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Berezin symbol inequalities via Grüss type inequalities and related questions

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Abstract: We prove some new inequalities for Berezin symbols of operators via classical Grüss type inequalities. Some other related questions are also discussed.

Key words: Berezin symbol, Grüss inequality, selfadjoint operator, reproducing kernel Hilbert space

1. Introduction

In this article, we prove some new inequalities for Berezin symbols of operators by using classical Grüss type inequalities.

1.1. The Berezin symbol

Recall that the reproducing kernel Hilbert space (briefly, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that :

(i) the evaluation functionals

$$\varphi_\lambda(f) := f(\lambda), \ \lambda \in \Omega,$$

are continuous on $\mathcal{H}$.

(ii) for every $\lambda \in \Omega$ there exists a function $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then, according to the Riesz representation theorem, for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The family $\{k_\lambda : \lambda \in \Omega\}$ is called the normalized reproducing kernel of the space $\mathcal{H}$ and family $\left\{\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}} : \lambda \in \Omega\right\}$ is called the normalized reproducing kernel of the space $\mathcal{H}$ (see Aronzajn [1]). For example, the Hardy space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the Bergman space $L^2_\alpha(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$ of analytic functions on $\mathbb{D}$ and the Fock space $\mathcal{F}(\mathbb{C})$ of entire functions are RKHSs. For any bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ (i.e. for any $A \in \mathcal{B}(\mathcal{H})$, the Banach algebra of all bounded linear operators on $\mathcal{H}$), its Berezin symbol $\widetilde{A}$ is defined by (see Berezin [2, 3])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle, \ \lambda \in \Omega.$$

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It is elementary to verify that $\widetilde{A}$ is the bounded complex-valued function on Ω, namely, $\sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right| \leq \|A\|$. The Berezin number of the operator A is defined by (see, Karaev [11])

$$\operatorname{ber}(A) := \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|.$$

So, $\operatorname{ber}(A) \leq \|A\|$ for all $A \in \mathcal{B}(\mathcal{H})$. Recall that the numerical radius $w(A)$ of operator A is the following number:

$$w(A) : \{\sup|\langle Ax, x\rangle| : x \in \mathcal{H} \text{ and } \|x\|_{\mathcal{H}} = 1\}.$$

Clearly, $\operatorname{ber}(A) \leq w(A) \leq \|A\|$.

We also define the following so-called Berezin norm of operators $A \in \mathcal{B}(\mathcal{H})$:

$$\|A\|_{\text{Ber}} := \sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\|.$$

It is easy to see that $\|A\|_{\text{Ber}}$ actually determines a new operator norm in $\mathcal{B}(\mathcal{H}(\Omega))$ (since the set of reproducing kernels $\{k_\lambda : \lambda \in \Omega\}$ span the space $\mathcal{H}(\Omega)$). It is also trivial that $\operatorname{ber}(A) \leq \|A\|_{\text{Ber}} \leq \|A\|$. The present paper is motivated mostly by [15]. In this paper, we also prove some new inequalities between $\|A\|_{\text{Ber}}$ and $\operatorname{ber}(A)$; moreover, we give upper estimates for the quantity $\left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\|$ which is important in some questions of operator theory [9].

1.2. Grüss and Grüss type inequalities

Following Dragomir [4], recall that in 1950, M. Biernacki et al. ([13], Chapter X) established the following discrete version of the Grüss integral inequality [4] :

Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ be two n-tuples of real numbers such that

$$r \leq a_{ij} \leq R \text{ and } s \leq b_i \leq S \text{ for } i = 1, ..., n.$$

Then, one has

$$\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i - \frac{1}{n}\sum_{i=1}^{n}a_i\frac{1}{n}\sum_{i=1}^{n}b_i\right| \leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1}$$

where $[x]$ denotes the integer part of x, $x \in \mathbb{R}$. Inequality (1) in abstract structures, has been generalized by Dragomir in the following theorem.

Theorem 1.1 *([5]) Let $(H, \langle . \rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - f, y - \gamma e\rangle \geq 0$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{4}|\Phi - \varphi||\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Let A be a self-adjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$, that is $A^* = A$. The Gelfand map established a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator I_H on H as follows (see for instance ([10], p. 3):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have:

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi\left(\overline{f}\right) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = I_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on H, that is $\langle f(A)x, x \rangle \geq 0$ for all $x \in H$. Moreover, if both f and g are real valued functions on $Sp(A)$, then the following important property holds :

$$f(t) \geq g(t) \text{ for all } t \in Sp(A) \Longrightarrow f(A) \geq g(A) \tag{2}$$

in the operator order of $\mathcal{B}(H)$.

2. Quasi Grüss type inequalities for Berezin symbols

In the present section, we prove some new vector Grüss type inequalities for the Berezin symbol of continuous functions of self-adjoint operators in Hilbert spaces. We need some facts concerning the spectral representation of such functions.

Let A be a self-adjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then, for any continuous function $f : [m, M] \to \mathbb{C}$, the following spectral representation in terms of the Riemann-Stieltjes integral:

$$f(A) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda,$$

which in terms of vectors can be written as

$$\langle f(A)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y \rangle \tag{3}$$

for all $x, y \in H$ is well known. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for all $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is monotonic nondecreasing and right continuous on $[m, M]$.

In this section, we provide various bounds for the magnitude of the difference

$$\widetilde{V^* f(A)}(\mu) - \widetilde{V^*}(\mu)\,\widetilde{f(A)}(\mu),$$

where $V \in B(\mathcal{H})$ is an isometry, under different assumptions on the continuous function and the self-adjoint operator $A : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$.

Note that, if $\mathcal{H}(\Omega) = H^2(\mathbb{D})$ (the Hardy space on the unit disc $\mathbb{D}$), then for every analytic Toeplitz operator $T_\varphi, T_\varphi f = \varphi f$, $\varphi \in H^\infty(\mathbb{D})$ (the space of all bounded analytic functions on $\mathbb{D}$) and any bounded linear operator A on $H^2(\mathbb{D})$, it is easy to see that

$$\widetilde{T_\varphi A}(\mu) = \widetilde{T}_\varphi(\mu)\,\widetilde{A}(\mu) \text{ for all } \mu \in \mathbb{D},$$

that is $\widetilde{T_\varphi A} = \widetilde{T}_\varphi \widetilde{A}$. However, in general, the Berezin symbol is not multiplicative, i.e. $\widetilde{AB} \neq \widetilde{A}\widetilde{B}$ (see Kılıç [12]). In the following theorem we show in particular that for some class of operators A and B, the Berezin symbol is asymptotically multiplicative.

Theorem 2.1 *Let A be a self-adjoint operator in the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, and let $\{E_\lambda\}_\lambda$ be its spectral family. Let $V : \mathcal{H} \to \mathcal{H}$ be an isometry. Assume that $\mu \in \Omega$ is such that there exists $\gamma, \Gamma \in \mathbb{C}$ with either $\operatorname{Re}\left\langle \Gamma\widehat{k}_\mu - V\widehat{k}_\mu, V\widehat{k}_\mu - \gamma\widehat{k}_\mu \right\rangle \geq 0$ or, equivalently*

$$\left\| V\widehat{k}_\mu - \frac{\gamma + \Gamma}{2}\widehat{k}_\mu \right\| \leq \frac{1}{2}|\Gamma - \gamma|.$$

(i) If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned}
&\left| \widetilde{V^* f(A)}(\mu) - \widetilde{V^*}(\mu)\,\widetilde{f(A)}(\mu) \right| \\
&\leq \max_{\lambda \in [m,M]} \left| \widetilde{V^* E_\lambda}(\mu) - \widetilde{V^*}(\mu)\,\widetilde{E_\lambda}(\mu) \right| \bigvee_m^M (f) \\
&\leq \max_{\lambda \in [m,M]} \left(\widetilde{E}_\lambda(\mu)\,\widetilde{(I_{\mathcal{H}} - E_\lambda)}(\mu) \right)^{\frac{1}{2}} \left(1 - \left| \widetilde{V}(\mu) \right|^2 \right)^{\frac{1}{2}} \bigvee_m^M (f) \\
&\leq \frac{1}{2}\left(1 - \left| \widetilde{V}(\mu) \right|^2 \right)^{\frac{1}{2}} \bigvee_m^M (f) \leq \frac{1}{4}|\Gamma - \gamma| \bigvee_m^M (f),
\end{aligned} \tag{4}$$

where $\bigvee_m^M (f)$ denotes the total variation of f on $[m, M]$.

(ii) If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the

inequality

$$\begin{aligned}
&\left|\widetilde{V^{*}f(A)}(\mu)-\widetilde{V^{*}}(\mu)\widetilde{f(A)}(\mu)\right| \\
&\leq L\int_{m-0}^{M}\left|\widetilde{V^{*}E_{\lambda}}(\mu)-\widetilde{V^{*}}(\mu)\widetilde{E_{\lambda}}(\mu)\right|d\lambda \\
&\leq L\left(1-\left|\widetilde{V}(\mu)\right|^{2}\right)^{\frac{1}{2}}\int_{m-0}^{M}\left(\widetilde{E_{\lambda}}(\mu)\left(1-\widetilde{E_{\lambda}}(\mu)\right)\right)^{1/2}d\lambda \\
&\leq L\left(1-\left|\widetilde{V}(\mu)\right|^{2}\right)^{\frac{1}{2}}\left(M-\widetilde{A}(\mu)\right)^{\frac{1}{2}}\left(\widetilde{A}(\mu)-m\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}(M-m)L\left(1-\left|\widetilde{V}(\mu)\right|^{2}\right)^{\frac{1}{2}}\leq\frac{1}{4}\left|\Gamma-\gamma\right|(M-m)L. \qquad (5)
\end{aligned}$$

(iii) If $f:[m,M]\to\mathbb{R}$ is a continuous monotonic nondecreasing function on $[m,M]$, then we have the inequality

$$\begin{aligned}
&\left|\widetilde{V^{*}f(A)}(\mu)-\widetilde{V^{*}}(\mu)\widetilde{f(A)}(\mu)\right| \\
&\leq \int_{m-0}^{M}\left|\widetilde{V^{*}E_{\lambda}}(\mu)-\widetilde{V^{*}}(\mu)\widetilde{E_{\lambda}}(\mu)\right|df(\lambda) \\
&\leq \left(1-\left|\widetilde{V}(\mu)\right|^{2}\right)^{\frac{1}{2}}\int_{m-0}^{M}\left(\widetilde{E_{\lambda}}(\mu)\left(1-\widetilde{E_{\lambda}}(\mu)\right)\right)^{\frac{1}{2}}df(\lambda) \\
&\leq \left(1-\left|\widetilde{V}(\mu)\right|^{2}\right)^{\frac{1}{2}}\left(f(M)-\widetilde{f(A)}(\mu)\right)^{\frac{1}{2}}\left(\widetilde{f(A)}(\mu)-f(m)\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left[f(M)-f(m)\right]\left(1-\left|\widetilde{V}(\mu)\right|^{2}\right)^{\frac{1}{2}}\leq\frac{1}{4}\left|\Gamma-\gamma\right|\left[f(M)-f(m)\right].
\end{aligned}$$

Proof (i) By the Schwarz inequality in $\mathcal{H}$ we have for any $u,v,e\in\mathcal{H}$ with $\|e\|=1$ that

$$\left|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle\right|\leq\left(\|u\|^{2}-\left|\langle u,e\rangle\right|^{2}\right)^{\frac{1}{2}}\left(\|v\|^{2}-\left|\langle v,e\rangle\right|^{2}\right)^{\frac{1}{2}}. \qquad (6)$$

By putting $e=\widehat{k}_{\mu}$, $u=E_{\lambda}\widehat{k}_{\mu}$ and $v=V\widehat{k}_{\mu}$, we have from (6) that

$$\begin{aligned}
&\left|\left\langle E_{\lambda}\widehat{k}_{\mu},V\widehat{k}_{\mu}\right\rangle-\left\langle E_{\lambda}\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle\left\langle\widehat{k}_{\mu},V\widehat{k}_{\mu}\right\rangle\right| \\
&\leq\left(\left\|E_{\lambda}\widehat{k}_{\mu}\right\|^{2}-\left|\left\langle E_{\lambda}\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle\right|^{2}\right)^{\frac{1}{2}}\left(1-\left|\left\langle V\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle\right|^{2}\right)^{\frac{1}{2}} \qquad (7)
\end{aligned}$$

for every $\lambda \in [m, M]$. Since $E_\lambda^2 = E_\lambda$ and $E_\lambda \geq 0$, we have that

$$\begin{aligned}\left\|E_\lambda \widehat{k}_\mu\right\|^2 - \left|\left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|^2 &= \left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle - \left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle^2 \\ &= \left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle \left\langle \left(I_{\mathcal{H}} - E_\lambda\right) \widehat{k}_\mu, \widehat{k}_\mu\right\rangle \leq \frac{1}{4}\end{aligned}$$

for all $\lambda \in [m, M]$ and $\mu \in \Omega$. Also, by making use of the Grüss type inequality in inner product spaces obtained by Dragomir in [5], we obtain that

$$\left(\left\|V \widehat{k}_\mu\right\|^2 - \left|\left\langle V \widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{2} \left|\Gamma - \gamma\right|,$$

therefore, since V is an isometry, we have

$$\left(1 - \left|\widetilde{V}(\mu)\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{2} \left|\Gamma - \gamma\right|. \tag{8}$$

Now, combining (7) and (8) we deduce the following inequality.

$$\begin{aligned}&\left|\left\langle E_\lambda \widehat{k}_\mu, V \widehat{k}_\mu\right\rangle - \left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle \left\langle \widehat{k}_\mu, V \widehat{k}_\mu\right\rangle\right| \\ &\leq \left(\left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle \left\langle \left(I_{\mathcal{H}} - E_\lambda\right) \widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right)^{\frac{1}{2}} \left(\left\|V \widehat{k}_\mu\right\|^2 - \left|\left\langle V \widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\left(1 - \left|\widetilde{V}(\mu)\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{4} \left|\Gamma - \gamma\right|,\end{aligned}$$

hence,

$$\begin{aligned}&\left|\widetilde{V^* E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu) \widetilde{V^*}(\mu)\right| \\ &\leq \left(\widetilde{E}_\lambda(\mu)\left(1 - \widetilde{E}_\lambda(\mu)\right)\right)^{\frac{1}{2}} \left(1 - \left|\widetilde{V}(\mu)\right|^2\right)^{\frac{1}{2}} \frac{1}{2}\left(1 - \left|\widetilde{V}(\mu)\right|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \left|\Gamma - \gamma\right|\end{aligned} \tag{9}$$

for any $\lambda \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\, dv(t)\right| \leq \max_{t \in [a,b]} \left|p(t)\right| \bigvee_m^M (v),$$

where $\overset{M}{\underset{m}{\vee}}(v)$ denotes the total variation of v on $[a,b]$. Utilizing this property of the Riemann-Stieltjes integral and the inequality (9), we have

$$\begin{aligned}
&\left|\int_{m-0}^{M}\left[\left\langle E_\lambda \widehat{k}_\mu, V\widehat{k}_\mu\right\rangle - \left\langle E_\lambda \widehat{k}_\mu, \widehat{k}_\mu\right\rangle \left\langle \widehat{k}_\mu, V\widehat{k}_\mu\right\rangle\right] df(\lambda)\right| \\
&\le \max_{\lambda\in[m,M]}\left|\widetilde{V^*E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu)\widetilde{V^*}(\mu)\right| \overset{M}{\underset{m}{\vee}}(f) \\
&\le \max_{\lambda\in[m,M]}\left(\widetilde{E}_\lambda(\mu)\left(1-\widetilde{E}_\lambda(\mu)\right)\right)^{\frac{1}{2}}\left(1-\left|\widetilde{V}(\mu)\right|^2\right)^{\frac{1}{2}} \overset{M}{\underset{m}{\vee}}(f) \\
&\le \frac{1}{2}\left(1-\left|\widetilde{V}(\mu)\right|^2\right)^{\frac{1}{2}} \overset{M}{\underset{m}{\vee}}(f) \le \frac{1}{4}\left|\Gamma-\gamma\right| \overset{M}{\underset{m}{\vee}}(f) \qquad (10)
\end{aligned}$$

for $\mu\in\Omega$ and V as in the assumptions of the theorem. Now, integrating by parts in the Riemann-Stieltjes integral and making use of the spectral representation (3), we have

$$\begin{aligned}
&\int_{m-0}^{M}\left[\widetilde{V^*E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu)\widetilde{V^*}(\mu)\right] df(\lambda) \\
&= \left[\widetilde{V^*E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu)\widetilde{V^*}(\mu)\right] f(\lambda)\Big|_{m-0}^{M} \\
&- \int_{m-0}^{M} f(\lambda)\, d\left[\widetilde{V^*E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu)\widetilde{V^*}(\mu)\right] \\
&= \widetilde{V^*}(\mu)\int_{m-0}^{M} f(\lambda)\, d\widetilde{E}_\lambda(\mu) - \int_{m-0}^{M} f(\lambda)\, d\widetilde{E}_\lambda(\mu) \\
&= \widetilde{V^*}(\mu)\widetilde{f(A)}(\mu) - \widetilde{V^*f(A)}(\mu) \qquad (11)
\end{aligned}$$

which together with (10) produces the desired result (4).

(ii) Recall that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e., $|f(s)-f(t)|\le L|s-t|$ for any $t,s\in[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt.$$

Therefore, applying this property of the Riemann-Stieltjes integral we see from (9) that

$$\begin{aligned}
&\left|\int_{m-0}^{M}\left[\widetilde{V^{*}E_{\lambda}}(\mu)-\widetilde{E}_{\lambda}(\mu)\widetilde{V^{*}}(\mu)\right]df(\lambda)\right| \\
&\leq L\int_{m-0}^{M}\left|\widetilde{V^{*}E_{\lambda}}(\mu)-\widetilde{E}_{\lambda}(\mu)\widetilde{V^{*}}(\mu)\right| \\
&\leq L\left(1-\left|\widetilde{V^{*}}(\mu)\right|^{2}\right)^{\frac{1}{2}}\int_{m-0}^{M}\left(\widetilde{E}_{\lambda}(\mu)\left(1-\widetilde{E}_{\lambda}(\mu)\right)\right)^{\frac{1}{2}}d\lambda. \qquad (12)
\end{aligned}$$

So, if we use the Cauchy-Bunyakovsky-Schwarz integral inequality and the spectral representation (3), we finally have

$$\begin{aligned}
&\int_{m-0}^{M}\left(\widetilde{E}_{\lambda}(\mu)\left(1-\widetilde{E}_{\lambda}(\mu)\right)\right)^{\frac{1}{2}}d\lambda \\
&\leq\left[\int_{m-0}^{M}\widetilde{E}_{\lambda}(\mu)\,d\lambda\right]^{\frac{1}{2}}\left[\int_{m-0}^{M}\left(1-\widetilde{E}_{\lambda}(\mu)\right)d\lambda\right]^{\frac{1}{2}} \\
&=\left[\widetilde{E}_{\lambda}(\mu)\,\lambda\,|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\widetilde{E}_{\lambda}(\mu)\right]^{\frac{1}{2}} \\
&\cdot\left[\left(1-\widetilde{E}_{\lambda}(\mu)\right)\lambda\,|_{m-0}^{M}-\int_{m-0}^{M}\lambda d\left(1-\widetilde{E}_{\lambda}(\mu)\right)\right] \\
&=\left(M-\widetilde{A}(\mu)\right)^{1/2}\left(\widetilde{A}(\mu)-m\right)^{1/2}. \qquad (13)
\end{aligned}$$

Now, by utilizing (13), (12), and (11), we have the first three inequalities in (5). The fourth inequality follows from the fact that

$$\begin{aligned}
\left(M-\widetilde{A}(\mu)\right)\left(\widetilde{A}(\mu)-m\right)&\leq\frac{1}{4}\left[\left(M-\widetilde{A}(\mu)\right)+\left(\widetilde{A}(\mu)-m\right)\right]^{2} \\
&=\frac{1}{4}(M-m)^{2}.
\end{aligned}$$

The last part is obtained from (8).

(iii) Further, from the theory of Riemann-Stieltjes integral, it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int\limits_a^b p(t)\,dv(t)$ and $\int\limits_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int\limits_a^b p(t)\,dv(t)\right| \leq \int\limits_a^b |p(t)|\,dv(t)\,.$$

By considering this property and the inequality (9), we have that

$$\left|\int\limits_{m-0}^{M} \left[\widetilde{V^* E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu)\,\widetilde{V^*}(\mu)\right] df(\lambda)\right|$$

$$\leq \int\limits_{m-0}^{M} \left|\widetilde{V^* E_\lambda}(\mu) - \widetilde{E}_\lambda(\mu)\,\widetilde{V^*}(\mu)\right| f(\lambda)$$

$$\leq \left(1 - \left|\widetilde{V}(\mu)\right|^2\right)^{1/2} \int\limits_{m-0}^{M} \left(\widetilde{E}_\lambda(\mu)\left(1 - \widetilde{E}_\lambda(\mu)\right)\right)^{1/2} df(\lambda)\,.$$

Finally, applying again the Cauchy-Bungakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic integrators and the spectral representation (3), we have

$$\int\limits_{m-0}^{M} \left(\widetilde{E}_\lambda(\mu)\left(1 - \widetilde{E}_\lambda(\mu)\right)\right)^{1/2} df(\lambda)$$

$$\leq \left[\int\limits_{m-0}^{M} \widetilde{E}_\lambda(\mu)\,df(\lambda)\right]^{1/2} \cdot \left[\int\limits_{m-0}^{M} \left(1 - \widetilde{E}_\lambda(\mu)\right) df(\lambda)\right]^{1/2}$$

$$= \left[\widetilde{E}_\lambda(\mu)\,f(\lambda)\,|_{m-0}^{M} - \int\limits_{m-0}^{M} f(\lambda)\,d\widetilde{E}_\lambda(\mu)\right]^{1/2} \cdot$$

$$\cdot \left[\left(1 - \widetilde{E}_\lambda(\mu)\right) f(\lambda)\,|_{m-0}^{M} - \int\limits_{m-0}^{M} f(\lambda)\,d\left(1 - \widetilde{E}_\lambda(\mu)\right)\right]^{1/2}$$

$$= \left(f(M) - \widetilde{f(A)}(\mu)\right)^{1/2} \left(\widetilde{f(A)}(\mu) - f(m)\right)^{1/2}$$

$$\leq \frac{1}{2}\left[f(M) - f(m)\right]$$

and the proof is complete. □

Remark 2.2 *By considering the proof of the previous theorem, it can be easily deduced that instead of an isometry, any bounded linear operator on* $\mathcal{H}(\Omega)$ *can be used. We considered an isometry purely for simplicity.*

Recall that the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ is called standard in the sense of Nordgren and Rosenthal [14] if $\hat{k}_\lambda$ weakly converges to zero whenever λ tends to the boundary point of Ω.

Proposition 2.3 *Let A, B be two self-adjoint operators on the standard RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ with spectra $Sp(A)$, $Sp(B) \subset [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be the spectral family of A. Assume that $g : [m, M] \to \mathbb{R}$ is a continuous function and denote*

$$n := \min_{t\in[m,M]} g(t) \ \text{ and } N := \max_{t\in[m,M]} g(t). \tag{14}$$

If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$ and $g(B)$ is a compact operator on $\mathcal{H}$, then

$$\lim_{\mu\to\partial\Omega} \left| \widetilde{g(B)^* f(A)}(\mu) - \widetilde{f(A)}(\mu)\,\widetilde{g(B)}(\mu) \right| = 0. \tag{15}$$

Proof It follows from (14) that $n \le \left\langle g(B)\widehat{k}_\mu, \widehat{k}_\mu \right\rangle \le N$ which implies that $\left\langle g(B)\widehat{k}_\mu - n\widehat{k}_\mu, M\widehat{k}_\mu - g(B)\widehat{k}_\mu \right\rangle \ge 0$ for all $\mu \in \Omega$. So, applying Theorem 2.1 for $y = B\widehat{k}_\mu$, $\Gamma = N$ and $r = n$, we deduce that

$$\begin{aligned} &\left| \widetilde{g(B)^* f(A)}(\mu) - \widetilde{f(A)}(\mu)\,\widetilde{g(B)}(\mu) \right| \\ &\le \frac{1}{2}\left(\left\| g(B)\widehat{k}_\mu \right\|^2 - \left| \widetilde{g(B)}(\mu) \right|^2 \right)^{1/2} \bigvee_m^M (f) \end{aligned}$$

for all $\mu \in \Omega$. Since $g(B)$ is a compact operator on the standard RKHS $\mathcal{H}$, we have that $\left\| g(B)\widehat{k}_\mu \right\| \to 0$ and $\left\| \widetilde{g(B)}(\mu) \right\| \to 0$ as $\mu \to \partial\Omega$, which implies the desired result (15). □

3. Reverses of Schwarz inequality and applications

A subset $\mathcal{M}(\Omega)$ in $\mathcal{H}(\Omega)$ is called the multiplier for the space $\mathcal{H}(\Omega)$ if $\mathcal{M}(\Omega)\mathcal{H}(\Omega) \subset \mathcal{H}(\Omega)$, i.e., $fg \in \mathcal{H}(\Omega)$ for all $f \in \mathcal{M}(\Omega)$ and $g \in \mathcal{H}(\Omega)$. Let $f \in \mathcal{M}(\Omega)$ be fixed. Then, it follows from the closed graph theorem that the multiplication operator M_f defined on $\mathcal{H}(\Omega)$ by $M_f g = fg$, $g \in \mathcal{H}(\Omega)$ is a bounded linear operator. Operator M_f is characterized by the property that $M_f^*\widehat{k}_\lambda = \overline{f(\lambda)}\widehat{k}_\lambda$, $\lambda \in \Omega$. It is elementary to see that $\overline{f(\lambda)} = \widetilde{M_f^*}(\lambda)$. However, this property is not satisfied for any operator A in $\mathcal{B}(\mathcal{H}(\Omega))$, and therefore it is natural and important to estimate the difference $\left\| A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda \right\|$. It is well known that for any Toeplitz operator T_φ, $\varphi \in L^\infty(\partial\mathbb{D})$, on the Hardy space $H^2(\mathbb{D})$ $\left\| T_\varphi\widehat{k}_\lambda - \widetilde{T}_\varphi(\lambda)\widehat{k}_\lambda \right\|$ vanishes on the boundary $\partial\mathbb{D}$ whenever λ radially tends to the point $e^{it} \in \partial\mathbb{D}$ for almost all $t \in [0, 2\pi]$ (see Engliš [9]). However, this is not true in general for Bergman space Toeplitz operators (see ([9], Question 1)).

Following Otachel [15], note that Dragomir (see [6], Theorem 1.6) obtained the following reverses of the Schwarz inequality in the Hilbert space H

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \le \frac{|\Gamma - \gamma|^2}{4}\|y\|^4, \ x, y \in H, \tag{16}$$

and

$$\|x\|\|y\| \le \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}|\langle x, y\rangle|, \ x, y \in H, \tag{17}$$

if scalars Γ, γ satisfy

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0 \tag{18}$$

or, equivalently,

$$\left\| x - \frac{\Gamma + \gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| . \tag{19}$$

In case of (17), it should be additionally assumed $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. Equivalently, one can reformulate (17) to

$$\begin{aligned} \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &\leq \left| \frac{\Gamma - \gamma}{\Gamma + \gamma} \right|^2 \|x\|^2 \|y\|^2 \\ &\leq \frac{|\Gamma - \gamma|^2}{4 \operatorname{Re}(\Gamma\overline{\gamma})} |\langle x, y\rangle|^2 , \ x, y \in H. \end{aligned} \tag{20}$$

Note that Grüss type inequalities are variants of Schwar's inequality for specific Hermitian forms. They provide upper bounds for the quantity

$$\left| \langle x, y\rangle - \left(\left(\langle x, v\rangle \langle v, y\rangle \right) / \|v\|^2 \right) \right| .$$

It is known ([7], Theorem 10) that for vectors x, y and $v \neq 0$

$$\left| \langle x, y\rangle - \frac{\langle x, v\rangle \langle v, y\rangle}{\|v\|^2} \right| \leq \frac{\sqrt{\left(\|x\|^2 \|v\|^2 - |\langle x, v\rangle|^2 \right) \left(\|y\|^2 \|v\|^2 - |\langle y, v\rangle|^2 \right)}}{\|v\|^2}. \tag{21}$$

In this way, reverse Schwarz inequalities automatically form Grüss type inequalities. The related Grüss type inequalities for (16) and (20) are as follows (see ([6], Th. $15, 22$) and ([8], Th. $4.1 - 4.2$)

$$\left| \langle x, y\rangle - \frac{\langle x, v\rangle \langle v, x\rangle}{\|v\|^2} \right| \leq \frac{1}{4} |\Phi - \varphi| \, |\Gamma - \gamma| \, \|v\|^2 , \tag{22}$$

$$\begin{aligned} \left| \langle x, y\rangle - \frac{\langle x, v\rangle \langle v, x\rangle}{\|v\|^2} \right| &\leq \left| \frac{\Phi - \varphi}{\Phi + \varphi} \right| \left| \frac{\Gamma - \gamma}{\Gamma + \gamma} \right| \|x\| \, \|y\| \\ &\leq \frac{1}{4} \frac{|\Phi + \varphi| \, |\Gamma + \gamma|}{\sqrt{\operatorname{Re}(\Phi\overline{\varphi}) \operatorname{Re}(\Gamma\overline{\gamma})}} \frac{\langle x, v\rangle \langle v, x\rangle}{\|v\|^2} \end{aligned} \tag{23}$$

respectively, if $\operatorname{Re}\langle \Phi v - x, x - \varphi v\rangle \geq 0$ and $\operatorname{Re}\langle \Gamma v - y, y - \gamma v\rangle \geq 0$, where $\Phi, \varphi, \Gamma, \gamma \in \mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) and $x, y, v \in H$ with $v \neq 0$. For the last inequality, additionally, $\operatorname{Re}(\Phi\overline{\varphi}), \operatorname{Re}(\Gamma\overline{\gamma}) > 0$.

Now, the following results are immediate from the above inequalities, in case $H = \mathcal{H}(\Omega)$.

Proposition 3.1 *Let $A : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ be an operator. Then*

$$\left\| A\widehat{k}_\lambda - \widetilde{A}(\lambda) \widehat{k}_\lambda \right\| \leq \frac{1}{2} |\Gamma - \gamma| , \tag{24}$$

for all $\lambda \in \Omega$ and all scalers Γ, γ satisfying condition (18) (or, equivalently, condition (19)).

Proof Since $A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda \perp \widehat{k}_\lambda$ for all $\lambda \in \Omega$, it is elementary to get that

$$0 \le \left\|A\widehat{k}_\lambda\right\|^2 - \left|\widetilde{A}(\lambda)\right|^2 = \left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\|^2, \lambda \in \Omega,$$

that is

$$\left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\| = \sqrt{\left\|A\widehat{k}_\lambda\right\|^2 - \left|\widetilde{A}(\lambda)\right|^2},\ \lambda \in \Omega. \tag{25}$$

Now, we put $x = A\widehat{k}_\lambda$ and $y = \widehat{k}_\lambda$ in (16) and get that

$$\left\|A\widehat{k}_\lambda\right\|^2 - \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \le \frac{|\Gamma-\gamma|^2}{4}\left\|\widehat{k}_\lambda\right\|^4 = \frac{|\Gamma-\gamma|^2}{4},$$

hence, from this by using (25) we deduce inequality (24), as desired. □

Proposition 3.2 *Let $H = \mathcal{H}(\Omega)$ and $A \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\| \le \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right| \|A\|_{\mathrm{Ber}}$$

and

$$\|A\|_{\mathrm{Ber}} \le \frac{|\Gamma+\gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\operatorname{ber}(A)$$

for all scalers Γ, γ satisfying (20).

Proof In fact, let us put $x = A\widehat{k}_\lambda$ and $y = \widehat{k}_\lambda$ in (20). Then by dividing both sides of the inequalities in (20) by $\|k_\lambda\|^2$ (which is nonzero by the assumption on the RKHS), we have that

$$\left\|A\widehat{k}_\lambda\right\|^2 - \left|\left\langle A\widehat{k}_\lambda, \frac{k_\lambda}{\|k_\lambda\|}\right\rangle\right|^2 \le \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right|^2 \left\|A\widehat{k}_\lambda\right\|^2$$

and

$$\left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right|^2 \left\|A\widehat{k}_\lambda\right\|^2 \le \frac{|\Gamma-\gamma|^2}{4\operatorname{Re}(\Gamma\overline{\gamma})}\left|\left\langle A\widehat{k}_\lambda, \frac{k_\lambda}{\|k_\lambda\|}\right\rangle\right|^2$$

for all $\lambda \in \Omega$. These imply that

$$\sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda\right\| \le \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right| \sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\| = \left|\frac{\Gamma-\gamma}{\Gamma+\gamma}\right| \|A\|_{\mathrm{Ber}}$$

and

$$\begin{aligned}\|A\|_{\mathrm{Ber}} = \sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\| &\le \frac{1}{2}\left|\frac{\Gamma+\gamma}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\right| \sup_{\lambda\in\Omega}\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\ &= \frac{1}{2}\frac{|\Gamma+\gamma|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\operatorname{ber}(A),\end{aligned}$$

which gives the desired result of the proposition. □

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