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On a class of generalized Humbert–Hermite polynomials via generalized Fibonacci polynomials

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Abstract: A unified presentation of a class of Humbert's polynomials in two variables which generalizes the well known class of Gegenbauer, Humbert, Legendre, Chebycheff, Pincherle, Horadam, Kinney, Horadam–Pethe, Djordjević, Gould, Milovanović and Djordjević, Pathan and Khan polynomials and many not so called 'named' polynomials has inspired the present paper. We define here generalized Humbert–Hermite polynomials of two variables. Several expansions of Humbert-Hermite polynomials, Hermite–Gegenbauer (or ultraspherical) polynomials and Hermite–Chebyshev polynomials are proved.

Key words: Hermite polynomials, generalized Humbert polynomials, generalized (p, q)-Fibonacci polynomials, generalized (p, q)-Lucas polynomials

1. Introduction

In 1965, Gould [23] introduced the Humbert polynomials $P_n(m, x, y, p, c)$ and studied the explicit expressions, recurrence relations, higher derivatives, operational expansion and related inverse relations. By specifying the parameters, the polynomials $P_n(m, x, y, p, c)$ reduce to some well-known ones such as those under the names Chebyshev, Gegenbauer, Humbert, Kinney, Legenedre, Liouville and Pincherle.

Because of the generality of the Humbert–Gould polynomials $P_n(m, x, y, p, c)$, many studies have been devoted to them, and various generalizations have been presented. For example, the readers may consult the works due to Agarwal and Parihar [3], Dilcher [13], Djordjević [14–18], Djordjević and Djordjević [19], Djordjević and Srivastava [20], Djordjević and Milovanović [21], He and Shiue [25, 26], Horadam [28, 29, 33], Horadam and Mohan [30, 31], Horadam and Pethe [32, 34], Hsu [35], Hsu and Shiue [36], Kruchinin and Kruchinin [39], Dave [8], Liu [43], Milovanović and Djordjević [45], Pathan and Khan [49, 50, 51], Olver et al. [47], Ramírez [52, 53], Ozdemir et al. [48], Dere and Simsek [22], Kilar and Simsek [40], Simsek [56], Sinha [55], Shreshta [54], Khan and Pathan [41], Wang [57] and Wang and Wang [58, 59]. We begin with the following definition.

Humbert–Gould polynomials [23] are defined by the following generating relation

$$(c - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, c)t^n, \tag{1.1}$$

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2010 *AMS Mathematics Subject Classification:* 0A115, 05A19, 11B39, 33C45, 33C55, 33C99 where m is a positive integer and other parameters are unrestricted in general. The explicit representation of $P_n(m,x,y,p,c)$ is defined by (see [23, p.699]):

$$P_n(m,x,y,p,c)=\sum_{k=0}^{[\frac{n}{m}]}\binom{p}{k}\binom{p-k}{n-mk}c^{p-n+(m-1)k}y^k(-mx)^{n-mk}. \tag{1.2}$$

Recently, Wang and Wang [59] introduced the generalized (p,q)-Fibonacci polynomials and (p,q)-Lucas polynomials defined as follows.

Let $m\geq 2$ be a fixed positive integer, and let $p(x)$ and $q(x)$ be polynomials with real coefficients. The generalized (p,q)-Fibonacci polynomials $u_{n,m}(x)$ are defined by (see [53])

$$u_{n,m}(x)=p(x)u_{n-1,m}(x)+q(x)u_{n-m,m}(x), n\geq m, \tag{1.3}$$

with the initial conditions $u_{0,m}(x)=0,\ u_{1,m}(x)=1,\ u_{2,m}(x)=p(x),\cdots,u_{m-1,m}(x)=p^{m-2}(x)$, and define the generalized (p,q)-Lucas polynomials $v_{n,m}(x)$ by

$$v_{n,m}(x)=p(x)v_{n-1,m}(x)+q(x)v_{n-m,m}(x), n\geq m, \tag{1.4}$$

with the initial conditions $v_{0,m}(x)=2,\ v_{1,m}(x)=p(x),\ v_{2,m}(x)=p^2(x),\cdots,v_{m-1,m}(x)=p^{m-1}(x)$. Thus the polynomial sequences $(u_{n,m}(x))$ and $(v_{n,m}(x))$ satisfy the same recurrence relation of order m and different initial conditions. As usual, we may also call them the generalized Lucas u-polynomial sequence and generalized v-polynomial sequence, respectively.

Many well-known polynomial sequences are special cases of $(u_{n,m}(x))$ and $(v_{n,m}(x))$. For example, when $m=2$, $u_{n,m}(x)$ and $v_{n,m}(x)$ turn into the classical (p,q)-Fibonacci polynomials $u_n(x)$ and (p,q)-Lucas polynomials $v_n(x)$ defined by (see [6, 7, 14–20, 27, 37, 38, 42, 43, 44, 46]), which further reduce to the polynomials named after Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, Jacobsthal–Lucas, etc. Moreover, by specifying $p(x)$ and $q(x)$, the polynomials $u_{n,m}(x)$ and $v_{n,m}(x)$ reduce to those introduced in [52, 57, 58, 59].

The generating functions of the sequences $(u_{n,m}(x))$ and $(v_{n,m}(x))$ are defined by

$$U_m(x,t)=\sum_{n=0}^{\infty}u_{n,m}(x)t^n=\frac{t}{1-p(x)t-q(x)t^m}, \tag{1.5}$$

and

$$V_m(x,t)=\sum_{n=0}^{\infty}v_{n,m}(x)t^n=\frac{2-p(x)t}{1-p(x)t-q(x)t^m}, \tag{1.6}$$

which further give us the following relation:

$$v_{n,m}(x)=u_{n+1,m}(x)+q(x)u_{n-m+1,m}(x), n\geq m-1. \tag{1.7}$$

The two variable Hermite Kampé de Fériet polynomials $H_n(x,y)$ [1, 2, 5, 9] are defined by

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}. \tag{1.8}$$

The Gould-Hopper polynomials $g_n^m(x,y)$ (see [24]) is a generalization of (1.8). The notation $H_n^m(x,y)$ or $g_n^m(x,y)$ was given by Dattoli et al. [11, 12]. These are specified by

$$e^{xt+yt^m}=\sum_{n=0}^{\infty}H_n^m(x,y)\frac{t^n}{n!},(m\geq 2). \tag{1.9}$$

Another generalization of Hermite polynomials which we wish to consider in this paper is given by $H_{n,m,\nu}(x,y)$ [10] in the form of the generating function

$$e^{\nu(x+y)t-(xy+1)t^m}=\sum_{n=0}^{\infty}H_{n,m,\nu}(x,y)\frac{t^n}{n!}, \tag{1.10}$$

which reduces to the ordinary Hermite polynomials $H_n(x)$ when $\nu=2,\quad x=0$ or $\nu=2,\quad y=0$.

This paper is organized as follows. In Section 2, we introduce generalized Humbert–Hermite polynomials ${}_HG^{(r)}_{n+1,m}(x,y,z)$ via generalized (p,q)-Fibonacci polynomials and present their properties, related generating relations, expressions and identities. In Section 3, various expansions of Hermite-Chebyshev and Hermite–Gegenbaurer polynomials are obtained.

2. Generalized Humbert–Hermite polynomials

In this section, we introduce generalized Humbert–Hermite polynomials ${}_HG^{(r)}_{n+1,m}(x,y,z)$ arising from generalized (p,q)-Fibonacci polynomials and derive some properties of these polynomials. We start by the following definition.

Definition 2.1. For each complex number r, the generalized convolved (p,q)-Fibonacci polynomials, or in other words, the generalized Humbert–Hermite polynomials ${}_HG^{(r)}_{n+1,m}(x,y,z)$ are defined by

$$(1-p(x)t-q(x)t^m)^{-r}e^{yt+zt^m}=\sum_{n=0}^{\infty}{}_HG^{(r)}_{n+1,m}(x,y,z)\frac{t^n}{n!}, \tag{2.1}$$

so that

$$(1-p(x)t-q(x)t^m)^{-r}e^{\nu(y+z)t-(yz+1)t^m}=\sum_{n=0}^{\infty}{}_HG^{(r)}_{n+1,m}\left(x,\nu(y+z),-(yz+1)\right)\frac{t^n}{n!},$$

where $m\in\mathbb{N}$, $r>0$ and the other parameters are unrestricted in general.

On setting $y=z=0$ in (2.1), the result reduces to known result of Wang and Wang [59] as follows:

$$(1-p(x)t-q(x)t^m)^{-r}=\sum_{n=0}^{\infty}u^{(r)}_{n+1,m}(x)t^n. \tag{2.2}$$

Further by taking $m=2$, (2.2) reduces to another known result of Wang and Wang [59].

Note that when $r=1$, the polynomials $u^{(r)}_{n,m}(x)$ turn into the generalized (p,q)-Fibonacci polynomials $u_{n,m}(x)$, i.e. $u^{(1)}_{n,m}(x)=u_{n,m}(x)$ for $n\geq 1$ (see [42, 43, 44]).

In generating function (2.1), we replace r by $-r$ and then set $p(x) = \frac{mx}{c}$, $q(x) = -\frac{wy}{c}$ and $y = u$. Thus for $c \neq 0$, we obtain

$$\left(1 - \frac{mx}{c}t + \frac{wy}{c}t^m\right)^r e^{ut+zt^m} = c^{-r}\left(c - mxt + wyt^m\right)^r e^{ut+zt^m}$$
$$= c^{-r}\sum_{n=0}^{\infty} {}_HP_n(m,x,y,z,u,w,r,c)\frac{t^n}{n!}, \tag{2.3}$$

where ${}_HP_n(m,x,y,z,w,r,c)$ are the Humbert–Hermite–Gould polynomials, which further reduces to Hermite–Gegenbauer polynomials, Pincherlo polynomials, etc., by specifying the parameters m, x, y, r, c. Moreover, by specifying m, $p(x)$ and $q(x)$ in (2.1), we can obtain a number of polynomial sequences (see Table 1).

Using the definitions of $H_n^m(y,z)$ and $u_{n+1,m}^{(r)}(x)$ given by (1.9) and (2.2) in (2.1), we find the representation

$$ {}_HG_{n+1,m}^{(r)}(x,y,z) = \sum_{k=0}^{n} \frac{n! H_k^m(y,z) u_{n-k+1,m}^{(r)}(x)}{k!}. \tag{2.4}$$

Some special cases of (2.4) are given below.

On replacing $q(x)$ by $-q(x)$ in (2.4), we get

$$ {}_HG_{n+1,m}^{(r)}(x,y,z) = {}_HC_n^{(r,m)}(x,y,z) = \sum_{k=0}^{n} \frac{n! H_k^m(y,z) C_{n-k}^{(r,m)}(x)}{k!}.$$

Here ${}_HC_n^{(r,m)}(x,y,z)$ are Hermite–Gegenbaurer polynomials of three variables.

$$ {}_HC_n^{(1,m)}(x,y,z) = {}_HU_n^m(x,y,z) = \sum_{k=0}^{n} \frac{n! H_k^m(y,z) U_{n-k}^m(x)}{k!},$$

where ${}_HU_n^m(x,y,z)$ are Hermite–Chebychev polynomials of three variables.

$$ {}_HC_n^{1/2,m}(x,y,z) = {}_HP_n^m(x,y,z) = \sum_{k=0}^{n} \frac{n! H_k^m(y,z) P_{n-k}^m(x)}{k!},$$

where ${}_HP_n^m(x,y,z)$ are Hermite–Legendre polynomials of three variables.

By writing $-q(x)$ for $q(x)$ and setting $z = -1$ and $y = 2x$, the generating function (2.1) can be put in the form

$$[1 - p(x)t + q(x)t^m]^{-r} e^{2xt-t^m} = \sum_{n=0}^{\infty} {}_HG_n^{(r,m)}(x)t^n. \tag{2.5}$$

Furthermore, the Hermite–Gegenbauer (or ultraspherical) polynomials ${}_HC_n^{(r,2)}(x) = {}_HC_n^r(x)$ of one variable, for nonnegative integer r are given by

$$e^{2xt-t^2}(1 - p(x)t + q(x)t^2)^{-r} = \sum_{n=0}^{\infty} {}_HC_n^r(x)\frac{t^n}{n!}. \tag{2.6}$$

Table . Special cases of the generalized Humbert–Hermite polynomials.

	p(x)	q(x)	m	r	Generating function $\sum_{n=0}^{\infty} {}_HG^{(r)}_{n+1,m}(x,y,z)\frac{t^n}{n!}$	${}_Hu^{(r)}_{n+1,m}(x,y,z)$	Polynomials
(1)	1	1	2	r	$(1-t-t^2)^{-r}e^{yt+zt^2}$	${}_HF^{(r)}_{n+1}(y,z)$	Hermite–Fibonacci–Hoggatt
(2)	k	1	2	r	$(1-kt-t^2)^{-r}e^{yt+zt^2}$	${}_HF^{(r)}_{k,n+1}(y,z)$	Hermite–Fibonacci–Ramèrez
(3)	1	x	2	r	$(1-t-xt^2)^{-r}e^{yt+zt^2}$	${}_HD^{(r)}_{n+1}(x,y,z)$	Hermite–Dilcher
(4)	2x	1	2	r	$(1-2xt-t^2)^{-r}e^{yt+zt^2}$	${}_H\mathbb{P}^{(r)}_{n+1}(x,y,z)$	Hermite–Pell–Horadam–Mohan
(5)	2x	-1	2	$\frac{1}{2}$	$(1-2xt-t^2)^{-\frac{1}{2}}e^{yt+zt^2}$	${}_HP_n(x,y,z)$	Hermite–Legendre
(6)	2x	-1	2	r	$(1-2xt-t^2)^{-r}e^{yt+zt^2}$	${}_HC^{(r)}_n(x,y,z)$	Hermite–Gegenbauer
(7)	2x	-(2x-1)	2	r	$(1-2xt+(2x-1)t^2)^{-r}e^{yt+zt^2}$	${}_HS^{(r)}_n(x,y,z)$	Hermite–Sinha
(8)	2x	p	2	r	$(1-2xt-pt^2)^{-r}e^{yt+zt^2}$	${}_HF^{(r)}_n(x,y,z)$	Hermite–Fibonacci–Liu
(9)	$h(x)$	1	2	r	$(1-h(x)t-t^2)^{-r}e^{yt+zt^2}$	${}_HF^{(r)}_{h,n+1}(x,y,z)$	Hermite–Fibonacci-Ramèrez
(10)	$p(x)$	$q(x)$	2	r	$(1-p(x)t-q(x)t^2)^{-r}e^{yt+zt^2}$	${}_Hu^{(r)}_{n+1}(x,y,z)$	Hermite–Fibonacci–Wang
(11)	$\frac{2q}{p^2}$	$\frac{1}{p^2}$	2	$\frac{1}{2}$	$p(p^2-2qt-t^2)^{-\frac{1}{2}}e^{yt+zt^2}$	${}_Hf^{(r)}_{n+1,p,q}(x,y,z)$	Hermite–Lioville
(12)	2x	-1	3	r	$(1-2xt+t^3)^{-r}e^{yt+zt^2}$	${}_Hp^{(r)}_{n+1}(x,y,z)$	Hermite–Horadam–Pethe
(13)	3x	-1	3	$\frac{1}{2}$	$(1-3xt+t^3)^{-\frac{1}{2}}e^{yt+zt^2}$	${}_H\mathbb{P}_n(x,y,z)$	Hermite–Pincherle
(14)	3x	-1	3	r	$(1-3xt+t^3)^{-r}e^{yt+zt^2}$	${}_H\mathbb{P}^{(r)}_n(x,y,z)$	Hermite–Pincherle–Humbert
(15)	1	2x	m	r	$(1-t-2xt^m)^{-r}e^{yt+zt^m}$	${}_HJ^{(r)}_{n+1,m}(x,y,z)$	Gould–Hopper–Jacobsthal–Djordjević
(16)	x	-1	m	r	$(1-xt+t^m)^{-r}e^{yt+zt^m}$	${}_HV^{(r)}_{n+1,m}(x,y,z)$	Hermite–Chebyshev–Djordjević
(17)	x	-2	m	r	$(1-xt+2t^m)^{-r}e^{yt+zt^m}$	${}_Ha^{(r)}_{n+1,m}(x,y,z)$	Hermite–Fermat–Djordjević
(18)	2x	-1	m	r	$(1-2xtt+t^m)^{-r}e^{yt+zt^m}$	${}_Hp^{(r)}_{n,m}(x,y,z)$	Hermite–Milovanović–Djordjević
(19)	m x	-1	m	$\frac{1}{m}$	$(1-mxt+t^m)^{-\frac{1}{m}}e^{yt+zt^m}$	${}_HP^{(r)}_{n,m}(x,y,z)$	Hermite–Kinney
(20)	m x	-1	m	r	$(1-mxt+t^m)^{-r}e^{yt+zt^m}$	${}_H\prod^{(r)}_{n,m}(x,y,z)$	Hermite–Humbert
(21)	$\frac{mx}{c}$	$-\frac{y}{c}$	m	-r	$c^{-r}(c-mxt+yt^m)^{-r}e^{yt+zt^m}$	$c^{-r}{}_HP_n(m,x,y,z,r,c)$	Hermite–Humbert–Gould
(22)	2+x	-1	m	r	$(1-(2+x)t+t^m)^{-r}e^{yt+zt^m}$	${}_HB^{(r)}_{n+1,m}(x,y,z)$	Hermite–Morgan–Voyce–Djordjević
(23)	$\frac{ax}{c}$	$-\frac{b(2x-1)^d}{c}$	m	r	$c^r(1-axt+b(2x-1)^dt^m)^{-r}e^{yt+zt^m}$	$c^r{}_H\Theta^{(r)}_n(x,y,z)$	Hermite–Pathan–Khan
(24)	$\frac{ax^\mu}{c}$	$-\frac{b(px^\nu+qx+s)^\rho}{c}$	m	r	$c^r(c-ax^\mu t+b(px^\nu+qx+s)^\rho t^m)^{-r}e^{yt+zt^m}$	$c^r{}_H\Phi^{(r)}_n(x,y,z)$	Hermite–Agarwal–Parihar

Letting $r = 1/2$ and $r = 1$ in (2.6) gives

$$e^{2xt-t^2}(1 - p(x)t + q(x)t^2)^{-1/2} = \sum_{n=0}^{\infty} {}_HP_n(x)\frac{t^n}{n!}, \tag{2.7}$$

where ${}_HP_n(x)$ are Hermite–Legendre polynomials and

$$e^{2xt-t^2}(1 - p(x)t + q(x)t^2)^{-1} = \sum_{n=0}^{\infty} {}_HU_n(x)\frac{t^n}{n!}, \tag{2.8}$$

respectively, where ${}_HU_n(x)$ are Hermite–Chebyshev polynomials.

Now, we establish the explicit expressions for the generalized Hermite–Humbert polynomials. We begin by the following theorem.

Theorem 2.1. The following explicit expressions for the generalized Hermite–Humbert polynomials ${}_HG^{(r)}_{n+1,m}(x, y, z)$ holds true:

$$\begin{aligned} {}_HG^{(r)}_{n+1,m}(x, y, z) = \sum_{s=0}^{n} \sum_{j=0}^{[\frac{n-s}{m}]} n! \binom{-r}{N} \binom{N}{j} (-p)^{n-s-mj}(-q)^j \\ \times H^m_s(y, z)\frac{1}{s!}, \end{aligned} \tag{2.9}$$

where $N = n - s - (m - 1)j$.

Proof From (2.1) can be expanded as

$$\begin{aligned} \sum_{n=0}^{\infty} {}_HG^{(r)}_{n+1,m}(x, y, z)\frac{t^n}{n!} &= \sum_{k=0}^{\infty} \binom{-r}{k} (-pt - qt^m)^k \sum_{s=0}^{\infty} H^m_s(y, z)\frac{t^s}{s!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{[\frac{n}{m}]} \binom{-r}{n-(m-1)j} \binom{n-(m-1)j}{j} (-p)^{n-mj}(-q)^j t^n \\ &\quad \times \sum_{s=0}^{\infty} H^m_s(y, z)\frac{t^s}{s!}. \end{aligned}$$

Replacing n by $n - s$ in above equation, we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_HG^{(r)}_{n+1,m}(x, y, z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{j=0}^{[\frac{n-s}{m}]} \binom{-r}{n-s-(m-1)j} \binom{n-s-(m-1)j}{j} (-p)^{n-s-mj}(-q)^j \\ \times H^m_s(y, z)\frac{t^n}{s!}. \end{aligned}$$

On comparing the coefficients of t^n on both sides, we get (2.9). □

Remark 2.1. On setting $y = z = 0$ in Theorem 2.1, the result reduces to the known result of Wang and Wang [59].

Remark 2.2. On setting $m=2, z=-1$ and replacing y by $2y$ in Theorem 2.1, it reduces to the following result involving Hermite numbers H_s

$$ {}_HG^{(r)}_{n+1,2}(x,2y,-1) = n!\sum_{s=0}^{n}\sum_{j=0}^{[\frac{n-s}{2}]} \left(\begin{array}{c} -rn-s-j \end{array} \right) \left(\begin{array}{c} n-s-jj \end{array} \right) (-p)^{n-s-2j}(-q)^j \times \frac{H_s}{s!}. \tag{2.10} $$

Theorem 2.2. The following explicit expressions for the generalized Hermite–Humbert polynomials ${}_HG^{(r)}_{n+1,m}(x,y,z)$ holds true:

$$ {}_HG^{(r)}_{n+1,m}(x,y,z) = \sum_{s=0}^{n}\frac{n!H^m_s(y,z)}{s!}\sum_{i=0}^{[\frac{n-s}{2}]} $$

$$ \times \sum_{0\le j\le i}\binom{r+i-1}{i}\binom{i}{j}\binom{M}{N}\left(\frac{p(x)}{2}\right)^{n-s-mj} q^j(x), \tag{2.11} $$

where $M=n+2r-s-(m-2)j-1,\ N=n-s-2i-(m-2)j$.

Proof Let $p(x)=p$ and $q(x)=q$ in (2.1). Then generating function (2.1) can be written as

$$ \sum_{n=0}^{\infty} {}_HG^{(r)}_{n+1,m}(x,y,z)\frac{t^n}{n!} = (1-pt-qt^m)^{-r}e^{yt+zt^m}. $$

Now

$$ (1-pt-qt^m)^{-r} = \left(1-pt+\left(\frac{pt}{2}\right)^2-\left(\frac{pt}{2}\right)^2-qt^m\right)^{-r} $$

$$ = \left(1-\frac{pt}{2}\right)^{-2r}\left(1-\frac{(\frac{pt}{2})^2+qt^m}{(1-\frac{pt}{2})^2}\right)^{-r} $$

$$ = \sum_{i=0}^{\infty}\binom{-r}{i}\left\{\left(\frac{pt}{2}\right)^2+qt^m\right\}^i\left(1-\frac{pt}{2}\right)^{-2r-2i} $$

$$ = \sum_{i=0}^{\infty}\binom{-r}{i}(-1)^i\left(\frac{pt}{2}\right)^{2i}\sum_{j=0}^{i}\binom{i}{j}\left\{\frac{qt^m}{(\frac{pt}{2})^2}\right\}\sum_{k=0}^{\infty}\binom{-2r-2i}{k}(-1)^k\left(\frac{pt}{2}\right)^k $$

$$ = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\infty}\binom{r+i-1}{i}\binom{i}{j}\binom{2r+2i+k-1}{k}\left(\frac{p}{2}\right)^{2i-2j+k} q^j t^{2i-2j+k+mj}. $$

From (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HG^{(r)}_{n+1,m}(x,y,z)\frac{t^n}{n!} $$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{[\frac{n}{2}]} \sum_{0\le j\le i} \begin{pmatrix} r+i-1 \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} n+2r-(m-2)j-1 \\ n-2i-(m-2)j \end{pmatrix} \left(\frac{p(x)}{2}\right)^{n-mj} q^j(x) t^n$$

$$\times \sum_{s=0}^{\infty} H_s^m(y,z) \frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{H_s^m(y,z)}{s!} \sum_{i=0}^{[\frac{n-s}{2}]} \sum_{0\le j\le i} \begin{pmatrix} r+i-1 \\ i \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \left(\frac{p(x)}{2}\right)^{n-s-mj} q^j(x) t^n.$$

On comparing the coefficients of t^n on both the sides, we get the desired result (2.11). □

Remark 2.3. On setting $y = z = 0$ in Theorem 2.2, it reduces to the known result of Wang and Wang [59].

3. On expansions of Hermite–Chebyshev and Hermite–Gegenbauer polynomials

In this section, we prove several theorems on the expansions of Hermite–Gegenbauer and Hermite–Chebyshev polynomials of three variables. We will start with (2.1), (2.3) and a special case of (2.1) by replacing $q(x)$ by $-q(x)$ and setting $r = 1$. Thus we get

$$(1 - p(x)t + q(x)t^m)^{-1} e^{yt+zt^m} = \sum_{n=0}^{\infty} {}_HU_{n,m}(x,y,z) \frac{t^n}{n!}, \tag{3.1}$$

which will be used in obtaining the corollaries of the following theorem.

Theorem 3.1. For $k \in \mathbb{N}$ and $x, y, z \in \mathbb{C}$,

$$\sum_{s=0}^{n} \frac{H_s^m(ky,kz) u_{n+1-s}^{(rk,m)}(x)}{s!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG_{n_1+1,m}^{(r)}(x,y,z)\,{}_HG_{n_2+1,m}^{(r)}(x,y,z)\cdots {}_HG_{n_k+1,m}^{(r)}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}. \tag{3.2}$$

Proof The definition of ${}_HG_{n+1,m}^{(r)}(x,y,z)$ given in (2.1) can be written as

$$\left[(1 - p(x)t - q(x)t^m)^{-r} e^{yt+zt^m}\right]^k$$

$$= (1 - p(x)t - q(x)t^m)^{-rk} e^{kyt+kzt^m} = \left[\sum_{n=0}^{\infty} {}_HG_{n+1,m}^{(r)}(x,y) \frac{t^n}{n!}\right]^k.$$

Using (1.9), we can write

$$e^{kyt+kzt^m} = \sum_{s=0}^{\infty} H_s^m(ky,kz) \frac{t^s}{s!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^{\infty} u_{n+1,m}^{(rk,m)}(x)t^n \sum_{s=0}^{\infty} H_s^m(ky,kz)\frac{t^s}{s!}$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1+1,m}^{(r)}(x,y,z){}_HG_{n_2+1,m}^{(r)}(x,y,z)\cdots {}_HG_{n_k+1,m}^{(r)}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n.$$

An application of manipulation of series yields

$$\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{H_s^m(ky,kz)u_{n+1-s}^{(rk,m)}(x)}{s!}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1+1,m}^{(r)}(x,y,z){}_HG_{n_2+1,m}^{(r)}(x,y,z)\cdots {}_HG_{n_k+1,m}^{(r)}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. □

Remark 3.1. On setting $r=1$ in Theorem 3.1, the result reduces to

Corollary 3.1. For $k\in\mathbb{N}$ and $x,y,z\in\mathbb{C}$,

$$\sum_{s=0}^{n}\frac{H_s^m(ky,kz)C_{n-s}^{k,m}(x)}{s!}$$

$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HU_{n_1}^m(x,y,z){}_HU_{n_2}^m(x,y,z)\cdots {}_HU_{n_k}^m(x,y,z)}{n_1!n_2!\cdots n_k!}. \tag{3.3}$$

Remark 3.2. On taking $r=0$ in Theorem 3.1, the result reduces to

Corollary 3.2. For $k\in\mathbb{N}$ and $y,z\in\mathbb{C}$,

$$\frac{H_n^m(ky,kz)}{n!}=\sum_{n_1+n_2+\cdots+n_k=n}\frac{H_{n_1}^m(y,z)H_{n_2}^m(y,z)\cdots H_{n_k}^m(y,z)}{n_1!n_2!\cdots n_k!}. \tag{3.4}$$

Remark 3.3. On setting $m=2,\ r=0,\ z=-1,\ y=2x$ in Theorem 3.1, it reduces to known result of Batahan and Shehata [4,p.50.,Eq. (2.1)].

Corollary 3.3. For $k\in\mathbb{N}$ and $x\in\mathbb{C}$,

$$\sum_{r=0}^{[\frac{n}{2}]}\frac{(-k)^r(2kx)^{n-2r}}{(n-2r)r!}=\sum_{n_1+n_2+\cdots+n_k=n}\frac{H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)}{n_1!n_2!\cdots n_k!}. \tag{3.5}$$

Theorem 3.2. For $k\in\mathbb{N}$ and $X,Y\in\mathbb{C}$,

$$\sum_{s=0}^{n}\frac{H_s^m(kY,kZ)u_{n+1-s}^{(rk,m)}(X)}{s!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG^{(r)}_{n_1+1,m}(X,Y,Z)\,{}_HG^{(r)}_{n_2+1,m}(X,Y,Z)\ldots{}_HG^{(r)}_{n_k+1,m}(X,Y,Z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}, \tag{3.6}$$

where $X = \sum_{i=0}^{k} x_i$ and $Y = \sum_{j=0}^{k} y_j$, $Z = \sum_{l=0}^{k} z_l$.

Proof The definition of ${}_HG^{(r)}_{n+1,m}(x,y,z)$ can be written as

$$\left[(1-p(X)t-q(X)t^m)^{-r}\, e^{Yt+Zt^m}\right]^k$$

$$= (1-p(X)t-q(X)t^m)^{-rk}\, e^{kYt+kZt^m}$$

$$= \left[\sum_{n=0}^{\infty} {}_HG^{(r)}_{n+1,m}(x_1+x_2+\cdots+x_k, y_1+y_2+\cdots+y_k, z_1+z_2+\cdots+z_k)\frac{t^n}{n!}\right]^k.$$

Using (1.9), we can write

$$e^{kYt+kZt^m} = \sum_{s=0}^{\infty} H^m_s(kY,kZ)\frac{t^s}{s!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^{\infty} u^{rk,m}_{n+1}(X)t^n \sum_{s=0}^{\infty} H^m_s(kY,kZ)\frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG^{(r)}_{n_1+1,m}(X,Y,Z)\,{}_HG^{(r)}_{n_2+1,m}(X,Y,Z)\ldots{}_HG^{(r)}_{n_k+1,m}(X,Y,Z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n.$$

An application of manipulation of series yields

$$\sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{H^m_s(kY,kZ)u^{(rk,m)}_{n+1-s}(X)}{s!}t^n$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HG^{(r)}_{n_1+1,m}(X,Y,Z)\,{}_HG^{(r)}_{n_2+1,m}(X,Y,Z)\ldots{}_HG^{(r)}_{n_k+1,m}(X,Y,Z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. □

Remark 3.4. On setting $r = 1$ in Theorem 3.2, the result reduces to

Corollary 3.4. For $k \in \mathbb{N}$ and $X, Y, Z \in \mathbb{C}$,

$$\sum_{s=0}^{n} \frac{H^m_s(kY,kZ)C^{k,m}_{n-s}(X)}{s!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_HU^m_{n_1}(X,Y,Z)\,{}_HU^m_{n_2}(X,Y,Z)\cdots{}_HU^m_{n_k}(X,Y,Z)}{n_1!n_2!\cdots n_k!}. \tag{3.7}$$

Remark 3.5. On setting $r = 0$ in Theorem 3.2, the result reduces to

Corollary 3.5. For $k \in \mathbb{N}$ and $Y, Z \in \mathbb{C}$,

$$\frac{H_n^m(kY, kZ)}{n!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^m(Y,Z)H_{n_2}^m(Y,Z)\cdots H_{n_k}^m(Y,Z)}{n_1!n_2!\cdots n_k!}. \tag{3.8}$$

Remark 3.6. On setting $m = 2$, $r = 0$, $Z = -1$, $Y = 2X$ in Theorem 3.2, the result reduces to known result of Batahan and Shehata [4,p.51.,Eq. (2.4)].

Corollary 3.6. For $k \in \mathbb{N}$

$$\sum_{s=0}^{[\frac{n}{2}]} \frac{(-k)^s(2k(x_1+x_2+\cdots+x_k))^{n-2s}}{(n-2s)!s!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x_1)H_{n_2}(x_2)\cdots H_{n_k}(x_k)}{n_1!n_2!\cdots n_k!}. \tag{3.9}$$

Theorem 3.3. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$,

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s(rk)_{n-(m-1)s}(p(x))^{n-ms}(q(x))^s}{s!\,(n-ms)!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} u_{n_1+1,m}^{(r)}(x)u_{n_2+1,m}^{(r)}(x)\cdots u_{n_k+1,m}^{(r)}(x). \tag{3.10}$$

Proof Using the power series of $[1-p(x)t-q(x)t^m]^{-r}$ and making the necessary series arrangements gives

$$[1-p(x)t-q(x)t^m]^{-rk} = \sum_{n=0}^{\infty}\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s(rk)_{n-(m-1)s}(p(x))^{n-ms}(q(x))^s}{s!\,(n-ms)!}t^n.$$

In addition to this, we can write

$$[1-p(x)t-q(x)t^m]^{-rk} = \left[[1-p(x)t-q(x)t^m]^{-r}\right]^k = \left[\sum_{n=0}^{\infty} u_{n+1,m}^{(r)}(x)t^n\right]^k$$

$$= \sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n} u_{n_1+1,m}^{(r)}(x)u_{n_2+1,m}^{(r)}(x)\cdots u_{n_k+1,m}^{(r)}(x)t^n.$$

Now equating coefficients of t on both sides of the resulting equation will give the required result. □

Remark 3.7. Setting $r = 1$ in Theorem 3.3, the result reduces to

Corollary 3.7. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$,

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s(k)_{n-(m-1)s}(p(x))^{n-ms}(q(x))^s}{s!\,(n-ms)!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} U_{n_1}^m(x)U_{n_2}^m(x)\cdots U_{n_k}^m(x). \tag{3.11}$$

Similarly, we can define generalized (p,q)-Lucas polynomials as follows:

Definition 3.1. For each complex number r, the generalized convolved (p,q)-Lucas polynomials, or in other words, the generalized Humbert–Hermite polynomials ${}_Hv^{(r)}_{n,m}(x,y,z)$ are defined by

$$\left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)^r e^{yt+zt^m}=\sum_{n=0}^{\infty}{}_Hv^{(r)}_{n,m}(x,y,z)\frac{t^n}{n!}, \tag{3.12}$$

where $m\in\mathbb{N},\ r>0$ and the other parameters are unrestricted in general.

On setting $y=z=0$ in (3.12), it reduces to known result of Wang and Wang [59] as follows:

$$\left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)^r=\sum_{n=0}^{\infty}v^{(r)}_{n,m}(x)t^n. \tag{3.13}$$

Theorem 3.4. For $k\in\mathbb{N}$ and $x,y,z\in\mathbb{C}$,

$$\sum_{s=0}^{n}v^{rk}_{n-s,m}(x)H^m_s(ky,kz)\frac{1}{s!}$$

$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_Hv^{(r)}_{n_1+1,m}(x,y,z)\,{}_Hv^{(r)}_{n_2+1,m}(x,y,z)\cdots{}_Hv^{(r)}_{n_k+1,m}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}. \tag{3.14}$$

Proof Using (1.9) and (3.12), we can write

$$\left[\left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)^r e^{yt+zt^m}\right]^k=\left(\sum_{n=0}^{\infty}{}_Hv^{(r)}_{n,m}(x,y,z)\frac{t^n}{n!}\right)^k$$

$$\left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)^{kr} e^{kyt+kzt^m}=\left(\sum_{n=0}^{\infty}{}_Hv^{(r)}_{n,m}(x,y,z)\frac{t^n}{n!}\right)^k$$

$$\sum_{n=0}^{\infty}v^{rk}_{n,m}(x)t^n\sum_{s=0}^{\infty}H^m_s(ky,kz)\frac{t^s}{s!}$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_Hv^{(r)}_{n_1+1,m}(x,y,z)\,{}_Hv^{(r)}_{n_2+1,m}(x,y,z)\cdots{}_Hv^{(r)}_{n_k+1,m}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n$$

$$\sum_{n=0}^{\infty}\sum_{s=0}^{n}v^{rk}_{n-s,m}(x)H^m_s(ky,kz)\frac{t^n}{s!}$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_Hv^{(r)}_{n_1+1,m}(x,y,z)\,{}_Hv^{(r)}_{n_2+1,m}(x,y,z)\cdots{}_Hv^{(r)}_{n_k+1,m}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n.$$

Comparing the coefficients of t^n on both sides, we get (3.14). □

Remark 3.8. On setting $y=z=0$ in Theorem 3.4, the result reduces to

Corollary 3.8. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$,

$$v_{n,m}^{rk}(x) = \sum_{n_1+n_2+\cdots+n_k=n} \frac{v_{n_1+1,m}^{(r)}(x) v_{n_2+1,m}^{(r)}(x) \cdots v_{n_k+1,m}^{(r)}(x)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}. \tag{3.15}$$

Theorem 3.5. For $r \in \mathbb{N}$ and $x, y, z \in \mathbb{C}$,

$$\sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_Hv_{n_1+1,m}(x,y,z)\,{}_Hv_{n_2+1,m}(x,y,z)\cdots {}_Hv_{n_k+1,m}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}$$

$$= \sum_{i=0}^{n} \binom{r}{n-i} 2^{r-n+i}(-p(x))^{n-i}\,{}_Hv_{i+1,m}^{(r)}(x,ry,rz)\frac{1}{i!}. \tag{3.16}$$

Proof From (3.12) can be written as

$$\left[\left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)e^{yt+zt^m}\right]^r = \left(\sum_{n=0}^{\infty} {}_Hv_{n,m}(x,y,z)\frac{t^n}{n!}\right)^r$$

$$(2-p(x)t)^r(1-p(x)t-q(x)t^m)^{-r}e^{ryt+rzt^m} = \left(\sum_{n=0}^{\infty} {}_Hv_{n,m}(x,y,z)\frac{t^n}{n!}\right)^r$$

$$\sum_{n=0}^{r} \binom{r}{n} 2^{r-n}(-p(x))^n t^n \sum_{i=0}^{\infty} {}_Hv_{i+1,m}^{(r)}(x,ry,rz)\frac{t^i}{i!}$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} \frac{{}_Hv_{n_1+1,m}(x,y,z)\,{}_Hv_{n_2+1,m}(x,y,z)\cdots {}_Hv_{n_k+1,m}(x,y,z)}{(n_1+1)!(n_2+1)!\cdots(n_k+1)!}t^n.$$

Comparing the coefficients of t^n on both sides, we get (3.16). □

Remark 3.9. On setting $y = z = 0$ in Theorem 3.5, the result reduces to

Corollary 3.9. For $k \in \mathbb{N}$ and $x \in \mathbb{C}$,

$$\sum_{n_1+n_2+\cdots+n_k=n} v_{n_1,m}(x)v_{n_2,m}(x)\cdots v_{n_k,m}(x) = \sum_{i=0}^{n} \binom{r}{n-i} 2^{r-n+i}(-p(x))^{n-i}u_{i+1,m}^{(r)}(x). \tag{3.17}$$

4. Conclusion

In concluding this discussion on a class of generalized Humbert–Hermite polynomials, Hermite–Gegenbaurer (or ultraspherical) polynomials and Hermite–Chebyshev polynomials, we want to offer here a few further considerations on the topic, and present some modifications by generalizing (2.1) and (3.12).

The formalism and the definition associated with Milne–Thomson type polynomials and special numbers proposed here may offer significant advantages due to the abundance of their applications in many branches of mathematics such as in p-adic analytic number theory, umbral calculus, special functions and mathematical analysis, numerical analysis, combinatorics and other related areas.

In [56], Simsek defined the three-variable polynomials $y_6(n;x,y,z;a,b,\nu)$ as follows:

$$(b+f(t,a))^z e^{xt+yh(t,\nu)} = \sum_{n=0}^{\infty} y_6(n;x,y,z;a,b,\nu)\frac{t^n}{n!}, \tag{4.1}$$

where $f(t,a)$ is a member of family of analytic functions or meromorphic functions, a and b are any real numbers, ν is positive integer. When $x=0$, $y=z=1$, Equation (4.1)

$$y_6(n;0,1,1;a,b,\nu) = y_6(n;a,b,\nu),$$

which is defined by means of the following generating function:

$$(b+f(t,a))^z e^{h(t,\nu)} = \sum_{n=0}^{\infty} y_6(n;a,b,\nu)\frac{t^n}{n!}, \text{(see [56])}.$$

When $b=0$, in the numbers $y_6(n;a,b,\nu)$ reduce to the Milne–Thomson numbers of order a, $\varphi_n^{(a)}$:

$$y_6(n;a,0,\nu) = \varphi_n^{(a)}.$$

Some special cases of these numbers give well-known numbers such as the Milne–Thomson numbers, the Hermite numbers, the Bernoulli and Euler numbers, the Lah numbers, and other numbers. For example, setting $f(t,a) = \frac{1}{q(a)t^k - p(a)t + 1}, b=0$, x is replaced by $2x$ and $y=1$ and $h(t,\nu) = -t^\nu$ in (4.1), we get

$$\left(\frac{1}{1-p(a)t+q(a)t^k}\right)^z e^{2xt-t^\nu} = \sum_{n=0}^{\infty} y_6(n;2x,1,z;a,0,\nu)\frac{t^n}{n!}, \tag{4.2}$$

where k is any positive integer. That is,

$$\frac{y_6(n;2x,1,z;a,0,\nu)}{n!} = {}_HG_n^{(z,\nu)}(z,a),$$

which is a special case of (2.1).

A similar type of the three-variable polynomials associated with Milne–Thomson type polynomials which can generalize (3.12) is possible to investigate. These polynomials presented by means of generating functions may be useful in developing extensive researches on various families of special polynomials such as the Bernoulli polynomials, Euler polynomials, Genocchi polynomials and Catalan polynomials and numbers, which will be investigated in a forthcoming article.

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