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On the blow-up of solutions to a fourth-order pseudoparabolic equation

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Abstract: In this note, we consider a fourth-order semilinear pseudoparabolic differential equation including a strong damping term together with a nonlocal source term. The problem is considered under the periodic boundary conditions and a finite time blow-up result is established. Also a lower bound estimate for the blow-up time is obtained.

Key words: Fourth order pseudoparabolic equation, positive initial energy, periodic boundary conditions, blow-up

1. Introduction

In this work, we consider the following fourth-order pseudoparabolic equation

$$u_t - \alpha\Delta u_t - \Delta u + \Delta^2 u = |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx, \quad x \in \Omega,\ 0 < t < T, \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x),\ \ u_0 \in \dot{H}^2_{per}(\Omega) \quad x \in \Omega, \quad u_0 \not\equiv 0 \tag{1.2}$$

and periodic boundary conditions

$$u(x,t) = u(x + L_i e_i, t),\ u_{x_i}(x,t) = u_{x_i}(x + L_i e_i, t) \text{ for all } x \in \Gamma_i \ \ 0 < t < T, \tag{1.3}$$

where

$$\Omega = (0, L_1) \times \cdots \times (0, L_n) \subset \mathbb{R}^n,\ n = 2 \text{ or } 3,\ \Gamma_i = \partial\Omega \cap \{x_i = 0\},$$

$\partial\Omega$ is the boundary of Ω, and $p > 1$, and for $u_0 \not\equiv 0$,

$$\int_\Omega u_0(x)dx = 0. \tag{1.4}$$

The second term on the left-hand side of the equation (1.1) is a strong damping term and the second term on the right-hand side is a nonlocal source term. When $\alpha = 0$, the fourth-order parabolic equation is a thin-film equation with nonlocal source term. Here we will take $\alpha = 1$. For the correlation between the parabolic and pseudoparabolic equations and their solutions, we refer to [25].

It is obvious from the equation (1.1) and the assumption (1.4) that $\frac{d}{dt}\int_\Omega u\,dx = 0$, which means the average zero periodic initial value functions produce average zero periodic solutions.

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Existence of blow-up solutions for the nonlinear models of partial differential equations is a long standing topic. Blow-up solutions for the parabolic boundary-initial value problems have been studied by many researchers; among them, we refer to [4]–[23]. The problems including $|u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx$ type nonlocal source terms allow sign changing solutions and has been studied by many autors in the last decade. The early work for this aspect is the study of the heat equation by Budd and Dold [2] when $p = 2$:

$$u_t = \Delta u + u^2 - -\frac{1}{|\Omega|}\int_\Omega |u|^2\,dx,$$

For evolution equations with nonlocal source terms while p is general we refer to [2]–[5].

In [21], Qu and Zhou considered the following initial boundary value problem for a fourth-order reaction-diffusion equation including a nonlocal source term:

$$u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{a}\int_0^a |u|^{p-1}u\,dx, \quad x \in \Omega = (0,a),\ t > 0, \tag{1.5}$$

$$u_x = u_{xxx} = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \qquad u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.6}$$

where $p > 1$ and $u_0 \in H^2(\Omega)$, $\int_0^a u_0(x)dx = 0$ with $u_0 \not\equiv 0$.

Using the potential well method Qu and Zhou derived a threshold result for the global existence and nonexistence of solutions for this problem. In [27], Zhou established a blow-up result for the same problem under the assumption that the initial energy is positive and also derived an upper bound for the blow-up time. In [19], Philippin studied the following initial value problem of a fourth-order variable coefficient reaction diffusion equation:

$$u_t + \Delta^2 u = k(x)|u|^{p-1}u, \quad x \in \Omega,\ \ 0 < t < T,$$

$$u(x,t) = 0, \quad \frac{\partial u}{\partial n} = 0 \ or\ \delta u = 0,\ x \in \partial\Omega,\ 0 < t < T,$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad \Omega \subset \mathbb{R}^n,\ n \geq 2,\ \ k(x) > 0.$$

In addition to a blow result, he obtained a lower bound for the blow-up time.

Let us denote the $L^2(\Omega)$-inner product and norm by $(u,v) = \int_0^a u(x)v(x)\,dx$ and $\|\cdot\|$, respectively. Let $\dot{H}^2_{per}(\Omega) := \{u \in H^2_{per}(\Omega) : \int_\Omega u\,dx = 0\}$.

The pair $\left(\dot{H}^2_{per}(\Omega), \|\cdot\|_{\dot{H}^2_{per}(\Omega)}\right)$ is a Hilbert space with the inner product $(u,v)_{\dot{H}^2_{per}(\Omega)} := \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega \Delta u \Delta v\,dx$, and the norm

$$\|u\|^2_{\dot{H}^2_{per}(\Omega)} := \|\nabla u\|^2 + \|\Delta u\|^2,$$

respectively. Also $\|u\|_p$ represents the norm $(\int_\Omega |u|^p\,dx)^{\frac{1}{p}}$ of the space $L^p(\Omega)$.

The rest of this note is organized as follows: In Section 2, the weak solution to (1.1)–(1.3) is defined and a local existence result is given. In Section 3, in addition to proving existence of a blow-up solution, a lower bound for the blow-up time is derived.

2. Local existence

We begin this section by defining mild solutions. For more detail, we refer to ([24], Chapter 4.2).

Definition 2.1 *Consider the following linear inhomogeneous equation with initial condition*

$$\partial_t u + Au = f(t), \quad u(0) = u_0 \in W, \tag{2.1}$$

where W is a Banach space, and $t \geq 0$. A function $u : [0,T) \to W$ is said to be a mild solution of the initial value problem (2.1) in the space W and on the interval $[0,T)$, provided that $u \in C([0,T); W)$ and u satisfies

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s) ds.$$

One can write the problem (1.1)–(1.3) in an abstract form

$$\partial_t u + Au = F(u), \quad u(0) = u_0, \tag{2.2}$$

where the operator $A = -\Delta$ is a positive, self-adjoint, linear operator with compact resolvent on the Hilbert space $L^2(\Omega)$ with domain $\mathcal{D}(A) = \dot{H}^2_{per}(\Omega)$.

Since for $p > 1$, $f(s) = |s|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |s|^{p-1} s\, ds$ is smooth from $\mathbb{R}$ to $\mathbb{R}$, The nonlinear operator $F(u) = (I - \Delta)^{-1}\big(|u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\, dx\big)$ has the property $F \in C_{Lip}(\dot{H}^1_{per}(\Omega), L^2(\Omega))$ and also $F \in C_{Lip}(\dot{H}^1_{per}(\Omega), \dot{H}^1_{per}(\Omega))$. For more details, we refer to [24](Chapter 5.1).

Now, the local existence theorem of the mild solutions may be given as:

Theorem 2.2

labellocal existence Assume that $p > 1$, $u_0 \in \dot{H}^2_{per}(\Omega)$ then the problem (1.1)–(1.3) has a unique local solution

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} F(u(s)) ds$$

with $0 \leq t < T$ for some positive T,

$$u \in C\big([0,T); \dot{H}^2_{per}(\Omega)\big) \cap C\big((0,T); \dot{H}^4_{per}(\Omega)\big),$$

and

$$u \in C^1\big((0,T); \dot{H}^2_{per}(\Omega)\big).$$

3. The blow-up solutions

We start this section by defining the following functionals:

$$\Phi(t) = \|u\|^2 + \|\nabla u\|^2,$$

$$\Psi(t) = \frac{1}{p+1}\|u\|^{p+1}_{p+1} - \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}\|\Delta u\|^2.$$

Here is the main result:

Theorem 3.1 *Assume that* $p>1$, $u_0 \in \dot{H}^2_{per}(\Omega)$, $u_0 \not\equiv 0$ *and* $\Psi(0)=\frac{1}{p+1}\|u_0\|_{p+1}^{p+1}-\frac{1}{2}\|\nabla u_0\|^2-\frac{1}{2}\|\Delta u_0\|^2>0$. *Then for the solution* $u(x,t)$ *of* (1.1)–(1.3)*, there exists some* $T^*>0$ *such that*

$$\lim_{t\to T^*}\Phi(t)=\infty,$$

where

$$T^*=\frac{\Phi^{\frac{1-p}{2}}(0)}{p^2-1}.$$

In the proof of this and next theorem, we adapted ideas of [19].

Proof Using (1.1)–(1.3), we can easily obtain

$$\begin{aligned}\Phi'(t)&=2\big[-\|\Delta u\|^2-\|\nabla u\|^2+\|u\|_{p+1}^{p+1}\big]=\\&2(p+1)\big[\frac{1}{p+1}\|u\|_{p+1}^{p+1}-\frac{1}{2}\|\nabla u\|^2-\frac{1}{2}\|\Delta u\|^2\big]\\&+(p-1)\big(\|\nabla u\|^2+\|\Delta u\|^2\big)\geq 2(p+1)\Psi(t),\end{aligned}\tag{3.1}$$

$$\textit{and}\quad \Psi'(t)=\|u'\|^2+\|\nabla u'\|^2.$$

By Schwartz inequality, we have

$$\Phi(t)\Psi'(t)=\big(\|u\|^2+\|\nabla u\|^2\big)\big(\|u'\|^2+\|\nabla u'\|^2\big)\geq\frac{1}{4}(\Phi'(t))^2.\tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\Phi(t)\Psi'(t)\geq\frac{1}{4}\Phi'(t)\Phi'(t)\geq\frac{p+1}{2}\Phi'(t)\Psi(t).\tag{3.3}$$

So

$$\Phi(t)\Psi'(t)-\frac{p+1}{2}\Phi'(t)\Psi(t)\geq 0.$$

Now consider

$$\Big(\Psi(t)\Phi(t)^{-\frac{(p+1)}{2}}\Big)'=\Phi(t)^{\frac{-p-3}{2}}\Big(\Psi'(t)\Phi(t)-\frac{p+1}{2}\Phi'(t)\Psi(t)\Big)\geq 0.\tag{3.4}$$

From this, we get

$$\Psi(t)\Phi(t)^{-\frac{(p+1)}{2}}\geq\Psi(0)\big(\Phi(0)\big)^{-\frac{(p+1)}{2}}:=M,\tag{3.5}$$

and

$$\Psi(t)\geq M\Phi(t)^{\frac{(p+1)}{2}}.$$

Now recalling (3.1), we get

$$\frac{\Phi'(t)\Phi(t)^{-\frac{(p+1)}{2}}}{2(p+1)}\geq M,\tag{3.6}$$

equivalently

$$\frac{\left(\Phi(t)^{\frac{1-p}{2}}\right)'}{1-p^2} \geq M,$$

and

$$\frac{1}{1-p^2}\left\{\left(\Phi(t)\right)^{\frac{1-p}{2}} - \left(\Phi(0)\right)^{\frac{1-p}{2}}\right\} \geq Mt.$$

Thus,

$$\Phi(t) \geq \left(\Phi^{\frac{1-p}{2}}(0) - (p^2-1)t\right)^{\frac{2}{1-p}}.$$

Thus, the solution blows up in a finite $T \leq T^* = \frac{\Phi^{\frac{1-p}{2}}(0)}{p^2-1}$. □

4. An estimate for the lower blow-up time

The aim of the following theorem is to determine a finite time interval $(0, T_*)$ on which the quantity $\|\nabla u\|^2 + \|\Delta u\|^2$ remains bounded and this is inspired by [15–17, 19], Indeed T_* is a lower bound for T^* since by the Poincaré inequality, we have

$$\|u\|^2 + \|\nabla u\|^2 \leq \frac{1}{\lambda_1}(\|\nabla u\|^2 + \|\Delta u\|^2), \quad t \in (0, T_*),$$

where λ_1 is the first eigenvalue of the $-\Delta u = \lambda u$ under the periodic periodic boundary conditions.

Theorem 4.1 *Let $u(x,t)$ be solution of the problem (1.1)–(1.3). Then there exists a positive number $T_* = \dfrac{\Theta^{1-p}(0)}{(p-1)C^{2p}(\Omega)}$ such that*

$$\Theta(t) = \int_\Omega (|\nabla u|^2 + |\Delta u|^2)\, dx$$

remains bounded in $(0, T_*)$.

Proof By help of the Green's identity and using (1.1)–(1.3)

$$\begin{aligned}
\Theta'(t) &= 2\int_\Omega \Delta u(-u_t + \Delta u_t)\, dx \\
&= 2\int_\Omega \Delta u\big(\Delta^2 u - \Delta u - |u|^{p-1}u\big)dx - \frac{1}{|\Omega|}\Big(\int_\Omega |u|^{p-1}u\, dx\Big)\int_\Omega \Delta u dx, \\
&= -2\|\nabla\Delta u\|^2 - 2\|\Delta u\|^2 - 2\int_\Omega \Delta u|u|^{p-1}u\, dx. \qquad (4.1)
\end{aligned}$$

We remind that the last integral in the middle line above is zero. By using the inequality $2|\delta u|u|^{p-1}u| \leq |\delta u|^2 + |u|^{2p}$ the last term above gives

$$|2\int_\Omega |\Delta u|u|^{p-1}u\, dx| \leq \|\Delta u\|^2 + \|u\|_{2p}^{2p}.$$

Plugging this into (4.1) $\Theta'(t) \leq \|u\|_{2p}^{2p}$. Now with the help of the Sobolev inequality

$$\|u\|_{2p}^{2p} \leq C^{2p}(\Omega)(\|\nabla u\|^2 + \|\Delta u\|^2)^p. \tag{4.2}$$

Hence

$$\Theta'(t) \leq C^{2p}(\Omega)(\|\Delta u\|^2 + \|\nabla u\|^2)^p = C^{2p}(\Omega)\Theta^p(t).$$

Solving this inequality, we get

$$\Theta^{1-p}(t) \geq \Theta^{1-p}(0) - (p-1)C^{2p}(\Omega)t,$$

Hence the desired $T_* = \dfrac{\Theta^{1-p}(0)}{(p-1)C^{2p}(\Omega)}$.

□

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