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Multiplicative conformable fractional Dirac system

Sertaç GÖKTAŞ[1], **Hikmet KEMALOĞLU**[2], **Emrah YILMAZ**[2,*]
[1]Department of Mathematics, Faculty of Science and Letters, Mersin University, Mersin, Turkey
[2]Department of Mathematics, Faculty of Science, Firat University, Elazığ, Turkey

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Abstract: In multiplicative fractional calculus, the well-known Dirac system in fractional calculus is redefined. The aim of this study is to analyze some spectral properties such as self-adjointness of the operator, structure of all eigenvalues, orthogonality of distinct eigenfunctions, etc. for this system. Moreover, Green's function in multiplicative case is reconstructed for this system.

Key words: Conformable Dirac system, Multiplicative conformable fractional derivative, Multiplicative conformable fractional integral, Multiplicative Dirac system, Non-Newtonian calculus

1. Introduction

This study is organized as follows:

In the present section, we'll introduce a multiplicative conformable fractional (CF) Dirac ($\alpha-{}^*$Dirac) system. Then, we research some studies closely related to the present system,

In Section 2, we'll give some basic definitions and properties of the multiplicative calculus, the CF calculus theory, and multiplicative CF calculus,

In Section 3, we'll calculate asymptotic estimates of the *eigenfunctions of $\alpha-{}^*$Dirac system,

In Section 4, we'll investigate some spectral properties as self-adjointness of operator, the structure of all eigenvalues, orthogonality of distinct eigenfunctions for $\alpha-{}^*$Dirac system,

In Section 5, we'll reconstruct the Green's function of this system.

Consider the following $\alpha-{}^*$Dirac system

$$(B \odot \tau y(t)) \oplus (Q(t) \odot y(t)) = e^{\lambda} \odot y(t), \qquad t \in [a,b], \tag{1.1}$$

with the conditions

$$U_1(y) := (e^{c_1} \odot y(a)) \oplus (e^{c_2} \odot \tau y(a)) = 1, \tag{1.2}$$

$$U_2(y) := \left(e^{d_1} \odot y(b)\right) \oplus \left(e^{d_2} \odot \tau y(b)\right) = 1, \tag{1.3}$$

where λ is a spectral parameter; $q(t)$, $r(t)$ are real-valued continuous and multiplicative CF integrable functions

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on $[a,b]$; $\left(c_1^2+c_2^2\right)\left(d_1^2+d_2^2\right)\neq 0$, $c_i,\, d_i\in\mathbb{R}$ $(i=1,2)$ and

$$B=\begin{bmatrix} 1 & e \\ e^{-1} & 1 \end{bmatrix},\quad Q(t)=\begin{bmatrix} e^{q(t)} & 1 \\ 1 & e^{r(t)} \end{bmatrix}.$$

Throughout this study, $\tau\cdot$ denotes ${}^{*}T_\alpha\cdot=\frac{d_\alpha^*\cdot}{d_\alpha^* t}$ (the multiplicative CF derivative of order $\alpha\in(0,1]$ on t) for brevity.

For the above system,

$$y(t,\lambda)=\left[\begin{array}{cc} y_1(t,\lambda) & y_2(t,\lambda) \end{array}\right]^{\mathrm{T}}$$

is the *eigenfunction of the system (1.1)-(1.3) corresponding to the *eigenvalue λ, where T denotes transpose of the matrix and y_j, $j=1,2$ are continuous and positive functions. If we simplify this system by using the properties of multiplicative CF calculus [2], following multiplicative system is obtained:

$$\boldsymbol{D}_\alpha[y]=e^\lambda\odot y=\left[\begin{array}{cc} y_1^\lambda & y_2^{-\lambda} \end{array}\right]^{\mathrm{T}}, \tag{1.4}$$

with the conditions

$$\begin{array}{l} U_1(y)=y_1^{c_1}(a)\,y_2^{c_2}(a)=1, \\ U_2(y)=y_1^{d_1}(b)\,y_2^{d_2}(b)=1, \end{array} \tag{1.5}$$

where

$$\boldsymbol{D}_\alpha[y]=\left\{\begin{array}{c} \tau y_2\, y_1^{q(t)}, \\ \tau y_1\, y_2^{-r(t)}. \end{array}\right.$$

Let's start with the necessity of developing and applying multiplicative (geometric) fractional calculus by reminding the importance of the polar coordinates while cartesian coordinates already exist. Furthermore, the theory of multiplicative fractional calculus is a combination of both fractional and multiplicative theories. Therefore, it is necessary to examine these two theories separately.

Firstly, let's consider the fractional calculus theory. Fractional calculus encountered in different fields of engineering and science extensively with a variety of applications[13, 34, 37, 39, 42] defines a generalization of classical calculus. Almost all fractional derivatives such as Grünwald–Letnikov, Riemann–Liouville, Caputo and Jumarie, Marchaud and Riesz used in the literature fail to satisfy some fundamental properties. Thus, we prefer CF derivative in the present study. It can be found fundamental properties and main results in [1, 33] and other results in[7, 15, 16, 36–38, 40, 45] on conformable fractional derivative.

Secondly, let's consider the multiplicative calculus theory. Multiplicative calculus was firstly presented as an alternative to usual calculus by Grossman and Katz [25, 26]. This is also entitled simultaneously Geometric calculus which is one of sub-branches of non-Newtonian calculus. Afterwards, the basic features of multiplicative calculus were explained by many authors, and important results were obtained [10, 11, 17, 43]. This calculus changes the roles of known operations such as multiplication instead of addition, division instead of subtraction due to the properties of the logarithm. It develops additive calculations circuitously. Although it has a relatively restrictive (includes only positive functions) applications compared to calculus of Newton and Leibnitz. Some difficult problems in usual calculus may be arranged incredibly easily here. In the usual calculus, each feature may be redefined in multiplicative calculus by specific rules.

Many phenomena where the logarithmic scale appears are given by means of multiplicative derivatives. Therefore, considering multiplicative calculus in place of usual calculus lets better physical appraisal for these events. This calculus yields better outcomes than the usual case in numerous fields as applied mathematics[24, 46–50] biology[9]; chaos theory[5, 6]; demography, earthquakes[14]; engineering[12]; economics[18, 21]; business[41] and medicine[22].

Finally, we refer to study [2] that encourages us and from which the main concepts of the multiplicative fractional calculus are set. Here, it has been defined Caputo, Riemann, Letnikov, and conformable multiplicative fractional derivatives and multiplicative fractional integrals and have been studied some of their properties.

In recent years, since it appears in solving many problems of natural, engineering, physics, and the social sciences in a natural manner, Dirac system theory as an attractive field of research has been received considerable attention [3, 8, 19, 20, 23, 30, 32, 35, 44]. When we take into account the form of the system (1.1)-(1.3), $\alpha-{}^*$Dirac system is obtained by replacing the multiplicative CF derivative with the fractional derivative. This method has been similarly applied by replacing the fractional derivative with the derivatives such as usual, delta, etc. by many authors [4, 27, 38].

2. Preliminaries

Here, we give some basic definitions and features of CF calculus, multiplicative and multiplicative CF theories that we'll use for the rest of present study.

First of all, let's focus on some of arithmetic operations we used throughout the study. The arithmetic operations occurred by exp-functions are known as multiplicative algebraic operations. Let's denote some features of these operations with below arithmetic table, where $p, r \in \mathbb{R}^+$:

$$p \oplus r = p\,r, \quad p \ominus r = \frac{p}{r}, \quad p \odot r = p^{\ln r} = r^{\ln p}, \quad p^{2_G} = p \odot p = p^{\ln p}.$$

Above operations construct several algebraic structures. If $\oplus : E \times E \to E$ is an operation for $E \neq \emptyset$ and $E \subset \mathbb{R}^+$, $(E, \oplus)$ is a *group. Analogously, $(E, \oplus, \odot)$ defines a ring in multiplicative sense [31].

Definition 2.1 *[1, 33] Let $f : [a, \infty) \to \mathbb{R}$. Then, left and right CF derivatives of f order $\alpha \in (0, 1]$ are defined by:*

$$T_\alpha^a f(t) := \lim_{h \to 0} \frac{f(t + h(t-a)^{1-\alpha}) - f(t)}{h},$$

$${}_\alpha^b T f(t) := -\lim_{h \to 0} \frac{f(t + h(b-t)^{1-\alpha}) - f(t)}{h},$$

respectively.

When $a = 0$, the left CF derivative can be written as T_α. If f is usual differentiable, then $T_\alpha f(t) = t^{1-\alpha} f'(t)$.

Definition 2.2 *[1, 33] Let $f : [a, \infty) \to \mathbb{R}$. Then, left and right CF integrals of f order $\alpha \in (0, 1]$ are defined by:*

$$I_\alpha^a f(t) := \int_a^t f(x) d_\alpha(x, a) = \int_a^t (x-a)^{\alpha-1} f(x) dx,$$

$$^{b}I_{\alpha}f(t) := \int_{t}^{b} f(x)d_{\alpha}(b,x) = \int_{t}^{b}(b-x)^{\alpha-1}f(x)dx,$$

respectively for $t>0$. *Where, the last integrals to the right of these equalities are the usual Riemann integrals.*

When $a=0$, the left CF integral can be written as I_{α} and $d_{\alpha}(x,a)=d_{\alpha}x$.

Definition 2.3 *[2] Let* $f:\mathbb{R}\to\mathbb{R}^{+}$. *Then, forward and backward multiplicative derivatives of* $f(t)$ *are defined by:*

$$\frac{d^{*}}{dt^{*}}f(t) = f^{*}(t) := \lim_{h\to 0}\left(\frac{f(t+h)}{f(t)}\right)^{\frac{1}{h}},$$

$$\frac{d_{*}}{dt_{*}}f(t) = f_{*}(t) := \lim_{h\to 0}\left(\frac{f(t)}{f(t-h)}\right)^{\frac{1}{h}},$$

respectively.

It can be easily shown that

$$f^{*(n)}(t) = f_{*}{}^{(n)}(t) = \exp\left(\frac{d^{n}}{dx^{n}}\ln f(t)\right).$$

Definition 2.4 *[2] Let* $f:[a,b]\to\mathbb{R}^{+}$. *Then, forward and backward multiplicative integrals of* $f(t)$ *are defined by:*

$$\int_{a}^{b} f(t)^{dt} = \int_{a}^{b} f(t)_{dt} = \exp\left(\int_{a}^{b}\ln f(t)dt\right).$$

Definition 2.5 *[2] Let* $f:[a,b]\to\mathbb{R}^{+}$. *Then, left and right side multiplicative CF derivatives of* f *order* $\alpha\in(0,1]$ *are defined by:*

$$^{*}T_{\alpha}^{a}f(t) := \lim_{h\to 0}\left(\frac{f(t+h(t-a)^{1-\alpha})}{f(t)}\right)^{\frac{1}{h}},$$

$${}_{\alpha}^{b}T^{*}f(t) := \lim_{h\to 0}\left(\frac{f(t+h(b-t)^{1-\alpha})}{f(t)}\right)^{-\frac{1}{h}},$$

respectively.

Proposition 2.6 *[2] Let* $f:[a,b]\to\mathbb{R}^{+}$ *and* $\alpha\in(0,1]$. *Then,*

$$\textbf{\textit{(i)}}\quad {}^{*}T_{\alpha}^{a}f(t) = \exp\left\{T_{\alpha}^{a}\ln f(t)\right\} = \exp\left\{\frac{T_{\alpha}^{a}f(t)}{f(t)}\right\}, \tag{2.1}$$

$$\textbf{\textit{(ii)}}\quad {}_{\alpha}^{b}T^{*}f(t) = \exp\left\{{}_{\alpha}^{b}T\ln f(t)\right\} = \exp\left\{\frac{{}_{\alpha}^{b}Tf(t)}{f(t)}\right\}.$$

Definition 2.7 *[2] Let $f:[a,b]\to\mathbb{R}^+$. Then, left and right side multiplicative CF integrals of f order $\alpha\in(0,1]$ are defined by:*

$$\begin{aligned}({}^{*}I_{\alpha}^{a}f)(t):&=\int_a^t f(x)_{d_\alpha^*(x,a)}=\exp\left\{\int_a^t \ln f(x)d_\alpha(x,a)\right\}\\&=\int_a^t f(x)_{dx}^{(x-a)^{\alpha-1}}=\exp\left\{\int_a^t (x-a)^{\alpha-1}\ln f(x)dx\right\},\end{aligned}\tag{2.2}$$

$$\begin{aligned}({}_{\alpha}^{b}I^{*}f)(t):&=\int_t^b f(x)_{d_\alpha^*(b,x)}=\exp\left\{\int_t^b \ln f(x)d_\alpha(b,x)\right\}\\&=\int_t^b f(x)_{dx}^{(b-x)^{\alpha-1}}=\exp\left\{\int_t^b (b-x)^{\alpha-1}\ln f(x)dx\right\},\end{aligned}$$

respectively for $t>0$, where last integrals to the right of these equalities are usual Riemann integrals.

When $a=0$, the left multiplicative CF integral can be written as ${}^{*}I_\alpha$, and $d_\alpha^*(x,a)=d_\alpha^*x$.

Proposition 2.8 *[2] Let $f:[a,b]\to\mathbb{R}^+$ and $\alpha\in(0,1]$. Then,*

$$\begin{aligned}&\textbf{\textit{(i)}}\quad ({}^{*}T_{\alpha}^{a}\,{}^{*}I_{\alpha}^{a}f)(t)=f(t),\quad \textit{if } f \textit{ is continuous},\\&\textbf{\textit{(ii)}}\quad ({}_{\alpha}^{b}T^{*}\,{}_{\alpha}^{b}I^{*}f)(t)=f(t),\quad \textit{if } f \textit{ is continuous},\\&\textbf{\textit{(iii)}}\quad ({}^{*}I_{\alpha}^{a}\,{}^{*}T_{\alpha}^{a}f)(t)=\frac{f(t)}{f(a)},\\&\textbf{\textit{(iv)}}\quad ({}_{\alpha}^{b}I^{*}\,{}_{\alpha}^{b}T^{*}f)(t)=\frac{f(t)}{f(b)}.\end{aligned}\tag{2.3}$$

Theorem 2.9 *Let $f,g:[0,b]\to\mathbb{R}^+$ be left multiplicative CF differentiable of order $\alpha\in(0,1]$ and h be CF differentiable order $\alpha\in(0,1]$ at t. Then,*

$$\begin{aligned}&\textbf{\textit{(i)}}\quad \tau\left(cf\right)(t)=\tau f(t),\\&\textbf{\textit{(ii)}}\quad \tau\left(fg\right)(t)=\tau f(t)\,\tau g(t),\\&\textbf{\textit{(iii)}}\quad \tau\left(\frac{f}{g}\right)(t)=\frac{\tau f(t)}{\tau g(t)},\\&\textbf{\textit{(iv)}}\quad \tau\left(f^h\right)(t)=\left\{\tau f(t)\right\}^{h(t)}f(t)^{T_\alpha h(t)},\\&\textbf{\textit{(v)}}\quad \tau\left(f\circ h\right)(t)=\left\{(\tau f)\left(h(t)\right)\right\}^{T_\alpha h(t)h(t)^{\alpha-1}},\\&\textbf{\textit{(vi)}}\quad \tau\left(f+g\right)(t)=\left[\tau f(t)\right]^{\frac{f(t)}{f(t)+g(t)}}\left[\tau g(t)\right]^{\frac{g(t)}{f(t)+g(t)}},\end{aligned}\tag{2.4}$$

where c is a positive constant.

Proof From [1, 33], all of the rules can be easily proved. For example, from (2.1) and

$$T_\alpha(fg)(t) = g(t)T_\alpha f(t) + f(t)T_\alpha g(t),$$

we obtain

$$\begin{aligned}\tau\,(fg)\,(t) &= \exp\left\{\frac{T_\alpha(fg)(t)}{(fg)(t)}\right\} = \exp\left\{\frac{g(t)T_\alpha f(t) + f(t)T_\alpha g(t)}{f(t)g(t)}\right\} \\ &= \exp\left\{\frac{T_\alpha f(t)}{f(t)}\right\}\exp\left\{\frac{T_\alpha g(t)}{g(t)}\right\} = \tau f(t)\,\tau g(t).\end{aligned}$$

In addition, (2.1) and

$$T_\alpha(f\circ h)(t) = T_\alpha f(h(t))T_\alpha h(t)h(t)^{\alpha-1},$$

yields

$$\begin{aligned}\tau\,(f\circ h)\,(t) &= \exp\left\{\frac{T_\alpha(f\circ h)(t)}{(f\circ h)(t)}\right\} = \exp\left\{\frac{T_\alpha f(h(t))T_\alpha h(t)h(t)^{\alpha-1}}{f(h(t))}\right\} \\ &= \left\{\exp\left\{\frac{T_\alpha f(h(t))}{f(h(t))}\right\}\right\}^{\left\{T_\alpha h(t)h(t)^{\alpha-1}\right\}} = \left\{(\tau f)\,(h(t))\right\}^{T_\alpha h(t)h(t)^{\alpha-1}}.\end{aligned}$$

□

Theorem 2.10 *Let $f,g:[0,b]\to\mathbb{R}^+$ be left multiplicative CF integrable of order $\alpha\in(0,1]$ at t. Then,*

$$\textit{(i)}\quad \int_0^b [f(t)]^k_{d^*_\alpha t} = \left[\int_0^b f(t)_{d^*_\alpha t}\right]^k,$$

$$\textit{(ii)}\quad \int_0^b [f(t)g(t)]_{d^*_\alpha t} = \int_0^b f(t)_{d^*_\alpha t}\int_0^b g(t)_{d^*_\alpha t},$$

$$\textit{(iii)}\quad \int_0^b \left[\frac{f(t)}{g(t)}\right]_{d^*_\alpha t} = \frac{\int_0^b f(t)_{d^*_\alpha t}}{\int_0^b g(t)_{d^*_\alpha t}}$$

$$\textit{(iv)}\quad \int_0^b f(t)_{d^*_\alpha t} = \int_0^c f(t)_{d^*_\alpha t}\int_c^b f(t)_{d^*_\alpha t},$$

$$\textit{(v)}\quad \int_0^b [\tau f(t)]^{g(t)}_{d^*_\alpha t} = \frac{f(b)^{g(b)}}{f(0)^{g(0)}}\left\{\int_0^b f(t)^{T_\alpha g(t)}_{d^*_\alpha t}\right\}^{-1}, \tag{2.5}$$

where $k\in\mathbb{R}$ and $c\in[a,b]$ is a positive constant. The last formula is called $\alpha-{}^*$ integration by parts formula.

Proof We only prove *(ii)* and *(v)*. The proofs of *(i)*, *(iii)* and *(iv)* are similar.

From (2.2) and the features of multiplicative derivative, we get

$$\int_0^b [f(t)g(t)]_{d_\alpha^* t} = \int_0^b [f(t)g(t)]_{dt}^{t^{\alpha-1}}$$

$$= \int_0^b [f(t)]_{dt}^{t^{\alpha-1}} \int_0^b [g(t)]_{dt}^{t^{\alpha-1}} = \int_0^b f(t)_{d_\alpha^* t} \int_0^b g(t)_{d_\alpha^* t}.$$

On the other hand, after applying multiplicative CF integral to both sides of the equality (2.4) and considering *(ii)*, we arrive at

$$\int_0^b [\tau f^g(t)]_{d_\alpha^* t} = \int_0^b [\tau f(t)]_{d_\alpha^* t}^{g(t)} \int_0^b f(t)_{d_\alpha^* t}^{T_\alpha g(t)}.$$

Finally, since $\int_0^b [\tau f^g(t)]_{d_\alpha^* t} = \frac{f(b)^{g(b)}}{f(0)^{g(0)}}$ by (2.3), we complete the proof. □

Definition 2.11 *Let $f : [0, b] \to \mathbb{R}^+$ and $\alpha \in (0, 1]$.*

$$^*L_\alpha^2[0, b] = \left\{ f : \int_0^b [f(t) \odot f(t)]_{d_\alpha^* t} < \infty \right\}$$

is an *inner product space with

$$<, >_*: {}^*L_\alpha^2[0, b] \times {}^*L_\alpha^2[0, b] \to \mathbb{R}^+,$$

$$< f, h >_*= \int_0^b [f(t) \odot h(t)]_{d_\alpha^* t},$$

where $f, h \in {}^*L_\alpha^2[0, b]$ are positive functions.

Definition 2.12 *For an arbitrary multiplicative square matrix $B = \left[e^{b_{ij}}\right]_{n\times n}$, let $\{\lambda_i\}_{i=1}^k$ be set of distinct *eigenvalues of B. The *characteristic polynomial of B, denoted $\Delta(\lambda)$, can be written in the factorized form*

$$\Delta(\lambda) = \det\left(B \ominus \left\{e^{\lambda_i} \odot I\right\}\right) = e^{(\lambda_1-\lambda)^{m_1}} \odot e^{(\lambda_2-\lambda)^{m_2}} \odot \cdots \odot e^{(\lambda_k-\lambda)^{m_k}}$$

with $m_1 + m_2 + \cdots + m_k = m$. When $\Delta(\lambda) = 1$, the exponent m_i, corresponding to each eigenvalue λ_i, is called the algebraic *multiplicity of λ_i. Moreover, dimension of the null space of $B \ominus \left\{e^{\lambda_i} \odot I\right\}$ is called the geometric *multiplicity of λ_i. If $m_i = 1$ for some $i \in \overline{1, k}$, then λ_i is said to be a simple eigenvalue of B (see [29] for examples).

Remark 2.13 *Let λ_0 be a root of the *characteristic polynomial $\Delta(\lambda)$. We say that is a root of the algebraic *multiplicity l if $\Delta(\lambda_0) = 1$, $^*T_\alpha\Delta(\lambda_0) = 1$, $^{*(2)}T_\alpha\Delta(\lambda_0) = 1, \ldots, {}^{*(l-1)}T_\alpha\Delta(\lambda_0) = 1$ but $^{*(l)}T_\alpha\Delta(\lambda_0) \neq 1$.*

3. Asymptotic estimates of *Eigenfunctions

Theorem 3.1 **eigenfunctions of the system (1.1)-(1.3) have below asymptotic estimates*

$$y_1(t,\lambda) = [y_1(0)]^{\cos\left(\lambda\frac{t^\alpha}{\alpha}\right)}[y_2(0)]^{-\sin\left(\lambda\frac{t^\alpha}{\alpha}\right)}\int_0^t\left[\left(y_1^{q(s)}\right)^{\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\left(y_2^{r(s)}\right)^{\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}, \tag{3.1}$$

$$y_2(t,\lambda) = [y_1(0)]^{\sin\left(\lambda\frac{t^\alpha}{\alpha}\right)}[y_2(0)]^{\cos\left(\lambda\frac{t^\alpha}{\alpha}\right)}\int_0^t\left[\left(y_1^{q(s)}\right)^{-\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\left(y_2^{r(s)}\right)^{\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}, \tag{3.2}$$

where $y(t,\lambda) = \begin{bmatrix} y_1(t,\lambda) & y_2(t,\lambda)\end{bmatrix}^T$.

Proof By the given $\alpha-{}^*$Dirac system,

$$\begin{aligned}
&\int_0^t\left[\left(y_1^{q(s)}\right)^{\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\left(y_2^{r(s)}\right)^{\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}\\
&=\int_0^t\left[\left(y_1^{\lambda}\,\{\tau y_2\}^{-1}\right)^{\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\left(y_2^{\lambda}\,\tau y_1\right)^{\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s} \qquad (3.3)\\
&=\int_0^t[\tau y_2]_{d_\alpha^* s}^{-\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\int_0^t[\tau y_1]_{d_\alpha^* s}^{\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\int_0^t\left[y_1^{\lambda\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}\int_0^t\left[y_2^{\lambda\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}.
\end{aligned}$$

If $\alpha-{}^*$integration by parts formula (2.5) is applied to first two multiplier in the right side of last equality, it yields that

$$\int_0^t[\tau y_2]_{d_\alpha^* s}^{-\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}} = [y_2(0)]^{\sin\left(\lambda\frac{t^\alpha}{\alpha}\right)}\left\{\int_0^t y_{2\,d_\alpha^* s}^{\,T_\alpha\left(\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}\right)}\right\}, \tag{3.4}$$

$$\int_0^t[\tau y_1]_{d_\alpha^* s}^{\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}} = y_1(t)\,[y_1(0)]^{-\cos\left(\lambda\frac{t^\alpha}{\alpha}\right)}\left\{\int_0^t y_{1\,d_\alpha^* s}^{\,T_\alpha\left(\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}\right)}\right\}^{-1}. \tag{3.5}$$

By considering (3.4) and (3.5) with the properties $T_\alpha f(s) = s^{1-\alpha}f'(s)$ in (3.3), we get

$$\int_0^t\left[\left(y_1^{q(s)}\right)^{\sin\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\left(y_2^{r(s)}\right)^{\cos\left\{\lambda\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s} = [y_2(0)]^{\sin\left(\lambda\frac{t^\alpha}{\alpha}\right)}\,y_1(t)\,[y_1(0)]^{-\cos\left(\lambda\frac{t^\alpha}{\alpha}\right)}.$$

The asymptotic estimates (3.1) is obtained. Similarly, the asymptotic estimates (3.2) can be obtained in the same way. It completes the proof. □

4. Some spectral properties of $\alpha-{}^*$Dirac system

In here, we examine some properties of $\alpha-{}^*$Dirac system such as reality, simplicity, orthogonality and self-adjointness.

Lemma 4.1 *($\alpha-{}^*$Lagrange Identity) Let $u, v \in {}^*L^2_\alpha[0,b]$. Then*

$$(\boldsymbol{D}_\alpha[u] \odot v) \ominus (u \odot \boldsymbol{D}_\alpha[v]) = \tau\left\{[u,v]_t\right\},$$

where $u = u(t,\lambda) = \begin{bmatrix} u_1(t,\lambda) & u_2(t,\lambda) \end{bmatrix}^T$, $v = v(t,\lambda) = \begin{bmatrix} v_1(t,\lambda) & v_2(t,\lambda) \end{bmatrix}^T$ *and*

$$[u,v]_t = (u_2 \odot v_1) \ominus (u_1 \odot v_2). \tag{4.1}$$

Proof From (1.4), we arrive at

$$\begin{aligned}(\boldsymbol{D}_\alpha[u] \odot v) \ominus (u \odot \boldsymbol{D}_\alpha[v]) &= \left\{\left(\tau u_2\, u_1^{q(t)} \odot v_1\right) \oplus \left(\tau u_1\, u_2^{-r(t)} \odot v_2\right)\right\} \ominus \left\{\left(\tau v_2\, v_1^{q(t)} \odot u_1\right) \oplus \left(\tau v_1\, v_2^{-r(t)} \odot u_2\right)\right\} \\ &= \left\{(\tau u_2 \odot v_1) \ominus (\tau u_1 \odot v_2)\right\} \oplus \left\{(\tau v_1 \odot u_2) \ominus (\tau v_2 \odot u_1)\right\} \\ &= \tau\left\{(u_2 \odot v_1) \ominus (u_1 \odot v_2)\right\}\end{aligned}$$

which proves the result. □

Lemma 4.2 *($\alpha-{}^*$Green's Formula) Let $u, v \in {}^*L^2_\alpha[0,b]$. Then*

$$\int_0^b \left[(\boldsymbol{D}_\alpha[u] \odot v) \ominus (u \odot \boldsymbol{D}_\alpha[v])\right]_{d^*_\alpha t} = [u,v]_t\Big|_0^b, \tag{4.2}$$

where $u = u(t,\lambda) = \begin{bmatrix} u_1(t,\lambda) & u_2(t,\lambda) \end{bmatrix}^T$, $v = v(t,\lambda) = \begin{bmatrix} v_1(t,\lambda) & v_2(t,\lambda) \end{bmatrix}^T$.

Proof The proof is easily proved by multiplicative CF integral on $[0,b]$ for both sides of $\alpha-{}^*$Lagrange Identity formula. □

Theorem 4.3 *$\alpha-{}^*$Dirac operator $\boldsymbol{D}_\alpha$ in (1.1) is formally self-adjoint on ${}^*L^2_\alpha[0,b]$.*

Proof From the boundary conditions (1.2), we arrive at

$$[u,v]_b = (u_2(b) \odot v_1(b)) \ominus (u_1(b) \odot v_2(b)) = \left(u_1^{\frac{-d_1}{d_2}}(b) \odot v_1(b)\right) \ominus \left(v_1^{\frac{-d_1}{d_2}}(b) \odot u_1(b)\right) = 1.$$

Similarly, from (1.3), we have $[u,v]_0 = 1$. So, by (4.2), we obtain

$$\int_0^b \left[(\boldsymbol{D}_\alpha[u] \odot v) \ominus (u \odot \boldsymbol{D}_\alpha[v])\right]_{d^*_\alpha t} = 1$$

or

$$< \boldsymbol{D}_\alpha[u], v >_* = < u, \boldsymbol{D}_\alpha[v] >_*, \tag{4.3}$$

which proves the theorem. □

Theorem 4.4 *All eigenvalues of $\alpha-{}^*$ Dirac system (1.1)-(1.3) are real.*

Proof Let λ be an eigenvalue with an eigenfunction $u = u(t,\lambda)$. Then,

$$< \boldsymbol{D}_\alpha [u], u >_* = < e^\lambda \odot u, u >_* = e^\lambda \odot < u, u >_* \tag{4.4}$$

and

$$< u, \boldsymbol{D}_\alpha [u] >_* = < u, e^\lambda \odot u >_* = e^{\bar{\lambda}} \odot < u, u >_* . \tag{4.5}$$

From (4.3)-(4.5), we arrive at

$$e^\lambda \odot < u, u >_* = e^{\bar{\lambda}} \odot < u, u >_* \quad \text{or} \quad < u, u >_*^{\lambda-\bar{\lambda}} = 1.$$

Since $u(t) \neq 1$, we get $\lambda = \bar{\lambda}$. It completes the proof. □

Theorem 4.5 **eigenfunctions $u = u(t,\lambda) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$ and $v = v(t,\mu) = \begin{bmatrix} v_1(t) & v_2(t) \end{bmatrix}^T$ corresponding to the distinct eigenvalues λ and μ are orthogonal, i.e.*

$$\int_0^b \left[u^T(t,\lambda) \odot v(t,\mu)\right]_{d_\alpha^* t} = 1.$$

Proof If $\boldsymbol{D}_\alpha [u] = e^\lambda \odot u$ and $\boldsymbol{D}_\alpha [v] = e^\lambda \odot v$ from (1.4) is taken into account in the equality (4.3), we arrive at

$$< e^\lambda \odot u, v >_* = < u, e^\lambda \odot v >_* \quad \text{or} \quad < u, v >_*^{\lambda-\mu} = 1.$$

Since $\lambda \neq \mu$, $< u, v >_* = 1$. Namely, we obtain that $u(t)$ and $v(t)$ are orthogonal. □

Let $u = u(t,\lambda) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$ and $v = v(t,\mu) = \begin{bmatrix} v_1(t) & v_2(t) \end{bmatrix}^T$. Then, we define $\alpha-{}^*$Wronskian of $u(t)$ and $v(t)$ by the formula (4.1).

Theorem 4.6 *$\alpha-{}^*$ Wronskian of any solution for equation (1.1) is independent of t.*

Proof Let $u(t)$ and $v(t)$ be two solutions the equation (1.1). From (4.2) and $\boldsymbol{D}_\alpha [u] = e^\lambda \odot u$ and $\boldsymbol{D}_\alpha [\bar{v}] = e^{\bar{\lambda}} \odot \bar{v}$, we arrive at

$$\int_0^t \left[u^T(x,\lambda) \odot v(x,\mu)\right]_{d_\alpha^* t}^{\lambda-\bar{\lambda}} = \frac{u\,[u,\bar{v}]_t}{u\,[u,\bar{v}]_0}.$$

On the other hand, since $\lambda = \bar{\lambda}$ (from Theorem 4.4) and $[u,\bar{v}]_0 = 1$, then $^*W_\alpha (u,v)(t) = [u,v]_t = 1$, i.e., $\alpha-{}^*$Wronskian is independent of t. □

Theorem 4.7 *Any two solutions of equation (1.1) are multiplicative linearly dependent iff their $\alpha-{}^*$ Wronskian is one.*

Proof Let $u(t)$ and $v(t)$ be two multiplicative linearly dependent solutions the equation (1.1). Then, there exists a constant $c \neq 1$ where $u(t) = v(t)^c$ [48]. Hence,

$$^{*}W_{\alpha}\left(u,v\right)(t) = [u,v]_t = (u_2(t) \odot v_1(t)) \ominus (u_1(t) \odot v_2(t)) = (v_2(t)^c \odot v_1(t)) \ominus (v_1(t)^c \odot v_2(t)) = 1.$$

Conversely, $^{*}W_{\alpha}\left(u,v\right)(t) = (u,v)(t) = 1$, and therefore, $u(t) = v(t)^c$, i.e., $u(t)$ and $v(t)$ be two multiplicative linearly dependent. □

Lemma 4.8 *All *eigenvalues of $\alpha-{}^{*}$Dirac system (1.1)-(1.3) are simple from the geometric point of view.*

Proof Let μ be an eigenvalue for *eigenfunctions $u(t)$ and $v(t)$.

By the condition (1.2), we get $^{*}W_{\alpha}\left(u,v\right)(0) = [u,v]_0 = 1$, then the set $\{u(t), v(t)\}$ is linearly dependent. Consequently, one *eigenfunction corresponds to one *eigenvalue. □

Theorem 4.9 *All *eigenvalues of $\alpha-{}^{*}$Dirac system (1.1)-(1.3) are simple.*

Proof Since $\varphi(t) = \varphi(t,\lambda)$ satisfies boundary conditions (1.2), to determine *eigenvalues of the system, $\varphi_1(t,\lambda)$, $\varphi_2(t,\lambda)$ should be substituted in boundary condition (1.3), and its roots should be determined. Put

$$V(\lambda) = \varphi_1^{d_1}(b)\,\varphi_2^{d_2}(b).$$

Then,

$$\eta V(\lambda) = \eta\varphi_1^{d_1}(b)\,\eta\varphi_2^{d_2}(b),$$

where $\eta\cdot$ denotes $\frac{\partial_{\alpha}^{*}\cdot}{\partial_{\alpha}^{*}\lambda}$ (partial multiplicative conformable derivative of order $\alpha \in (0,1]$ on λ) for brevity.

Let $\tilde{\lambda}$ be a double *eigenvalue, and $\tilde{\varphi}(t) = \tilde{\varphi}(t,\tilde{\lambda})$ be one of the corresponding *eigenfunctions. Then,

$$V(\tilde{\lambda}) = 1, \qquad \eta V(\tilde{\lambda}) = 1$$

should be fulfilled simultaneously, i.e.,

$$\tilde{\varphi}_1^{d_1}(b)\,\tilde{\varphi}_2^{d_2}(b) = 1,$$

$$\eta\tilde{\varphi}_1^{d_1}(b)\,\eta\tilde{\varphi}_2^{d_2}(b) = 1.$$

Since $\left(d_1^2 + d_2^2\right) \neq 0$, it follows from the last two equalities that

$$\ln\tilde{\varphi}_1(b)\,\ln\eta\tilde{\varphi}_2(b) - \ln\tilde{\varphi}_2(b)\,\ln\eta\tilde{\varphi}_1(b) = 0. \tag{4.6}$$

By the multiplicative CF differentiating with respect to λ both sides of the system (1.4), we have

$$\begin{array}{r} \tau(\eta y_2)\,(\eta y_1)^{q(t)-\lambda} = y_1^{\lambda^{1-\alpha}}, \\ \tau(\eta y_1)\,(\eta y_2)^{\lambda-r(t)} = y_2^{-\lambda^{1-\alpha}}. \end{array} \tag{4.7}$$

After using the natural logarithm of the equations in the system (1.4), multiplying by $\ln\eta y_1$, $-\ln\eta y_2$, respectively, we arrive at

$$\begin{array}{r} \ln\tau y_2\,\ln\eta y_1 + (q(t)-\lambda)\ln y_1\,\ln\eta y_1 = 0, \\ -\ln\tau y_1\,\ln\eta y_2 + (r(t)-\lambda)\ln y_2\,\ln\eta y_2 = 0. \end{array} \tag{4.8}$$

On the other hand, after using the natural logarithm of the equations in the system (4.7), multiplying by $-\ln y_1$, $\ln y_2$, respectively, we arrive at

$$\begin{array}{c} -\ln y_1 \ \ln\tau(\eta y_2) - (q(t)-\lambda)\ln y_1 \ \ln\eta y_1 = -\lambda^{1-\alpha}\ln^2 y_1, \\ \ln y_2 \ \ln\tau(\eta y_1) + (\lambda - r(t))\ln y_2 \ \ln\eta y_2 = -\lambda^{1-\alpha}\ln^2 y_2. \end{array} \tag{4.9}$$

After the equalities (4.8) and (4.9) are added side by side, from (2.1), we obtain

$$T_\alpha\left(\ln y_1 \ \ln\eta y_2 - \ln y_2 \ \ln\eta y_1\right) = \lambda^{1-\alpha}\left(\ln^2 y_1 + \ln^2 y_2\right), \tag{4.10}$$

where $T_\alpha f$ is defined by left CF derivative of f order $\alpha\in(0,1]$.

By the left CF integration with respect to t on $[0,b]$ to both sides of the equality (4.10), we have

$$\lambda^{1-\alpha}\int_0^b\left[\ln^2 y_1 + \ln^2 y_2\right]d_\alpha t = \left\{\ln y_1 \ \ln\eta y_2 - \ln y_2 \ \ln\eta y_1\right\}_{t=0}^b. \tag{4.11}$$

By using the equality (4.6) and

$$\eta\tilde\varphi_1(0,\tilde\lambda) = \eta\tilde\varphi_2(0,\tilde\lambda) = 1$$

from Theorem (3.1) when $\lambda=\tilde\lambda$ on the equality (4.11), we get

$$\int_0^b\left[\ln^2\tilde\varphi_1 + \ln^2\tilde\varphi_2\right]d_\alpha t = 0.$$

Consequently, $\tilde\varphi_1(t,\tilde\lambda) = \tilde\varphi_2(t,\tilde\lambda) = 1$ or $\tilde\varphi(t,\tilde\lambda) = 1$. This is a contradiction. It completes the proof. $\square$

Let

$$\varphi_1 := \varphi_1(t,\lambda) = \begin{bmatrix}\varphi_{11}(t,\lambda)\\ \varphi_{12}(t,\lambda)\end{bmatrix}, \quad \varphi_2 := \varphi_2(t,\lambda) = \begin{bmatrix}\varphi_{21}(t,\lambda)\\ \varphi_{22}(t,\lambda)\end{bmatrix},$$

be linearly independent solutions of (1.1) which satisfies initial conditions

$$\varphi_{i,j}(0,\lambda) = \delta^*_{ij} \quad i,j=1,2 \quad \lambda\in\mathbb{C},$$

where $\delta^*_{ij} = \left\{\begin{array}{ll} e, & i=j \\ 1, & i\neq j\end{array}\right.$ is the *Kronecker delta. Then, each solution of (1.1) has the form

$$y(t,\lambda) = \varphi_1^{\xi_1}\varphi_2^{\xi_2},$$

where ξ_1 and ξ_2 do not depend on t. If we can obtain a non-trivial solution of linear system

$$\begin{array}{c} U_1(\varphi_1)^{\xi_1}\,U_1(\varphi_2)^{\xi_2} = 1, \\ U_2(\varphi_1)^{\xi_1}\,U_2(\varphi_2)^{\xi_2} = 1, \end{array}$$

or

$$\begin{array}{c} \xi_1\ln U_1(\varphi_1) + \xi_2\ln U_1(\varphi_2) = 0, \\ \xi_1\ln U_2(\varphi_1) + \xi_2\ln U_2(\varphi_2) = 0, \end{array}$$

$y(t,\lambda)$ is called an *eigenfunction of the equation (1.1), where U_1 and U_2 defined by (1.5). Hence, $\lambda \in \mathbb{R}$ is an *eigenvalue iff

$$^*\Delta_\alpha(\lambda) = \begin{vmatrix} \ln U_1(\varphi_1) & \ln U_1(\varphi_2) \\ \ln U_2(\varphi_1) & \ln U_2(\varphi_2) \end{vmatrix} = 0.$$

$^*\Delta_\alpha(\lambda)$ is called the characteristic determinant associated with $\alpha-{}^*$Dirac system defined by (1.1)-(1.3).

Corollary 4.10 **eigenvalues of (1.1)-(1.3) are zeros of $^*\Delta_\alpha(\lambda)$. Since $\varphi_1(t,\lambda)$ and $\varphi_2(t,\lambda)$ are entire in λ for each fixed $x \in [0,b]$, $^*\Delta_\alpha(\lambda)$ is an entire function in λ. Hence, *eigenvalues of the system (1.1)-(1.3) are at most countable with no finite limit points.*

5. $\alpha-{}^*$Green's function

In this section, $\alpha-{}^*$Green's function will be constructed for $\alpha-{}^*$Dirac system which is not homogeneous and some of its properties will be given. We consider the system

$$\tau y_2\, y_1^{q(t)-\lambda} = e^{f_1(t)}, \tag{5.1}$$

$$\tau y_1\, y_2^{\lambda-r(t)} = e^{f_2(t)}, \tag{5.2}$$

with the conditions

$$U_1(y) = y_1^{c_1}(0)\, y_2^{c_2}(0) = 1, \tag{5.3}$$

$$U_2(y) = y_1^{d_1}(b)\, y_2^{d_2}(b) = 1, \tag{5.4}$$

where λ is a spectral parameter; $q(t)$, $r(t)$ real-valued continuous and multiplicative CF integrable functions on $[0,b]$; $\left(c_1^2+c_2^2\right)\left(d_1^2+d_2^2\right) \neq 0$; $f(t) = \left[\begin{array}{cc} e^{f_1(t)} & e^{f_2(t)} \end{array}\right]^T$.

Theorem 5.1 *Let's admit that λ is not an *eigenvalue of (1.1)-(1.3) and $y(\cdot,\lambda)$ satisfies system (5.1),(5.2) and conditions (5.3), (5.4). Then,*

$$y(t,\lambda) = \int_0^b \left({}^*G_\alpha(t,x,\lambda) \odot e^{f(x)}\right)_{d_\alpha^* x}, \quad x \in [0,b] \tag{5.5}$$

where $e^{f(t)} \in {}^*L_\alpha^2[0,b]$ and $^*G_\alpha(t,x,\lambda)$ is $\alpha-{}^*$Green's function for (5.1)-(5.4) defined by

$$^*G_\alpha(t,x,\lambda) = e^{\frac{-1}{^*\Delta_\alpha(\lambda)}} \odot \begin{cases} \theta_2(t,\lambda) \odot [\theta_1(x,\lambda)]^T, & 0 \le x \le t \\ \theta_1(t,\lambda) \odot [\theta_2(x,\lambda)]^T, & t < x \le b \end{cases}. \tag{5.6}$$

Conversely, $y(\cdot,\lambda)$ is defined by the system (5.1),(5.2) and the conditions (5.3), (5.4). Furthermore, $^*G_\alpha(t,x,\lambda)$ is unique. Here, θ_1 and θ_2 are linearly independent solutions of the system (5.1),(5.2) and the conditions (5.3), (5.4).

Proof By notion of $\alpha-{}^*$Green's function, we get

$$ {}^*G_\alpha(t,x,\lambda)\odot e^{f(x)} = \begin{cases} \begin{bmatrix} \left\{\theta_{11}(x)^{f_1(x)}\,\theta_{12}(x)^{f_2(x)}\right\}^{\ln\theta_{21}(t)\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \\ \left\{\theta_{11}(x)^{f_1(x)}\,\theta_{12}(x)^{f_2(x)}\right\}^{\ln\theta_{22}(t)\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \end{bmatrix}, & 0\le x\le t \\ \begin{bmatrix} \left\{\theta_{21}(x)^{f_1(x)}\,\theta_{22}(x)^{f_2(x)}\right\}^{\ln\theta_{11}(t)\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \\ \left\{\theta_{21}(x)^{f_1(x)}\,\theta_{22}(x)^{f_2(x)}\right\}^{\ln\theta_{12}(t)\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \end{bmatrix}, & t<x\le b \end{cases}. \tag{5.7} $$

From (5.5), applying multiplicative CF integration to (5.7) with respect to x on $[0,b]$, it follows that

$$ y_1(t,\lambda) = \left\{\int_0^t \left[\theta_{11}(x)^{f_1(x)}\,\theta_{12}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{-\ln\theta_{21}(t)}{{}^*\Delta_\alpha(\lambda)}} \left\{\int_t^b \left[\theta_{21}(x)^{f_1(x)}\,\theta_{22}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{-\ln\theta_{11}(t)}{{}^*\Delta_\alpha(\lambda)}}, \tag{5.8} $$

$$ y_2(t,\lambda) = \left\{\int_0^t \left[\theta_{11}(x)^{f_1(x)}\,\theta_{12}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{-\ln\theta_{22}(t)}{{}^*\Delta_\alpha(\lambda)}} \left\{\int_t^b \left[\theta_{21}(x)^{f_1(x)}\,\theta_{22}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{-\ln\theta_{12}(t)}{{}^*\Delta_\alpha(\lambda)}}. \tag{5.9} $$

Now, by differentiating both sides of the equation (5.8) in multiplicative CF sence with respect to t, we have

$$ \begin{aligned} \tau y_1(t,\lambda) &= \left\{\int_0^t \left[\theta_{11}(x)^{f_1(x)}\,\theta_{12}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{-T_\alpha\ln\theta_{21}(t)}{{}^*\Delta_\alpha(\lambda)}} \left\{\int_t^b \left[\theta_{21}(x)^{f_1(x)}\,\theta_{22}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{-T_\alpha\ln\theta_{11}(t)}{{}^*\Delta_\alpha(\lambda)}} \\ &\quad \left[{}^*W_\alpha\left(\theta_1,\theta_2\right)\right]^{\frac{f_2(t)}{{}^*\Delta_\alpha(\lambda)}} \\ &= \left\{\int_0^t \left[\theta_{11}(x)^{f_1(x)}\,\theta_{12}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{(\lambda-r(t))\ln\theta_{22}(t)}{{}^*\Delta_\alpha(\lambda)}} \left\{\int_t^b \left[\theta_{21}(x)^{f_1(x)}\,\theta_{22}(x)^{f_2(x)}\right]_{d^*_\alpha x}\right\}^{\frac{(\lambda-r(t))\ln\theta_{12}(t)}{{}^*\Delta_\alpha(\lambda)}} e^{f_2(t)} \\ &= \left\{y_2(t,\lambda)\right\}^{(r(t)-\lambda)}\, e^{f_2(t)} \end{aligned} $$

It is proved similarly that the validity of (5.1).

Now, let us prove uniqueness of $\alpha-{}^*$Green's function for the given system. Let's face it, there is another $\alpha-{}^*$Green's function $\widetilde{{}^*G_\alpha}(t,x,\lambda)$ for same system. Then, we get

$$ y\left(t,\lambda\right) = \int_0^b \left({}^*G_\alpha(t,x,\lambda)\odot e^{f(x)}\right)_{d^*_\alpha x}, $$

and

$$y\,(t,\lambda) = \int\limits_0^b \left({}^*\widetilde{G_\alpha}(t,x,\lambda) \odot e^{f(x)} \right)_{d_\alpha^* x}.$$

Thence, we get

$$\int\limits_0^b \left(\left\{ {}^*G_\alpha(t,x,\lambda) \ominus {}^*\widetilde{G_\alpha}(t,x,\lambda) \right\} \odot e^{f(x)} \right)_{d_\alpha^* x} = 1.$$

By setting $f(t) = \ln \overline{{}^*G_\alpha(t,x,\lambda) \ominus {}^*\widetilde{G_\alpha}(t,x,\lambda)}$, it yields

$$\int\limits_0^b \left(\left\{ {}^*G_\alpha(t,x,\lambda) \ominus {}^*\widetilde{G_\alpha}(t,x,\lambda) \right\} \odot \left\{ \overline{{}^*G_\alpha(t,x,\lambda) \ominus {}^*\widetilde{G_\alpha}(t,x,\lambda)} \right\} \right)_{d_\alpha^* x} = 1,$$

or

$${}^*G_\alpha(t,x,\lambda) \ominus {}^*\widetilde{G_\alpha}(t,x,\lambda) = 1 \quad \text{or} \quad {}^*G_\alpha(t,x,\lambda) = {}^*\widetilde{G_\alpha}(t,x,\lambda).$$

This completes the proof. □

Theorem 5.2 *$\alpha-{}^*$ Green's function of (1.1)-(1.3) has below properties:*

i) *${}^*G_\alpha(t,x,\lambda)$ is continuous at $(0,0)$.*

ii) *${}^*G_\alpha(t,x,\lambda) = {}^*G_\alpha^T(x,t,\lambda)$.*

iii) *${}^*G_\alpha(t,x,\lambda)$ satisfies the equation (1.1) and conditions (1.2)-(1.3) for all $x \in \mathbb{R}$ as a function of t.*

iv) *Let λ_0 be an eigenvalue of ${}^*\Delta_\alpha(\lambda)$. Then, λ_0 is simple pole point of ${}^*G_\alpha(t,x,\lambda)$ and*

$${}^*G_\alpha(t,x,\lambda) = \left[\psi_0(t)^{-\psi_0(x)} \right]^{\frac{1}{\lambda-\lambda_0}} {}^*\widetilde{G_\alpha}(t,x,\lambda).$$

Here, ${}^*\widetilde{G_\alpha}(t,x,\lambda)$ is * type holomorfic function of λ in the neighbourhood of λ_0, ψ is normalized eigenfunction related to λ_0.

Proof ***i)*** Proof is obtained by continuity of $\theta_1(.,\lambda)$ and $\theta_2(.,\lambda)$ for all $\lambda \in \mathbb{C}$.

ii) It can be easily proved if some basic features of multiplicative calculus are used.

iii) Let $x \in [0,t]$. Then,

$${}^*G_\alpha(t,x,\lambda) = \begin{bmatrix} {}^*G_{\alpha 1}(t,x,\lambda) \\ {}^*G_{\alpha 2}(t,x,\lambda) \end{bmatrix} = \begin{bmatrix} \{\theta_{11}(x)\,\theta_{12}(x)\}^{\ln\theta_{21}(t)\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \\ \{\theta_{11}(x)\,\theta_{12}(x)\}^{\ln\theta_{22}(t)\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \end{bmatrix}$$

or

$$\boldsymbol{D}_\alpha {}^*G_\alpha(t,x,\lambda) = e^\lambda \odot {}^*G_\alpha(t,x,\lambda).$$

Similarly, proof can be made for $x \in [t,b]$.

$$[e^{c_1} \odot {}^*G_{\alpha 1}(0,x,\lambda)] \oplus [e^{c_2} \odot {}^*G_{\alpha 2}(0,x,\lambda)] = \left[\{\theta_{11}(x)\,\theta_{12}(x)\}^{\ln\left(\theta_{21}^{c_1}(0)\theta_{22}^{c_2}(0)\right)} \right]^{\frac{-1}{{}^*\Delta_\alpha(\lambda)}} = 1$$

and

$$\left[e^{d_1} \odot {}^*G_{\alpha 1}(b,x,\lambda)\right] \oplus \left[e^{d_2} \odot {}^*G_{\alpha 2}(b,x,\lambda)\right] = \left[\{\theta_{11}(x)\,\theta_{12}(x)\}^{\ln\left(\theta_{21}^{d_1}(b)\theta_{22}^{d_2}(b)\right)}\right]^{\frac{-1}{{}^*\Delta_\alpha(\lambda)}} = 1$$

iv) Let λ_0 be pole point of ${}^*G_\alpha(t,x,\lambda)$ and $R(t,x)$ be *residue of ${}^*G_\alpha(t,x,\lambda)$ at $\lambda=\lambda_0$. By Corollary 4.10, we get

$$R(t,x) = \psi_0(t,\lambda_0)^{-\psi_0(x,\lambda_0)}.$$

It completes the proof. □

6. Conclusion

We established the multiplicative CF Dirac system. Actually, this system is a fractional generalization of the multiplicative Dirac system[28] in case $\alpha = 1$. Firstly, we obtained the *eigenfunctions of that system. Later, we proved that *eigenvalues are real and simple, and the *eigenfunctions are orthogonal in ${}^*L^2_\alpha[0,b]$. We obtained Green's function on the multiplicative case. We think that this system, which is extremely important for quantum physics and effective in both classical and fractional cases, will make great contributions to mathematical physics in multiplicative cases. In fact, this equation, which we examined in the multiplicative case, corresponds to an equation that is much more difficult and tiring to analyze in classical and fractional calculus. This situation increases the importance of the results we obtained and the different calculus we used.

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