-nilpotent, G is finite-by-nilpotent; as claimed. \square

2.2. The nilpotent case

Proposition 2.2 *Let G be a finitely generated soluble group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are nilpotent for all $(x, a) \in G \times A$. Then G is nilpotent.*

We omit the proof of Proposition 2.2 which can be extracted from that of Proposition 2.1.

2.3. The periodic-by-nilpotent case

In order to prove the main result of the periodic-by-nilpotent case, we need a preliminary result.

Lemma 2.3 *Let G be a finitely generated Abelian-by-nilpotent group having an Abelian normal subgroup A such that G/A is periodic-by-nilpotent and $\langle x, a \rangle$ is nilpotent for all $(x, a) \in G \times A$. Then G is periodic-by-nilpotent.*

Proof Let G and A be as stated. Clearly, one can assume that $A \neq 1$. Since G is Abelian-by-nilpotent and finitely generated, it satisfies the maximal condition on normal subgroups [10, Theorem 5.34]. As the hypotheses on G are inherited by quotients, we may therefore assume that G is not periodic-by-nilpotent but that every proper quotient of G is. Let $1 \neq a \in A$ and $x \in G$; since $\langle x, a \rangle$ is nilpotent, there exists a positive integer c such that $[a, c x] = 1$. We deduce that a is a right Engel element of G . By a result of Brookes [2], $a \in Z_n(G)$ for some positive integer n , because G satisfies the maximal condition on normal subgroups. Now, since $Z_n(G) \neq 1$, the center $Z(G)$ is nontrivial and therefore $G/Z(G)$ is periodic-by-nilpotent. We deduce by [3, Lemma 4.2] that G is periodic-by-nilpotent, which is a contradiction. This means that G is periodic-by-nilpotent. \square

Proposition 2.4 *Let G be a finitely generated soluble group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are periodic-by-nilpotent for all $(x, a) \in G \times A$. Then G is periodic-by-nilpotent.*

Proof Let G and A be as stated. One can assume that $A \neq 1$. As the hypotheses on G are inherited by quotients, we may therefore assume that A is torsion-free. Let x be in G and a in A ; since $\langle x, a \rangle$ is both periodic-by-nilpotent and (torsion-free)-by-cyclic, it is nilpotent. On the other hand, since G/A is periodic-by-nilpotent, there is a normal subgroup T of G containing A such that T/A is periodic and $G/T \simeq (G/A)/(T/A)$ is nilpotent. Let H be a finitely generated subgroup of T . We have $H/A \cap H \simeq AH/A \leq T/A$, so $H/A \cap H$ is finite and hence any infinite part of H contains two distinct elements x, y such that $x(A \cap H) = y(A \cap H)$, that is $y = xa$ for some element a of A , and thus $\langle x, y \rangle = \langle x, a \rangle$ is nilpotent as noted above. By [6, Theorem A], H is finite-by-nilpotent and so T is locally (finite-by-nilpotent). Let P be the torsion subgroup of T . Since T/P is torsion-free, locally nilpotent and Abelian-by-periodic, it is Abelian by [9, Lemma 6.33]. Hence G/P is Abelian-by-nilpotent. Moreover, as $AP/P \simeq A/(A \cap P) \simeq A$, the hypotheses on G are satisfied by the quotient G/P , it follows, by Lemma 2.3, that G/P , and therefore G , is periodic-by-nilpotent. \square

2.4. The nilpotent-by-finite case

We will consider in this subsection the nilpotent-by-finite case and as a consequence the supersoluble-by-finite case.

Proposition 2.5 *Let G be a finitely generated soluble group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are nilpotent-by-finite for all $(x, a) \in G \times A$. Then G is nilpotent-by-finite.*

Proof Let G and A be as stated. Clearly, G/A and $\langle x, a \rangle$ are polycyclic for all $(x, a) \in G \times A$, so by [6, Lemma], G is polycyclic and thus it satisfies the maximal condition on subgroups. As the hypotheses on G are inherited by quotients, we may therefore assume that A is torsion-free. Since G/A is nilpotent-by-finite, there is a normal subgroup N of G containing A such that N/A is nilpotent and G/N is finite. As the hypotheses on G are inherited by subgroups, there is no loss of generality if we assume that G/A is nilpotent. Let $(x, a) \in G \times A$,

so $\langle x, a \rangle$ is nilpotent-by-finite, and thus there are two positive integers $n = n(x, a)$ and $c = c(x, a)$ such that $[a^n, c x^n] = 1$. We have that $[a^n, c x^n] = [a, c x^n]^n$ and A is torsion-free, so $[a, c x^n] = 1$. Since G is a finitely generated (torsion-free Abelian)-by-nilpotent group, by [7, Theorem A] there exists a subgroup H of finite index in G such that for all $x \in H$ and for all positive integer k we have $C_H(x^k) \leq C_H(x)$. Clearly one can assume that $G = H$. We deduce that $[[a, c_{-1} x^n], x] = 1$, which implies that $[[a, x], c_{-1} x^n] = 1$ as $A \langle x \rangle$ is metabelian, hence $[[a, x], c_{-1} x] = 1$ by induction on c . Therefore $A \leq R(G)$ the set of right Engel elements of G . It follows, by [9, Theorem 7.21], that $A \leq Z_n(G)$ for some integer $n > 0$. Finally, since $G/Z_n(G) \simeq (G/A)/(Z_n(G)/A)$ is nilpotent, G is nilpotent; as required. \square

Since supersoluble-by-finite groups are nilpotent-by-finite, Proposition 2.5 has the following consequence.

Corollary 2.6 *Let G be a finitely generated soluble group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are supersoluble-by-finite for all $(x, a) \in G \times A$. Then G is nilpotent-by-finite and hence supersoluble-by-finite.*

2.5. The supersoluble case

In this subsection, we give two examples showing that Theorem 1.1 is not true for the class of supersoluble groups.

Example 2.7 *Let $P = \langle a_1 \rangle \times \langle a_2 \rangle$ be an elementary Abelian 7-group and let α and β be two automorphisms of P defined by $a_1^\beta = a_1^4$, $a_2^\beta = a_2^2$, $a_1^\alpha = a_2$, $a_2^\alpha = a_1$, and let $G = P \rtimes \langle \alpha, \beta \rangle$. We have that $o(\beta) = 3$, $o(\alpha) = 2$ and $\beta^\alpha = \beta^2$ so that $\langle \alpha, \beta \rangle = \langle \alpha \rangle \langle \beta \rangle \simeq S_3$. First, we prove that $\langle \alpha, \beta \rangle$ acts irreducibly on P , that is there is no subgroup of order 7 of P which is $\langle \alpha, \beta \rangle$ -invariant. Let $Q = \langle a_1^i a_2^j \rangle$, where $0 \leq i, j \leq 6$, be a subgroup of P of order 7, we show that either $\alpha \notin N_G(Q)$ or $\beta \notin N_G(Q)$. Assume that $\alpha \in N_G(Q)$, so $i, j \neq 0$ and $Q^\alpha = \langle a_1^i a_2^j \rangle = \langle a_1^i a_2^j \rangle$, which gives that $a_2^i a_1^j = (a_1^i a_2^j)^k$ for some $1 \leq k \leq 6$. It follows that $a_1^{j-kj} = a_2^{kj-i} = 1$, we deduce that $7|j - ki$ and $7|kj - i$, so $7|j^2 - i^2$, hence $7|j - i$ or $7|j + i$. We obtain that either $i = j$ or $7|j + i$ and the latter gives $(i, j) \in \{(6, 1); (5, 2); (4, 3); (3, 4); (2, 5); (1, 6)\}$. For such subgroups Q , we have to show that $\beta \notin N_G(Q)$. Assume, for a contradiction, that $\beta \in N_G(Q)$ and consider first the subgroup $Q = \langle a_1^i a_2^i \rangle$. So $Q^\beta = \langle a_1^{4i} a_2^{2i} \rangle = \langle a_1^i a_2^i \rangle$, hence $a_1^{4i} a_2^{2i} = a_1^{ki} a_2^{ki}$ for some $1 \leq k \leq 6$. We deduce that $a_1^{4i-ki} = a_2^{ki-2i} = 1$, so $7|4i - ki$ and $7|ki - 2i$, hence $7|2i$, which gives the contradiction that $7|i$. Now take $(i, j) = (6, 1)$, so $Q = \langle a_1^6 a_2 \rangle$ and $Q^\beta = \langle a_1^6 a_2 \rangle^\beta = \langle a_1^{24} a_2 \rangle = \langle a_1^3 a_2 \rangle$, hence $a_1^3 a_2 = a_1^{6k} a_2^k$ for some $1 \leq k \leq 6$. We deduce that $a_1^{3-6k} = a_2^{k-2} = 1$, so $7|3 - 6k$ and $7|k - 2$, which gives the contradiction that $7|-9$. In the same way, we show that the remaining cases cannot occur. Therefore, Q cannot be $\langle \alpha, \beta \rangle$ -invariant. We deduce that G has no subgroup of order 7.3.2, which implies that G is not supersoluble as it is well known [1, Theorem 1.4.1] that supersoluble groups satisfy the converse of Lagrange's Theorem. Now P is a normal Abelian subgroup of G such that $G/P \simeq S_3$ is supersoluble, and for every $(x, a) \in G \times P$, we have $\langle x, a \rangle \leq \langle x \rangle P \neq G$. But it is proved in [1, Lemma 3.4.3] that proper subgroups of each group of order $7^2 \cdot 3 \cdot 2$ are supersoluble and hence so is $\langle x, a \rangle$. It follows that Theorem 1.1 is not true for the class of finite supersoluble groups.*

Example 2.8 *Let $G_1 = \mathbb{Z} \times G = \mathbb{Z} \times (P \rtimes \langle \alpha, \beta \rangle)$, where G is the previous example. So G_1 is not supersoluble.*

Put $A = \mathbb{Z} \times P$, so A is a normal Abelian subgroup of G_1 such that $G_1/A \simeq S_3$ is supersoluble, and for every $(x, a) \in G_1 \times A$, we have $\langle x, a \rangle \leq \langle x \rangle A = \mathbb{Z} \langle x \rangle P$ which is cyclic-by-supersoluble, hence supersoluble. Therefore, Theorem 1.1 is not true for the class of infinite supersoluble groups.

2.6. The finite-by-supersoluble case

Before proving the result on the finite-by-supersoluble case, we need the following preliminary result.

Lemma 2.9 *Let G be a finitely generated (torsion-free nilpotent)-by-(finite cyclic) group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are finite-by-supersoluble for all $(x, a) \in G \times A$. Then G is supersoluble.*

Proof Let G and A be as stated. Since G is polycyclic, we will show, by induction on the Hirsch length $h(G)$ of G , that G is supersoluble. Note that, as G is (torsion-free)-by-cyclic, any finite subgroup of G is cyclic, and hence any finite-by-supersoluble subgroup of G is supersoluble. It follows that our claim is true if $h(G) = 0$. Suppose now that $h(G) > 0$ and that any polycyclic group H , of Hirsch length less than $h(G)$, which is (torsion-free nilpotent)-by-(finite cyclic) and having an Abelian normal subgroup B such that H/B and $\langle x, b \rangle$ are finite-by-supersoluble for all $(x, b) \in H \times B$, is supersoluble. We will first show that G has an infinite cyclic normal subgroup C . Indeed, let N be a torsion-free normal nilpotent subgroup of G and let $x \in G$ such that $G = N \langle x \rangle$ and G/N is finite. If $A \cap N = 1$, then A is isomorphic to a subgroup of G/N , that is A is finite. Since G/A is finite-by-supersoluble, we get that G is finite-by-supersoluble and hence supersoluble; which proves the existence of C . So one can assume that $A \cap N \neq 1$. Since N is nilpotent and $1 \neq A \cap N \triangleleft N$, we deduce that $A \cap Z(N) = (A \cap N) \cap Z(N) \neq 1$. Put $K := \langle x, a \rangle$ for some $1 \neq a \in A \cap Z(N)$, so K is finite-by-supersoluble, hence supersoluble. As $1 \neq A \cap Z(N) \cap K \triangleleft K$, $A \cap Z(N) \cap K$ is a term of a cyclic normal series of K , so $A \cap Z(N) \cap K$ contains a nontrivial cyclic subgroup, say $C = \langle c \rangle$, which is normal in K . Let $g \in G$ and $t \in C$, so $g = bx^k$ and $t = c^{k'}$ for some $b \in N$ and $k, k' \in \mathbb{Z}$. Hence $t^g = (c^{k'})^{bx^k} = (c^{bx^k})^{k'} = (c^{x^k})^{k'}$, as $c \in Z(N)$. But $c^{x^k} = c^{k''}$ for some $k'' \in \mathbb{Z}$, because $C \triangleleft K$, so $t^g = c^{k'k''}$ and therefore C is an infinite cyclic normal subgroup of G . Now, let T/C be the torsion subgroup of N/C , which is finite, so T is an (infinite cyclic)-by-finite normal subgroup of G and N/T is torsion-free. Since T is torsion-free, we deduce that T is infinite cyclic. If $h(G/T) = 0$, then G is (infinite cyclic)-by-finite, so G is either finite-by-(infinite cyclic) or finite-by-(infinite dihedral) (see for instance [12, Lemma 4.1]) and hence G is supersoluble. It follows that G/T is a (torsion-free nilpotent)-by-(finite cyclic) group whose Hirsch length is less than $h(G)$, so by the inductive hypothesis, G/T is supersoluble. As T is cyclic, we obtain that G is supersoluble, as required. \square

Proposition 2.10 *Let G be a finitely generated soluble group having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are finite-by-supersoluble for all $(x, a) \in G \times A$. Then G is finite-by-supersoluble.*

Proof Let G and A be as stated. Since finite-by-supersoluble groups are nilpotent-by-finite, we deduce, by Proposition 2.5, that G is nilpotent-by-finite and hence it is (torsion-free nilpotent)-by-finite. Let N be a normal torsion-free nilpotent subgroup of G such that G/N is finite. Since the hypotheses of the proposition are satisfied by subgroups of G , we deduce, by Lemma 2.9, that for all g in G , $N \langle g \rangle$ is supersoluble. Now let X be an infinite subset of G . As G/N is finite, there are two distinct elements x, y in X such that $xN = yN$,

so $\langle x, y \rangle = \langle x, a \rangle$ for some $a \in N$ and hence $\langle x, y \rangle \leq N \langle x \rangle$ and thus $\langle x, y \rangle$ is supersoluble. Consequently, every infinite part of G contains two distinct elements generating a supersoluble group. It follows, by a result of Groves [5, Theorem], that G is finite-by-supersoluble, as required. \square

2.7. Proof of Corollary 1.2

Corollary 1.2 is a particular case of the following general result.

Proposition 2.11 *Let \mathfrak{X} be a subgroup and quotient closed class of groups and let \mathfrak{Y} be a class of groups that contain all finitely generated soluble group G having an Abelian normal subgroup A such that G/A and $\langle x, a \rangle$ are in \mathfrak{Y} for all $(x, a) \in G \times A$.*

1. *If all finitely generated Abelian-by-cyclic groups of \mathfrak{X} are in \mathfrak{Y} , then all finitely generated soluble groups of \mathfrak{X} are in \mathfrak{Y} .*
2. *Suppose that \mathfrak{Y} contains all finitely generated Abelian groups. If all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are in \mathfrak{Y} , then all finitely generated soluble groups of \mathfrak{X} are in \mathfrak{Y} .*

Proof Let \mathfrak{X} and \mathfrak{Y} be as stated.

1. Suppose that all finitely generated Abelian-by-cyclic groups of \mathfrak{X} are in \mathfrak{Y} and let G be a finitely generated soluble group in the class \mathfrak{X} . We will show by induction on the derived length d of G that G is in the class \mathfrak{Y} . If $d = 1$, then G is Abelian and thus, by hypothesis, it is in \mathfrak{Y} . Assume that $d > 1$ and that any finitely generated soluble group of \mathfrak{X} of derived length less than d is in the class \mathfrak{Y} . Put $A := G^{(d-1)}$, so A is an Abelian normal subgroup of G and by the inductive hypothesis, G/A belongs to \mathfrak{Y} . Let $(x, a) \in G \times A$, so $\langle x, a \rangle$ is a finitely generated Abelian-by-cyclic group in the class \mathfrak{X} , hence, by hypothesis, $\langle x, a \rangle$ is in \mathfrak{Y} . It follows, by hypothesis, that G is in the class \mathfrak{Y} .
2. Suppose that \mathfrak{Y} contains all finitely generated Abelian groups and that all 2-generated Abelian-by-cyclic groups of \mathfrak{X} are in \mathfrak{Y} . Let G be a finitely generated soluble group in the class \mathfrak{X} . Again by induction on the derived length d of G . If $d = 1$, then G is Abelian and thus it is in \mathfrak{Y} . Assume that $d > 1$ and put $A := G^{(d-1)}$, so A is an Abelian normal subgroup of G and by the inductive hypothesis, G/A belongs to \mathfrak{Y} . Let $(x, a) \in G \times A$, so $\langle x, a \rangle$ is a 2-generated Abelian-by-cyclic group of the class \mathfrak{X} and hence $\langle x, a \rangle$ is in the class \mathfrak{Y} and consequently G will be in \mathfrak{Y} , as required.

\square

2.8. Application

If \mathfrak{X} is a class of groups, then denote by (\mathfrak{X}, ∞) the class of groups in which every infinite subset contains a pair of distinct elements generating an \mathfrak{X} -group. In [5], it was proved that any finitely generated soluble group in the class (\mathfrak{U}, ∞) is finite-by-supersoluble, where \mathfrak{U} denotes the class of supersoluble groups. As an application of our results, we will improve this result as follows.

Proposition 2.12 *If G is a finitely generated soluble group in the class $(\mathfrak{F}\mathfrak{U}, \infty)$, then it is finite-by-supersoluble; \mathfrak{F} being the class of finite groups.*

Proof Let G be as stated. Since $(\mathfrak{F}\mathfrak{U}, \infty)$ is a subgroup and quotient closed class, one can assume, by Corollary 1.2, that G is a 2-generated Abelian-by-cyclic group. Let A be a normal Abelian subgroup of G and let $x \in G$ such that $G = A \langle x \rangle$. Since supersoluble groups are nilpotent-by-finite, G belongs to the class $(\mathfrak{N}\mathfrak{F}, \infty)$, where \mathfrak{N} denotes the class of nilpotent groups. It follows, by [11], that G is nilpotent-by-finite and hence it satisfies the maximal condition on subgroups. We deduce that the torsion subgroup T of A is finite. So factoring G by T , one can assume that A is torsion-free. Therefore every finite-by-supersoluble subgroup of G is cyclic-by-supersoluble, so supersoluble. Consequently, G belongs to the class (\mathfrak{U}, ∞) and hence G is finite-by-supersoluble by [5], as claimed. \square

Acknowledgment

This work was completed by the support of the General Directorate of Scientific Research and Technological Development (DGRSDT, Algeria).

References

- [1] Bray HG, Deskins WE, Johnson D. Between Nilpotent and Solvable. Poly-gonal, Washington, 1982.
- [2] Brookes CJB. Engel elements of soluble groups. Bulletin of the London Mathematical Society 1986; 18 (1): 7-10. doi: 10.1112/blms/18.1.7
- [3] Endimioni G, Traustason G. On torsion-by-nilpotent groups. Journal of Algebra 2001; 241 (2): 669-676. doi: 10.1006/jabr.2001.8772
- [4] Groves JRJ. Soluble groups in which every finitely generated subgroup is finitely presented. Journal of the Australian Mathematical Society (Series A) 1978; 26 (1): 115-125. doi: 10.1017/S1446788700011599
- [5] Groves JRJ. A conjecture of Lennox and Wiegold concerning supersoluble groups. Journal of the Australian Mathematical Society (Series A) 1983; 35 (2): 218-220. doi: 10.1017/S1446788700025702
- [6] Lennox JC, Wiegold J. Extensions of a problem of Paul Erdős on groups. Journal of the Australian Mathematical Society (Series A) 1981; 31 (4): 459-463. doi: 10.1017/S1446788700024253
- [7] Lennox JC. On a centrality property of finitely generated torsion free soluble groups. Journal of Algebra 1971; 18 (4): 541-548. doi: 10.1016/0021-8693(71)90137-2
- [8] Robinson DJS. A course in the theory of groups. Springer-Verlag, New York, 1982.
- [9] Robinson DJS. Finiteness Conditions and Generalized Soluble Groups Part 2. Springer-Verlag, New York, 1972.
- [10] Robinson DJS. Finiteness Conditions and Generalized Soluble Groups Part 1. Springer-Verlag, New York, 1972.
- [11] Trabelsi N. Soluble groups with a condition on infinite subsets. Algebra Colloquium 2002; 9 (4): 427-432.
- [12] Wall CTC. Poincaré complexes I. Annals of Mathematics (Second Series) 1967; 86 (2): 213-245. doi: 10.2307/1970688