Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 699 – 709
© TÜBİTAK
doi:10.3906/mat-2110-41

A refinement of Newton and Maclaurin inequalities through abstract convexity

Gültekin TINAZTEPE[1*], **Ramazan TINAZTEPE**[2]
[1]Vocational School of Technical Sciences, Akdeniz University, Antalya, Turkey
[2]Department of Basic Sciences, Deanship of Preparatory Year and Supporting Studies, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia

Received: 10.10.2021 • **Accepted/Published Online:** 01.01.2022 • **Final Version:** 11.03.2022

Abstract: In this study, the refinements of Maclaurin's and Newton's inequalities are given. These refinements are obtained by applying the results on optimality conditions of abstract convex functions. When doing this, we obtain lower bounds for the solutions of some special rational equations.

Key words: Abstract convexity, Maclaurin inequality, Newton inequality

1. Introduction

Elementary symmetric polynomials are one of the special topics in mathematics. They were first introduced in Newton's studies [5]. Each elementary symmetric polynomial $E_k(x)$ is the sum of the products of all k combinations of the components of $x = (x_1, x_2, \ldots, x_n)$, i.e. for $k \geq 1$

$$E_k(x) = \sum_{\substack{J \subseteq \{1,\cdots,n\} \\ |J|=k}} \prod_{j \in J} x_j$$

where $|J|$ is the cardinality of J. We assume $E_0(x) = 1$ and $E_k(x) = 0$ if $k > n$. They attract special interest in algebra in relation to Viete's theorem

$$P(\lambda) = \prod_{i=1}^{n} (\lambda - x_i) = \sum_{i=0}^{n} (-1)^n \lambda^{n-i} E_i(x).$$

Also, they are used in the representation theory of symmetric groups and general linear groups over finite fields [13].

Newton and Maclaurin gave the first inequalities about the elementary symmetric polynomials. Their inequalities establish the relation between the averages of elementary symmetric polynomials, namely, normalized symmetric functions,

$$d_k(x) = \frac{E_k(x)}{\binom{n}{k}}.$$

Newton showed in [7] that for positive real numbers $x_1, x_2, \ldots, x_n$ and any natural numbers k, n with $0 < k < n$,

$$d_{k-1}(x) d_{k+1}(x) \leq d_k^2(x).$$

*Correspondence: gtinaztepe@akdeniz.edu.tr
2010 *AMS Mathematics Subject Classification:* 26D07, 52A40

Afterwards, Maclaurin gave the following namesake inequality in [6]:

Let $x_1, x_2, ..., x_n$ be positive real numbers and k be positive integer with $k < n$. Then

$$\sqrt[n]{d_n(x)} \leq \cdots \leq \sqrt[k]{d_k(x)} \leq \cdots \leq d_1(x).$$

The equality occurs in both inequalities if and only if all x_i s are equal [3, 21]. The first and last terms in the inequality are equal to the geometric and arithmetic means of $x_1, x_2, \ldots, x_n$, respectively. Studies on different aspects of these inequalities can be found in [2, 17] and the references therein.

A generalization of the Newton inequality has been studied in [16] and its extensions have been given in [8, 10]. Some generalizations of the Maclaurin inequalities are given in [3] and [9]. Although there are a lot of extensions of these inequalities in many different contexts (see [4, 14, 15]), no refinement has been provided for these inequalities.

In this article, we present a sharper version of these inequalities based on the results related to optimality conditions of abstract convex functions. This approach has been used in [1, 18–20]. The framework of the abstract convexity notion used in this work was introduced by Rubinov's work in [11]. Simply put, an abstract convex (or concave) function is defined as the supremum (or infimum) of a certain class of functions. Its definition can be formalized in the following way.

Let $\mathcal{F}$ be the family of real-valued functions defined on a subset A of a Hilbert space X and $f : A \to \mathbb{R}$. If there exists a subfamily $\mathcal{B}$ of $\mathcal{F}$ such that for all $x \in A$,

$$f(x) = \inf_{h \in \mathcal{B}} h(x),$$

then f is said to be an abstract concave function with respect to $\mathcal{F}$. An abstract convex function is defined in the same way except that the supremum is taken.

In [11], it is shown that if an upper semicontinuous finite function f defined on a subset A of a Hilbert space X is bounded from above by a function in the form of $h(x) = \alpha \|x\|^2 + \langle l, x \rangle + \beta$, for $x \in X$ where $\alpha > 0$, $l \in X$, $\beta \in \mathbb{R}$, then f is abstract convex with respect to quadratic functions. In [12], the following proposition provides a sufficient condition for a differentiable function to be abstract convex with respect to quadratic functions.

Proposition 1.1 *[12]Let A be a convex subset of X and f be a differentiable mapping on an open set including A. Consider that $\nabla f(x)$ is Lipschitz continuous on A, i.e.*

$$s := \sup_{x,y \in A, x \neq y} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < \infty.$$

Let $a \geq s$ and $f_t(x) = f(t) + \langle \nabla f(t), x - t \rangle + a\|x - t\|^2$, $x \in X$ for $t \in A$. Then

$$f(x) = \min_{t \in \Omega} f_t(x), x \in A.$$

In addition to the proposition above, it is also stated in [12] that a function $f : X \to \mathbb{R}$ with the property $\|\nabla f(x) - \nabla f(y)\| \leq a \|x - y\|$ for all $x, y \in X$ satisfies the following inequality:

$$\frac{1}{4a} \|\nabla f(x)\|^2 \leq f(x) - f(x^*) \tag{1.1}$$

for all $x \in X$ where x^* is a global minimum point of f over X.

The following theorem localizes the Lipschitz requirement on $\nabla f(x)$ for (1.1) and enables us to determine a accordingly.

Theorem 1.2 *[12]Let $\|.\|$, $\|.\|_\circ$ be Euclidian and any norm on $\mathbb{R}^n$, respectively. Let $\Omega \subset \mathbb{R}^n$ be a set with nonempty interior and let $f \in C^1(\Omega)$. Suppose that f has global minimum at $x^* \in int(\Omega)$ over Ω and the mapping $x \longmapsto \nabla f(x)$ is Lipschitz continuous on Ω with the Lipschitz constant s. Let*

$$m := \max \left\{ \|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r) \right\}$$

where

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leq r\} \subset int(\Omega).$$

Let a, q be positive real numbers such that $B_\circ(x^*, r+q) \subset \Omega$ and $a \geq \max\left(s, \frac{m}{2q}\right)$. Then, for $x \in B_\circ(x^*, r)$,

$$\frac{1}{4a} \|\nabla f(x)\|^2 \leq f(x) - f(x^*). \tag{1.2}$$

Before giving the main result, we define the following notation on vectors: for $x \in \mathbb{R}^n$

$$x^{[k]} = x - x_k \mathbf{e}_k \text{ and } x^{[j,k]} = x - x_j \mathbf{e}_j - x_k \mathbf{e}_k$$

where $\mathbf{e}_j$ and $\mathbf{e}_k$ are j^{th} and k^{th} standard unit vectors, respectively.

The notation above allows us to state the following facts about the partial derivatives of elementary symmetric functions:

$$\frac{\partial E_p(x)}{\partial x_k} = E_{p-1}(x^{[k]}) \text{ and } \frac{\partial E_p(x)}{\partial x_j \partial x_k} = E_{p-2}(x^{[j,k]}) \quad .$$

2. Main results

Theorem 2.1 *Let $\mu > r > 0$ and let p, q be natural numbers such that $\frac{n}{2} + 1 \geq q$ and $q > p \geq 2$. Suppose that $a \in (r, \mu)$ is the solution of the following equation*

$$2\sqrt{4n^2 - 3n} \frac{(\mu + x)^{2q-2}}{(\mu - x)^{2q-1}} = \frac{1}{x - r} \left(\frac{\mu + r}{\mu - r} \right)^{q-1} . \tag{2.1}$$

Then the following inequality holds for all $x \in \mathbb{R}^n_{++}$ satisfying $\|x - (\mu, \ldots, \mu)\|_\infty \leq r$:

$$\frac{E_q^{\frac{1}{q}}(x)}{\binom{n}{q}^{\frac{1}{q}}} + \frac{a - r}{4} \left(\frac{\mu - r}{\mu + r} \right)^{q-1} \sum_{k=1}^{n} \varphi_k^2(x) \leq \frac{E_p^{\frac{1}{p}}(x)}{\binom{n}{p}^{\frac{1}{p}}}$$

where

$$\varphi_k(x) = \frac{E_p^{1/p-1}(x) E_{p-1}\left(x^{[k]}\right)}{p\binom{n}{p}^{\frac{1}{p}}} - \frac{E_q^{1/q-1}(x)\, E_{q-1}\left(x^{[k]}\right)}{q\binom{n}{q}^{\frac{1}{q}}}.$$

Proof Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_{++}$ and let us define the following functions for $x \in \mathbb{R}^n_{++}$:

$$f(x) = \frac{E_p^{\frac{1}{p}}(x)}{\binom{n}{p}^{\frac{1}{p}}} - \frac{E_q^{\frac{1}{q}}(x)}{\binom{n}{q}^{\frac{1}{q}}}.$$

It is clear that for any positive real number μ, $x^* = \mu^* = (\mu, \ldots, \mu) \in \mathbb{R}^n$ is a global minimum point of f. So we can apply Theorem 1.2 on a set containing a neighborhood of x^*. First, we find the Lipschitz constant on the following set

$$\begin{aligned} V_{\mu,d} = B_\infty(\mu,d) &= \{x \in \mathbb{R}^n : \|x-\mu\|_\infty \le d, d \in \mathbb{R}_{++}\} \\ &= \{x \in \mathbb{R}^n : \mu - d \le x_i \le \mu + d, i = 1, ..., n\}. \end{aligned}$$

A direct calculation on f gives

$$\nabla f(x) = \left[\frac{E_p^{1/p-1}(x)\, E_{p-1}\left(x^{[1]}\right)}{p\binom{n}{p}^{\frac{1}{p}}} - \frac{E_q^{1/q-1}(x)\, E_{q-1}\left(x^{[1]}\right)}{q\binom{n}{q}^{\frac{1}{q}}}, \cdots, \frac{E_p^{1/p-1}(x)\, E_{p-1}\left(x^{[n]}\right)}{p\binom{n}{p}^{\frac{1}{p}}} - \frac{E_q^{1/q-1}(x)\, E_{q-1}\left(x^{[n]}\right)}{q\binom{n}{q}^{\frac{1}{q}}}\right],$$

so

$$\|\nabla f(x)\|^2 = \sum_{k=1}^{n} \varphi_k^2(x),$$

where

$$\varphi_k(x) = \frac{E_p^{1/p-1}(x)\, E_{p-1}\left(x^{[k]}\right)}{p\binom{n}{p}^{\frac{1}{p}}} - \frac{E_q^{1/q-1}(x)\, E_{q-1}\left(x^{[k]}\right)}{q\binom{n}{q}^{\frac{1}{q}}}.$$

Since we need later, let us search for an upper bound for $\|\nabla \varphi_k(x)\|$ on $x \in V_{\mu,d}$. Below we find the upper bounds for $\left|\frac{\partial \varphi_k}{\partial x_k}(x)\right|$, and $\left|\frac{\partial \varphi_k}{\partial x_j}(x)\right|$ for $j \ne k$.

$$\left|\frac{\partial \varphi_k}{\partial x_k}(x)\right| \le \left|\frac{\frac{1}{p}-1}{p\binom{n}{p}^{\frac{1}{p}}} E_p^{1/p-2}(x)\, E_{p-1}^2\left(x^{[k]}\right)\right| + \left|\frac{\frac{1}{q}-1}{q\binom{n}{q}^{\frac{1}{q}}} E_q^{1/q-2}(x)\, E_{q-1}^2\left(x^{[k]}\right)\right|.$$

$$\begin{aligned} \left|\frac{\partial \varphi_k}{\partial x_j}(x)\right| \le &\left|\frac{\frac{1}{p}-1}{p\binom{n}{p}^{\frac{1}{p}}} E_p^{1/p-2}(x)\, E_{p-1}\left(x^{[j]}\right) E_{p-1}\left(x^{[k]}\right) + \frac{1}{p\binom{n}{p}^{\frac{1}{p}}} E_p^{1/p-1}(x)\, E_{p-2}\left(x^{[j,k]}\right)\right| \\ &+ \left|\frac{\frac{1}{q}-1}{q\binom{n}{q}^{\frac{1}{q}}} E_q^{1/q-2}(x)\, E_{q-1}\left(x^{[j]}\right) E_{q-1}\left(x^{[k]}\right) + \frac{1}{q\binom{n}{q}^{\frac{1}{q}}} E_q^{1/q-1}(x)\, E_{q-2}\left(x^{[j,k]}\right)\right|. \end{aligned}$$

For $x \in V_{\mu,d}$, the following inequalities are valid:

$$\begin{aligned} \binom{n}{p}(\mu-d)^p \le E_p(x) \le \binom{n}{p}(\mu+d)^p, \\ \binom{n}{p-1}(\mu-d)^{p-1} \le E_{p-1}\left(x^{[j]}\right) \le \binom{n}{p-1}(\mu+d)^{p-1}, \\ \binom{n}{p-2}(\mu-d)^{p-2} \le E_{p-2}\left(x^{[j,k]}\right) \le \binom{n}{p-2}(\mu+d)^{p-2}. \end{aligned} \tag{2.2}$$

Thus, using the inequalities (2.2) and $\frac{n}{2}+1 \geq q > p$, for k and $j \neq k$ we have

$$\left|\frac{\partial \varphi_k}{\partial x_k}(x)\right| \leq \frac{1-\frac{1}{p}}{p\binom{n}{p}^{\frac{1}{p}}}\binom{n}{p}^{\frac{1}{p}-2}(\mu-d)^{1-2p}\binom{n}{p-1}^2(\mu+d)^{2p-2} + \frac{1-\frac{1}{q}}{q\binom{n}{q}^{\frac{1}{q}}}\binom{n}{q}^{\frac{1}{q}-2}(\mu-d)^{1-2q}\binom{n}{q-1}^2(\mu+d)^{2q-2}$$

$$= \sum_{z\in\{p,q\}} \frac{(z-1)}{(n-z+1)^2}(\mu-d)^{1-2z}(\mu+d)^{2z-2}$$

$$\leq \frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2p} + \frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q}$$

$$\leq 2\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q},$$

$$\left|\frac{\partial \varphi_k}{\partial x_j}(x)\right| \leq \frac{\frac{1}{p}-1}{p\binom{n}{p}^{\frac{1}{p}}}\binom{n}{p}^{\frac{1}{p}-2}(\mu-d)^{1-2p}\binom{n}{p-1}^2(\mu+d)^{2p-2} + \frac{1}{p\binom{n}{p}^{\frac{1}{p}}}\binom{n}{p}^{\frac{1}{p}-1}(\mu-d)^{1-p}\binom{n}{p-2}(\mu+d)^{p-2}$$

$$+ \frac{\frac{1}{q}-1}{q\binom{n}{q}^{\frac{1}{q}}}\binom{n}{q}^{\frac{1}{q}-2}(\mu-d)^{1-2q}\binom{n}{q-1}^2(\mu+d)^{2q-2} + \frac{1}{q\binom{n}{q}^{\frac{1}{q}}}\binom{n}{q}^{\frac{1}{q}-1}(\mu-d)^{1-q}\binom{n}{q-2}(\mu+d)^{q-2}$$

$$= \sum_{z\in\{p,q\}}\left(\frac{(z-1)}{(n-z+1)^2}(\mu-d)^{1-2z}(\mu+d)^{2z-2} + \frac{z(z-1)}{(n-z+2)(n-z+1)}(\mu-d)^{1-z}(\mu+d)^{z-2}\right)$$

$$\leq 2\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q} + 2\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{q}$$

$$\leq 4\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q},$$

so

$$\|\nabla\varphi_i(x)\| \leq 2\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q}\sqrt{4n-3}. \tag{2.3}$$

Let $x, z \in V_{\mu,d}$. The mean value theorem implies that there exist numbers $\theta_i \in (0,1)$, $i = 1, ..., n$ such that

$$\varphi_k(x) - \varphi_k(z) = \nabla\varphi_k(x + \theta_k(z-x))(x-z).$$

Also applying the Cauchy-Schwarz inequality, we obtain

$$\|\nabla f(x) - \nabla f(z)\| = \left(\sum_{k=1}^{n}[\varphi_k(x)-\varphi_k(z)]^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=1}^{n}[\nabla\varphi_k(x+\theta_k(z-x))(x-z)]^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^{n}\|\nabla\varphi_k(x+\theta_k(z-x))\|^2\right)^{\frac{1}{2}}\|x-z\|$$

$$\leq 2\sqrt{4n^2-3n}\,\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q}\|z-x\|.$$

Using (2.3), we get

$$\|\nabla f(x)-\nabla f(z)\| \leq a_1(\mu,d)\,\|x-z\|\,, \quad x,z\in V_{\mu,d}$$

where

$$a_1(\mu,d)=2\sqrt{4n^2-3n}\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q}.$$

This shows the existence of a Lipschitz constant K for ∇f on $V_{\mu,d}$ such that $K\leq a_1(\mu,d)$. In order to apply Theorem 1.2, we assume $\Omega=V_{\mu,d}$ where $d<\mu$, the point $x^*=\mu$ as the global minimum point of the function f and $\|.\|_\circ=\|.\|_\infty$. Taking $r\in(0,d)$ with $q=d-r$, we estimate $M=\max\{\|\nabla f(x)\|_\infty : x\in V_{\mu,r}\}$ as follows:

$$\begin{aligned}
M &= \max_{x\in V_{\mu,r}}\{\|\nabla f(x)\|_\infty\}=\max_{1\leq k\leq n}\left\{\max_{x\in V_{\mu,r}}\left|\frac{E_p^{1/p-1}(x)\,E_{p-1}\left(x^{[k]}\right)}{p\binom{n}{p}^{\frac{1}{p}}}-\frac{E_q^{1/q-1}(x)\,E_{q-1}\left(x^{[k]}\right)}{q\binom{n}{q}^{\frac{1}{q}}}\right|\right\}\\
&\leq \max_{1\leq k\leq n}\left\{\frac{\binom{n}{p-1}}{p\binom{n}{p}}\left(\frac{\mu+r}{\mu-r}\right)^{p-1}+\frac{\binom{n}{q-1}}{q\binom{n}{q}}\left(\frac{\mu+r}{\mu-r}\right)^{q-1}\right\}\\
&= \frac{1}{(n-p+1)}\left(\frac{\mu+r}{\mu-r}\right)^{p-1}+\frac{1}{(n-q+1)}\left(\frac{\mu+r}{\mu-r}\right)^{q-1}\\
&\leq 2\left(\frac{\mu+r}{\mu-r}\right)^{q-1}:=M_0.
\end{aligned}$$

Let

$$a_2(\mu,d,r)=\frac{M_0}{2(d-r)}$$

and

$$a(\mu,d,r)=\max\left\{a_1(\mu,d),a_2(\mu,d,r)\right\}.$$

Since $a(\mu,d,r)$ is continuous with respect to d on (r,μ) and $\lim\limits_{d\to\mu^-}a(\mu,d,r)=\lim\limits_{d\to r^+}a(\mu,d,r)=+\infty$, the function $d\longmapsto a(\mu,d,r)$ has minimum value on this interval. For convenience, we define $a_{\mu,r}$ as the following:

$$a_{\mu,r}=\min_{r<d<\mu}a(\mu,d,r).$$

Since $a_1(\mu,d)$ and $a_2(\mu,d,r)$ are increasing and decreasing functions on (r,μ), respectively, there exists only one value of d minimizing $a(\mu,d,r)$ on (r,μ). This real number, call a, is the solution of the following equation

$$2\sqrt{4n^2-3n}\frac{\mu-d}{(\mu+d)^2}\left(\frac{\mu+d}{\mu-d}\right)^{2q}=\frac{1}{d-r}\left(\frac{\mu+r}{\mu-r}\right)^{q-1}.$$

So we have

$$\min_{r<d<\mu}a(\mu,d,r)=\frac{1}{a-r}\left(\frac{\mu+r}{\mu-r}\right)^{q-1}.$$

Applying (1.2), we conclude that

$$\frac{E_q^{1/q}(x)}{\binom{n}{q}^{\frac{1}{q}}}+\frac{1}{4a_{\mu,r}}\sum_{k=1}^{n}\varphi_k^2(x)\leq\frac{E_p^{1/p}(x)}{\binom{n}{p}^{\frac{1}{p}}}\text{ for }x\in V_{\mu,r}.$$

□

Note that Theorem 2.1 has two major drawbacks. First, Theorem 2.1 entails the solution of the equation (2.1) or at least a lower bound for the solution to apply the inequality. Secondly, the inequality is applicable to a limited set of x values that depend on μ and r. These drawbacks are handled below.

We deal with the first drawback with the following lemma, which provides a lower bound for the solution of the equation (2.1).

Lemma 2.2 *Let $\mu > r > 0$ and let p, q be natural numbers such that $\frac{n}{2} + 1 \geq q$ and $q > p \geq 2$. If $A \in (r, \mu)$ is the solution of the equation*

$$\frac{(\mu + x)^{2q-2}(x - r)}{(\mu - x)^{2q-1}} = \frac{1}{2\sqrt{4n^2 - 3n}} \left(\frac{\mu + r}{\mu - r}\right)^{q-1},$$

then $A > r + B$ where

$$B = \begin{cases} \left(\frac{1}{2\sqrt{4n^2-3n}} \left(\frac{\mu+r}{\mu-r}\right)^{q-1} \frac{2\mu}{\mu-r} + \frac{\mu+r}{\mu-r}\right)^{-(2q-2)} \frac{(\mu-r)^2}{2(\mu+r)}, & \text{if } \frac{1}{2\sqrt{4n^2-3n}} \left(\frac{\mu+r}{\mu-r}\right)^{q-1} > 1 \\ \frac{1}{2\sqrt{4n^2-3n}} \left(\frac{\mu+r}{\mu-r}\right)^{q-1} \left(\frac{3\mu+r}{\mu-r}\right)^{-(2q-2)} \frac{(\mu-r)^2}{2(\mu+r)}, & \text{if } \frac{1}{2\sqrt{4n^2-3n}} \left(\frac{\mu+r}{\mu-r}\right)^{q-1} < 1 \end{cases}.$$

Proof

Letting $A = r + \epsilon$ and substituting $K = \dfrac{1}{2\sqrt{4n^2 - 3n}} \left(\dfrac{\mu + r}{\mu - r}\right)^{q-1}$ transforms the equation into

$$\frac{(\mu + r + \epsilon)^{2q-2}}{(\mu - r - \epsilon)^{2q-2}} \frac{\epsilon}{(\mu - r - \epsilon)} = K.$$

Letting $\dfrac{\mu + r + \epsilon}{\mu - r - \epsilon} = L$ transforms the equation into

$$L^{2q-2} \left(\frac{\mu - r}{2\mu} L - \frac{\mu + r}{2\mu}\right) = K.$$

After necessary algeraic manipulations, we get

$$L - \frac{\mu + r}{\mu - r} = \frac{2\mu}{\mu - r} K L^{-(2q-2)}. \tag{2.4}$$

Using the fact that $L > 1$ on the right hand side yields

$$L < \frac{2\mu}{\mu - r} K + \frac{\mu + r}{\mu - r}.$$

Hence this implies $L^{-(2q-2)} > \left(\dfrac{2\mu}{\mu - r} K + \dfrac{\mu + r}{\mu - r}\right)^{-(2q-2)}$ and

$$\frac{2\mu}{\mu - r} K L^{-(2q-2)} > \frac{2\mu}{\mu - r} K \left(\frac{2\mu}{\mu - r} K + \frac{\mu + r}{\mu - r}\right)^{-(2q-2)}.$$

Using (2.4) on the left hand side of the preceding inequality gives

$$L - \frac{\mu + r}{\mu - r} > \frac{2\mu}{\mu - r} K \left(\frac{2\mu}{\mu - r} K + \frac{\mu + r}{\mu - r} \right)^{-(2q-2)}.$$

After letting $U = \left(\frac{2\mu}{\mu - r} K + \frac{\mu + r}{\mu - r} \right)^{-(2q-2)}$, the above inequality can be written as

$$L > \frac{2\mu}{\mu - r} KU + \frac{\mu + r}{\mu - r}.$$

Now we use the following fact to get back to ϵ. Its proof directly follows from the definition of L.
Claim: If $L > M$ for some $M > 0$, then

$$\epsilon > \left(\frac{M(\mu - r) - (\mu + r)}{M + 1} \right) \frac{\mu - r}{\mu + r}. \tag{2.5}$$

Letting $M = \frac{2\mu}{\mu - r} KU + \frac{\mu + r}{\mu - r}$ in the inequality (2.5) and using the fact that $U < 1$ yields

$$\epsilon > \left(\frac{2\mu KU}{\frac{2\mu}{\mu - r} KU + \frac{\mu + r}{\mu - r} + 1} \right) \frac{\mu - r}{\mu + r} = \left(\frac{KU}{KU + 1} \right) \frac{(\mu - r)^2}{\mu + r} > \left(\frac{KU}{K + 1} \right) \frac{(\mu - r)^2}{\mu + r} > \left(\frac{KU}{2 \max\{K, 1\}} \right) \frac{(\mu - r)^2}{\mu + r}.$$

Thus, there are two cases:
If $K \geq 1$ then $\epsilon > \frac{U}{2} \frac{(\mu - r)^2}{\mu + r}$.

If $K < 1$ then $U > \left(\frac{3\mu + r}{\mu - r} \right)^{-(2q-2)}$ which implies $\epsilon > \frac{K \left(\frac{3\mu + r}{\mu - r} \right)^{-(2q-2)}}{2} \frac{(\mu - r)^2}{\mu + r}$. Consequently, B follows from these cases. □

Remark 2.3 *Note that it is possible to extend the inequality in Theorem 2.1 to all* $x \in \mathbb{R}_{++}$ *by taking* $\mu = \max\{x\}$ *and* $r = \max\{x\} - \min\{x\}$*.*

This modification along with Lemma 2.2 gives a general inequality below, which is a refinement of MacLaurin's inequality.

Corollary 2.4 *Let* p, q *be natural numbers such that* $\frac{n}{2} + 1 \geq q$ *and* $q > p \geq 2$*. For any* $x \in \mathbb{R}^n_{++}$*, let* $t = \frac{\max\{x\}}{\min\{x\}}$ *and*

$$\theta(t) = \begin{cases} \left(\frac{t((2t-1)^{q-1} + 2t - 1)}{\sqrt{4n^2 - 3n}} \right)^{-(2q-2)}, & \text{if } t > \frac{(2\sqrt{4n^2-3n})^{\frac{1}{q-1}} - 1}{2} \\ \frac{1}{2\sqrt{4n^2 - 3n}} \left(\frac{2t-1}{(4t+1)^2} \right)^{q-1}, & \text{if } t < \frac{(2\sqrt{4n^2-3n})^{\frac{1}{q-1}} - 1}{2}. \end{cases}$$

Then the following inequality holds for all $x \in \mathbb{R}^n_{++}$*:*

$$\frac{E_q^{\frac{1}{q}}(x)}{\binom{n}{q}^{\frac{1}{q}}} + \frac{\theta(t) \min\{x\}}{4} \left(\frac{1}{2t - 1} \right)^q \sum_{k=1}^n \varphi_k^2(x) \leq \frac{E_p^{\frac{1}{p}}(x)}{\binom{n}{p}^{\frac{1}{p}}}$$

where

$$\varphi_k(x) = \frac{E_p^{1/p-1}(x)E_{p-1}\left(x^{[k]}\right)}{p\binom{n}{p}^{\frac{1}{p}}} - \frac{E_q^{1/q-1}(x)\,E_{q-1}\left(x^{[k]}\right)}{q\binom{n}{q}^{\frac{1}{q}}}.$$

By following the same path in the proof of Theorem 2.1, the following theorem can be obtained, which leads to a refinement of Newton's inequality.

Theorem 2.5 *Let $\mu > r > 0$ and p, q be natural numbers such that $n > p+1$. Suppose that $a \in (r, \mu)$ is the solution of the equation*

$$4\sqrt{4n^2-3n}(x+\mu)^{2p-1} = \frac{3p}{n}\frac{(\mu+r)^{2p-1}}{(x-r)}. \tag{2.6}$$

Then the following inequality holds for all $x \in \mathbb{R}^n_{++}$ such that $\|x-(\mu,\ldots,\mu)\|_\infty \leq r$:

$$\frac{E_{p-1}(x)}{\binom{n}{p-1}} \cdot \frac{E_{p+1}(x)}{\binom{n}{p+1}} + \frac{n}{12p}\frac{a-r}{(\mu+r)^{2p-1}}\sum_{k=1}^{n}\varphi_k^2(x) \leq \left[\frac{E_p(x)}{\binom{n}{p}}\right]^2 \quad \text{for } x \in V_{\mu,r}$$

where

$$\varphi_k(x) = \frac{2E_p(x)\,E_{p-1}\left(x^{[k]}\right)}{\binom{n}{p}^2} - \frac{E_{p-2}\left(x^{[k]}\right)E_{p+1}(x) + E_{p-1}(x)\,E_p\left(x^{[k]}\right)}{\binom{n}{p-1}\binom{n}{p+1}}.$$

It is clear that Theorem 2.5 has the same drawbacks as Theorem 2.1. We overcome these drawbacks in the same way as in Theorem 2.1. The following lemma provides a lower bound for the solution of the equation (2.6).

Lemma 2.6 *Let $\mu > r > 0$ and p, q be natural numbers such that $n > p+1$. Suppose that $A \in (r, \mu)$ is the solution of the equation*

$$4\sqrt{4n^2-3n}(x+\mu)^{2p-1} = \frac{3p}{n}\frac{(\mu+r)^{2p-1}}{(x-r)}.$$

Then A satisfies the following inequality:

$$A > r + \frac{3p}{4n\sqrt{4n^2-3n}}\left(\frac{\mu+r}{\mu+r+\frac{3p}{4n\sqrt{4n^2-3n}}}\right)^{2p-1}.$$

Proof Let $A = r+\epsilon$ for some $\epsilon > 0$. After substituting it in the equation, we rewrite the equation in the following way:

$$\epsilon = \frac{3p}{4n\sqrt{4n^2-3n}}\left(\frac{\mu+r}{\mu+r+\epsilon}\right)^{2p-1}.$$

Thus, it is clear that $\epsilon < \dfrac{3p}{4n\sqrt{4n^2-3n}}$ and this implies that

$$\left(\frac{1}{\mu+r+\frac{3p}{4n\sqrt{4n^2-3n}}}\right)^{2p-1} < \left(\frac{1}{\mu+r+\epsilon}\right)^{2p-1}.$$

After multiplying appropriate terms, we obtain

$$\frac{3p}{4n\sqrt{4n^2-3n}}\left(\frac{\mu+r}{\mu+r+\frac{3p}{4n\sqrt{4n^2-3n}}}\right)^{2p-1}<\frac{3p}{4n\sqrt{4n^2-3n}}\left(\frac{\mu+r}{\mu+r+\epsilon}\right)^{2p-1}.$$

The result follows from the fact that the expression on the right-hand side is equal to ϵ.

□

Remark 2.7 *Taking $\mu=\max\{x\}$ and $r=\max\{x\}-\min\{x\}$ for any $x\in\mathbb{R}_{++}$ in Theorem 2.5, one can extend the inequality to all $x\in\mathbb{R}_{++}$.*

Applying this remark to Lemma 2.6 yields the following refinement of Newton's inequality.

Corollary 2.8 *Let p,q be natural numbers such that $n>p+1$. Then the following inequality holds for all $x\in\mathbb{R}^n_{++}$:*

$$\frac{E_{p-1}(x)}{\binom{n}{p-1}}\cdot\frac{E_{p+1}(x)}{\binom{n}{p+1}}+\frac{1}{16n\sqrt{4n^2-3n}}\left(\frac{1}{2\max\{x\}+\min\{x\}+\frac{3p}{4n\sqrt{4n^2-3n}}}\right)^{2p-1}\sum_{k=1}^{n}\varphi_k^2(x)\le\left[\frac{E_p(x)}{\binom{n}{p}}\right]^2$$

where

$$\varphi_k(x)=\frac{2E_p(x)E_{p-1}\left(x^{[k]}\right)}{\binom{n}{p}^2}-\frac{E_{p-2}\left(x^{[k]}\right)E_{p+1}(x)+E_{p-1}(x)E_p\left(x^{[k]}\right)}{\binom{n}{p-1}\binom{n}{p+1}}$$

References

[1] Adilov G, Tinaztepe G. The Sharpening of Some Inequalities via Abstract Convexity. Mathematical Inequalities and Applications 2009; 12: 1-33.

[2] Ben-Ari I, Conrad K. Maclaurin's Inequality and a Generalized Bernoulli Inequality. Mathematics Magazine 2017; 87: 14-24. doi:10.4169/math.mag.87.1.143

[3] Favaro S, Walker S. On a general Maclaurin's inequality. Proceedings of the American Mathematical Society 2018; 146: 175-88. doi:10.1090/proc/136734

[4] Holtz O. M-Matrices satisfy Newton's inequalities. Proceedings of the American Mathematical Society 2005; 133: 711-777. doi:10.1090/s0002-9939-04-07576-85

[5] MacDonald IG. Symmetric Functions and Orthogonal Polynomials. Rhode Island: American Mathematical Society 1998.

[6] Maclaurin C. IV. A second letter from Mr. Colin McLaurin, Professor of Mathematics in the University of Edinburgh and F. R. S. to Martin Folkes, Esq; concerning the roots of equations, with the demonstration of other rules in algebra; being the continuation of the le. Philosophical Transactions of the Royal Society of London 1730; 36: 59-96. doi:10.1098/rstl.1729.00117

[7] Newton I. Arithmetica Universalis: Sive De Compositione Et Resolutione Arithmetica Liber. 1707 (in French).

[8] Niculescu CP. A new look at Newton's inequalities. Journal of Inequalities in Pure and Applied Mathematics 2000; 1: 1-14.

[9] Pecaric J, Wen J, Wang W, Lu T. A generalization of Maclaurin's inequalities and its applications. Mathematical Inequalities and Applications 2005; 8: 583-598.

[10] Rosset S. Normalized Symmetric Functions, Newton's Inequalities, and a New Set of Stronger Inequalities. The American Mathematical Monthly 2018; 96: 815-819. doi:10.1080/00029890.1989.1197228611

[11] Rubinov AM. Abstract Convexity and Global Optimization. New York: Springer; 2000.

[12] Rubinov AM, Wu ZY. Optimality conditions in global optimization and their applications. Mathematical Programming 2009; 120 (1): 101-123. doi:10.1007/s10107-007-0142-413

[13] Sagan B. The symmetric group: representations, combinatorial algorithms, and symmetric functions. New York: Springer; 2001.

[14] Sharma R, Bhandari R. Skewness, Kurtosis and Newton's Inequality. The Rocky Mountain Journal of Mathematics 2015; 45: 1639-1643.

[15] Sharma R, Sharma A, Saini R. Means, moments and Newton's inequalities. The Rocky Mountain Journal of Mathematics 2019; 49: 1667-1677.

[16] Simic S. A Note On Newton's Inequality. Journal of Inequalities in Pure and Applied Mathematics 2009; 10: 1-4.

[17] Steele JM. The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities. New York: Cambridge University Press, 2004.

[18] Tinaztepe G. The sharpening Hölder inequality via abstract convexity. Turkish Journal of Mathematics 2016; 40: 438-441.

[19] Tınaztepe G, Kemali S, Sezer S, Eken Z. The sharper form of a Brunn-Minkowski type inequality for boxes. Hacettepe Journal of Mathematics and Statistics 2021; 50: 377-386. doi:10.15672/hujms.65783919

[20] Tinaztepe R, Tinaztepe G. A sharpened version of Aczél inequality by abstract convexity. Mathematical Inequalities and Applications 2021; 24 (3): 635-643.

[21] Whiteley JN. A Generalization of a Theorem of Newton. Proceedings of the American Mathematical Society 1962; 13: 15120.