Clairaut invariant Riemannian maps with Kähler structure

Akhilesh YADAV, Kiran MEENA
Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India

Received: 04.08.2021 • Accepted/Published Online: 23.02.2022 • Final Version: 11.03.2022

Abstract: In this paper, we study Clairaut invariant Riemannian maps from Kähler manifolds to Riemannian manifolds, and from Riemannian manifolds to Kähler manifolds. We find necessary and sufficient conditions for the curves on the total spaces and base spaces of invariant Riemannian maps to be geodesic. Further, we obtain necessary and sufficient conditions for invariant Riemannian maps from Kähler manifolds to Riemannian manifolds to be Clairaut invariant Riemannian maps. Moreover, we obtain a necessary and sufficient condition for invariant Riemannian maps from Riemannian manifolds to Kähler manifolds to be Clairaut invariant Riemannian maps. We also give nontrivial examples of Clairaut invariant Riemannian maps whose total manifolds or base manifolds are Kähler manifolds.

Key words: Riemannian manifold, Kähler manifold, Clairaut Riemannian submersion, invariant Riemannian map, Clairaut Riemannian map

1. Introduction

In 1992, Fischer introduced Riemannian map between Riemannian manifolds in [6] as a generalization of the notion of an isometric immersion and Riemannian submersion. The geometry of Riemannian submersions have been discussed in [5]. We note that a remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation \( \|F_t\|^2 = \text{rank}F \), which is a bridge between geometric optics and physical optics [6]. The eikonal equation of geometrical optics was solved by using Cauchy’s method of characteristics. In [6] Fischer also proposed an approach to build a quantum model and he pointed out the success of such a program of building a quantum model of nature using Riemannian maps would provide an interesting relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side, and Maxwell’s equation, Shrödinger’s equation and their proposed generalization on the physical side.

In [20], as an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases and also obtained some fundamental properties of this map. In [9], B. Şahin introduced holomorphic Riemannian maps as a generalization of holomorphic submersions and holomorphic submanifolds and in [12], B. Şahin studied Riemannian submersions from almost Hermitian manifolds. The notion of invariant Riemannian maps has been introduced by B. Şahin in [10] as a generalization of invariant immersion of almost Hermitian manifolds and holomorphic Riemannian submersions. In [10], B. Şahin introduced invariant and anti-invariant Riemannian maps to Kähler manifolds. The notion of invariant Riemannian maps to Kähler manifolds is a generalization of submanifolds of Kähler manifolds. It can be see
that every holomorphic Riemannian map is an invariant Riemannian map but an invariant Riemannian map may not be holomorphic Riemannian map. Further, holomorphic, antiinvariant and semiinvariant Riemannian maps with Kähler structure were studied in [1–3, 10, 11, 15, 17–19].

In elementary differential geometry, if θ is the angle between the velocity vector of a geodesic and a meridian, and r is the distance to the axis of a surface of revolution, then Clairaut’s relation states that \( r \sin \theta \) is constant. In 1972 ([5], p. 29), Bishop defined Clairaut Riemannian submersion with connected fibers and gave a necessary and sufficient condition for Riemannian submersion to be Clairaut Riemannian submersion, also in [7], B. Şahin studied Clairaut submersions. Further in [16], B. Şahin and H. M. Taştan studied Clairaut submersions from almost Hermitian manifolds. In [13], B. Şahin introduced Clairaut Riemannian maps, in which he obtained necessary and sufficient conditions for Riemannian map to be Clairaut Riemannian map. In [22], present authors studied Clairaut antiinvariant Riemannian maps from Kähler manifolds and investigated some geometric properties. In [21], we defined Clairaut Riemannian maps by using geodesic curves [4] on base spaces and obtained necessary and sufficient conditions for Riemannian maps to be Clairaut Riemannian maps. Also we obtained interesting result that such Clairaut Riemannian maps are umbilical maps.

In this paper we study Clairaut invariant Riemannian maps from Kähler manifolds, and to Kähler manifolds. In Section 2, we give some basic information about Riemannian map which is needed for this paper. In Section 3, we obtain necessary and sufficient conditions for invariant Riemannian maps to be Clairaut invariant Riemannian maps whose total manifolds are Kähler manifolds and investigate geometric properties of such maps with a nontrivial example. Finally, in Section 4, we obtain a necessary and sufficient condition for invariant Riemannian maps to be Clairaut invariant Riemannian maps whose base manifolds are Kähler manifolds and give a nontrivial example of such maps.

2. Preliminaries

In this section, we recall the notion of Riemannian maps between Riemannian manifolds and give a brief review of basic facts of Riemannian maps.

Let \( F : (M^m, g_M) \to (N^n, g_N) \) be a smooth map between Riemannian manifolds such that \( 0 < \text{rank} F \leq \min\{m, n\} \), where \( \dim(M) = m \) and \( \dim(N) = n \). Then we denote the kernel space of \( F_p \) by \( \ker F_p \) at \( p \in M \) and consider the orthogonal complementary space \( \mathcal{H}_p = (\ker F_p) \perp \) to \( \ker F_p \) in \( T_p M \). Then the tangent space \( T_p M \) of \( M \) at \( p \) has the decomposition \( T_p M = (\ker F_p) \oplus (\ker F_p) \perp = \nu_p \oplus \mathcal{H}_p \). We denote the range of \( F \) by \( \text{range} F \) at \( p \in M \) and consider the orthogonal complementary space \( (\text{range} F_p) \perp \) to \( \text{range} F_p \) in the tangent space \( T_{F(p)} N \) of \( N \) at \( F(p) \in N \). Since \( \text{rank} F \leq \min\{m, n\} \), we have \( (\text{range} F_p) \perp \neq \{0\} \). Thus the tangent space \( T_{F(p)} N \) of \( N \) at \( F(p) \in N \) has the decomposition \( T_{F(p)} N = (\text{range} F_p) \oplus (\text{range} F_p) \perp \).

Now, the map \( F : (M^m, g_M) \to (N^n, g_N) \) is called Riemannian map at \( p \in M \) if the horizontal restriction \( F_p^h : (\ker F_p) \perp \to (\text{range} F_p) \) is a linear isometry between the inner product spaces \( ((\ker F_p) \perp, g_M(p)) ((\ker F_p) \perp) \) and \( (\text{range} F_p, g_N(F(p))))((\text{range} F_p)) \), where \( F(p) = p_1 \). In other words, \( F \) satisfies the equation

\[
g_N(F_* X, F_* Y) = g_M(X, Y),
\]

for all \( X, Y \) vector field tangent to \( \Gamma(\ker F_p) \perp \). It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with \( \ker F_* = \{0\} \) and \( (\text{range} F_*) \perp = \{0\} \).

1021
The O’Neill tensors $A$ and $T$ defined in [5] as

\[
A_E F = \mathcal{H}\nabla^M_{\nu_E} v F + \nu \nabla^M_{\nu_E} \mathcal{H} F, \tag{2.2}
\]

\[
T_E F = \mathcal{H}\nabla^M_{\nu_E} v F + \nu \nabla^M_{\nu_E} \mathcal{H} F, \tag{2.3}
\]

for all vector fields $E, F$ on $M$, where $\nabla^M$ is the Levi–Civita connection of $g_M$ and $\nu$, $\mathcal{H}$ denote the projections to vertical subbundle and horizontal subbundle, respectively. For any $E \in \Gamma(TM)$, $T_E$ and $A_E$ are skew-symmetric operators on $(\Gamma(TM), g_M)$ reversing the horizontal and the vertical distributions. It is also easy to see that $T$ is vertical, $T_E = T_{\nu E}$ and $A$ is horizontal, $A_E = A_{\mathcal{H} E}$. We note that the tensor field $T$ satisfies $T_U W = T_W U$, for all $U, W \in \Gamma(\ker F_*)$.

Now, from (2.2) and (2.3), we have

\[
\nabla^M_v W = T_v W + \hat{\nabla}_v W, \tag{2.4}
\]

\[
\nabla^M_X V = A_X V + \nu \nabla^M_X V, \tag{2.5}
\]

\[
\nabla^M_Y V = A_X Y + \mathcal{H} \nabla^M_X Y, \tag{2.6}
\]

for all $X, Y \in \Gamma(\ker F_*)^\perp$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_v W = \nu \nabla^M_v W$.

Also, a Riemannian map is a Riemannian map with totally umbilical fibers if [11]

\[
T_U V = g_M(U, V) H, \tag{2.7}
\]

for all $U, V \in \Gamma(\ker F_*)$, where $H$ is the mean curvature vector field of fibers.

Let $F : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Then the differential $F_*$ of $F$ can be viewed as a section of bundle Hom$(TM, F^{-1}TN) \to M$, where $F^{-1}TN$ is the pullback bundle whose fibers at $p \in M$ is $(F^{-1}TN)_p = T_{F(p)} N$, $p \in M$. The bundle Hom$(TM, F^{-1}TN)$ has a connection $\nabla$ induced from the Levi–Civita connection $\nabla^M$ and the pullback connection $\nabla^N$. Then the second fundamental form of $F$ is given by [8]

\[
(\nabla^N F_*)(X_1, Y_1) = \nabla^N_{F_* X_1} F_* Y_1 - F_*(\nabla^M_{X_1} Y_1), \tag{2.8}
\]

for all $X_1, Y_1 \in \Gamma(TM)$, where $\nabla^N_{F_* X_1} F_* Y_1 \circ F = \nabla^N_{F_*, X_1} F_* Y_1$. It is known that the second fundamental form is symmetric. In [10] B. Şahin proved that $(\nabla^N F_*)(X, Y)$ has no component in $\text{range} F_*$, for all $X, Y \in \Gamma(\ker F_*)^\perp$.

More precisely, we have

\[
(\nabla^N F_*)(X, Y) \in \Gamma(\text{range} F_*)^\perp. \tag{2.9}
\]

For any vector field $X_1$ on $M$ and any section $V$ of $(\text{range} F_*)^\perp$, we have $\nabla^N_{X_1} V$, which is the orthogonal projection of $\nabla^N_{X_1} V$ on $(\text{range} F_*)^\perp$, where $\nabla^F$ is linear connection on $(\text{range} F_*)^\perp$ such that $\nabla^F g_N = 0$.

Now, for a Riemannian map $F$ we define $\mathcal{S}_V$ as ([14], p. 188)

\[
\nabla^N_{F_* X_1} V = -\mathcal{S}_V F_* X + \nabla^F_{X_1} V, \tag{2.10}
\]
where $\nabla^N$ is the Levi–Civita connection on $N$, $S_V F_* X$ is the tangential component (a vector field along $F$) of $\nabla^N_{F_* X} V$. Thus at $p \in M$, we have $\nabla^N_{F_* X} V(p) \in T_{F(p)} N$, $S_V F_* X \in F_p (T_p M)$ and $\nabla^N_{F_* X} V(p) \in (F_p (T_p M))^\perp$.

It is easy to see that $S_V F_* X$ is bilinear in $V$, and $F_* X$ at $p$ depends only on $V$ and $F_p X_p$.

**Lemma 2.1** [15] Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then $F$ isumbilical Riemannian map if and only if

$$\langle \nabla F_*, (X, Y) \rangle = g_M(X, Y) H_2, \quad \text{(2.11)}$$

for $X, Y \in \Gamma(\ker F_*)$ and $H_2$ is, nowhere zero, vector field on $(\text{range} F_*)^\perp$.

Let $(M, g_M)$ be an almost Hermitian manifold [23], then $M$ admits a tensor $J$ of type $(1, 1)$ on $M$ such that $J^2 = -I$ and

$$g_M(J X_1, Y_1) = g_M(X_1, Y_1), \quad \text{(2.12)}$$

for all $X_1, Y_1 \in \Gamma(TM)$. An almost Hermitian manifold $M$ is called Kähler manifold if

$$\langle \nabla^M_{X_1} J Y_1 \rangle = 0, \quad \text{(2.13)}$$

for all $X_1, Y_1 \in \Gamma(TM)$, where $\nabla^M$ is a Levi–Civita connection on $M$.

### 3. Clairaut invariant Riemannian maps from Kähler manifolds

In this section, we introduce Clairaut invariant Riemannian maps from Kähler manifolds and investigate the geometry of such maps with a nontrivial example. For arbitrary Clairaut invariant Riemannian maps, we present the following definitions:

**Definition 3.1** [9] Let $F : (M, g_M, J) \to (N, g_N)$ be a Riemannian map from an almost Hermitian manifold $M$ to a Riemannian manifold $N$ with almost complex structure $J$. We say that $F$ is an invariant Riemannian map at $p \in M$ if $J(\ker F_* p) = \ker F_* p$. If $F$ is an invariant Riemannian map for every $p \in M$ then $F$ is called an invariant Riemannian map, i.e. the vertical distribution is invariant with respect to $J$, or $J(\ker F_*) = \ker F_*$. In this case, we have the horizontal distribution $(\ker F_*)^\perp$ is also invariant with respect to $J$, i.e. $J(\ker F_*)^\perp = (\ker F_*)^\perp$.

The notion of Clairaut Riemannian map was defined by B. Şahin in [13]. According to the definition, a Riemannian map $F : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $r : M \to \mathbb{R}^+$ such that for every geodesic $\alpha$ on $M$, the function $(r \circ \alpha) \sin \theta$ is constant, where, for all $t$, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

**Theorem 3.2** [13] Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian map with connected fibers, then $F$ is Clairaut Riemannian map with $r = e^f$ if and only if each fiber is umbilical and has mean curvature vector field $H = -\text{grad} f$, where $f$ is a smooth function on $M$ and $\text{grad} f$ is the gradient of the function $f$ with respect to $g_M$.

**Definition 3.3** An invariant Riemannian map from Kähler manifold to a Riemannian manifold is called Clairaut invariant Riemannian map if it satisfies the condition of Clairaut Riemannian map.
The origin of the notion of Clairaut Riemannian maps comes from geodesic curves on a surface. Therefore we are going to find necessary and sufficient conditions for the curves on the total spaces and the base spaces to be geodesic.

**Lemma 3.4** Let $F: (M, g_M, J) \rightarrow (N, g_N)$ be an invariant Riemannian map from a Kähler manifold $M$ to a Riemannian manifold $N$. Let $\alpha : I \rightarrow M$ be a regular curve and $X(t), U(t)$ denote the horizontal and vertical components of its tangent vector field. Then $\alpha$ is a geodesic curve on $M$ if and only if

\[
\nu \nabla^M_{\dot{\alpha}} JU + A_X JX + T_U JX = 0, \tag{3.1}
\]

\[
\mathcal{H} \nabla^M_{\dot{\alpha}} JX + A_X JU + T_U JU = 0. \tag{3.2}
\]

**Proof** Let $\alpha : I \rightarrow M$ be a regular curve on $M$. Then from (2.13), we get $\nabla^M_{\dot{\alpha}} \dot{\alpha} = -J \nabla^M_{\dot{\alpha}} \dot{J} \dot{\alpha}$. Since $\dot{\alpha}(t) = X(t) + U(t)$, for all $t \in I$, so we can write $\nabla^M_{\dot{\alpha}} \dot{\alpha} = -J \nabla^M_{X + U}(X + U)$ and obtain

\[
\nabla^M_{\dot{\alpha}} \dot{\alpha} = -J(\nabla^M_N JX + \nabla^M_U JX + \nabla^M_X JU + \nabla^M_U JU). \tag{3.3}
\]

Using (2.4), (2.5), (2.6) in (3.3), we get

\[
\nabla^M_{\dot{\alpha}} \dot{\alpha} = -J(A_X JX + T_U JX + \nu \nabla^M_{\dot{\alpha}} JU + \nu \nabla^M_U JU + \mathcal{H} \nabla^M_{\dot{\alpha}} JX + \mathcal{H} \nabla^M_U JX + A_X JU + T_U JU),
\]

which implies

\[
\nabla^M_{\dot{\alpha}} \dot{\alpha} = -J(A_X JX + T_U JX + \nu \nabla^M_{\dot{\alpha}} JU + \nu \nabla^M_U JU + \mathcal{H} \nabla^M_{\dot{\alpha}} JX + A_X JU + T_U JU). \tag{3.4}
\]

Applying $J$ both sides of (3.4) and take the vertical and horizontal components, we get

\[
\nu J \nabla^M_{\dot{\alpha}} \dot{\alpha} = A_X JX + T_U JX + \nu \nabla^M_{\dot{\alpha}} JU,
\]

\[
\mathcal{H} J \nabla^M_{\dot{\alpha}} \dot{\alpha} = A_X JU + T_U JU + \mathcal{H} \nabla^M_{\dot{\alpha}} JX.
\]

Now, $\alpha$ is a geodesic on $M$ if and only if $\nu J \nabla^M_{\dot{\alpha}} \dot{\alpha} = 0$ and $\mathcal{H} J \nabla^M_{\dot{\alpha}} \dot{\alpha} = 0$, which completes the proof. \hfill \Box

**Theorem 3.5** Let $F: (M, g_M, J) \rightarrow (N, g_N)$ be an invariant Riemannian map from a Kähler manifold $M$ to a Riemannian manifold $N$. Then $F$ is a Clairaut invariant Riemannian map with $r = e^f$ if and only if $g_{\alpha(t)}(A_X JU + T_U JU, JX) + g_{\alpha(t)}(\nabla^M f, X) g_{\alpha(t)}(U(t), U(t)) = 0$, where $\alpha : I \rightarrow M$ is a geodesic curve on $M$ with $X(t), U(t)$ are horizontal and vertical components of $\dot{\alpha}(t)$ and $f$ is a smooth function on $M$.

**Proof** Let $\alpha : I \rightarrow M$ be a geodesic on $M$ with $U(t) = \nu \dot{\alpha}(t)$ and $X(t) = \mathcal{H} \dot{\alpha}(t)$, $\theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $X(t)$. Assuming $a = \|\dot{\alpha}(t)\|^2$, then we get

\[
g_{\alpha(t)}(X(t), X(t)) = a \cos^2 \theta(t), \tag{3.5}
\]

\[
g_{\alpha(t)}(U(t), U(t)) = a \sin^2 \theta(t). \tag{3.6}
\]
Now, differentiating (3.5), we get
\[ \frac{d}{dt}g_{M_\alpha(t)}(X(t), X(t)) = -2acost(t)sin\theta(t)\frac{d\theta}{dt} \]  
(3.7)

On the other hand using (2.12), we get
\[ \frac{d}{dt}g_{M_\alpha(t)}(X(t), X(t)) = \frac{d}{dt}g_{M_\alpha(t)}(JX, JX). \]  
(3.8)

Since \( F \) is Riemannian map, then using (2.1) in (3.8), we get
\[ \frac{d}{dt}g_{M_\alpha(t)}(X(t), X(t)) = 2g_N(F_*(\nabla^M_{\alpha JX}), F_*(JX)). \]  
(3.9)

Using (2.8) in (3.9), we obtain
\[ \frac{d}{dt}g_{M_\alpha(t)}(X(t), X(t)) = 2g_N(\nabla^N_{\alpha JX}, F_*(JX)). \]  
(3.10)

By putting \( \dot{\alpha} = X + U \) in (3.10) and then using (2.9), we get
\[ \frac{d}{dt}g_{M_\alpha(t)}(X(t), X(t)) = 2g_N(\nabla^N_{\alpha X JU}, F_*(JX)). \]  
(3.11)

By using (3.2) in (3.11), we obtain
\[ \frac{d}{dt}g_{M_\alpha(t)}(X(t), X(t)) = -2g_M(A_X JU + T_{U JU}, JX). \]  
(3.12)

Now from (3.7) and (3.12), we get
\[ g_M(A_X JU + T_{U JU}, JX) = acost(t)sin\theta(t)\frac{d\theta}{dt}. \]  
(3.13)

Moreover, \( F \) is a Clairaut Riemannian map with \( r = e^f \) if and only if \( \frac{d}{dt}(e^{f(\alpha)}sin\theta(t)) = 0 \), that is, \( e^{f(\alpha)}(cos\theta(t)\frac{d\theta}{dt} + sin\theta(t)\frac{df(\alpha)}{dt}) = 0 \). By multiplying this with nonzero factor \( asin\theta(t) \), we get
\[ -acos\theta(t)sin\theta(t)\frac{d\theta}{dt} = asin^2\theta(t)\frac{df(\alpha)}{dt}. \]  
(3.14)

Using (3.6) in (3.14), we get
\[ g_{M_\alpha(t)}(U(t), U(t))\frac{df(\alpha)}{dt} = -acos\theta(t)sin\theta(t)\frac{d\theta}{dt}. \]  
(3.15)
Thus from (3.13) and (3.15), we get
\[ g_M(A_X JU + T_U JU, JX) + g_{M_{\alpha(t)}}(U(t), U(t)) \frac{d(f \circ \alpha)}{dt} = 0, \]
which implies
\[ g_M(A_X JU + T_U JU, JX) + g_{M_{\alpha(t)}}(U(t), U(t)) g_{M_{\alpha(t)}}(\nabla^M f, \dot{\alpha}) = 0. \]
Since \( \nabla^M f \in \Gamma(\ker F^*)^\perp \) therefore above equation can be written as
\[ g_M(A_X JU + T_U JU, JX) + g_{M_{\alpha(t)}}(U(t), U(t)) g_{M_{\alpha(t)}}(\nabla^M f, X) = 0, \]
which completes the proof.

**Theorem 3.6** Let \( F : (M, g_M, J) \to (N, g_N) \) be a Clairaut invariant Riemannian map with \( r = e^f \) from a Kähler manifold \( M \) to a Riemannian manifold \( N \). Then \( T_U V = 0 \), for \( U \neq V \in \Gamma(\ker F_*) \).

**Proof** Let \( F \) be a Clairaut invariant Riemannian map from a Kähler manifold to a Riemannian manifold. Then by using (2.7) in Theorem 3.2, we get
\[ T_U V = -g_M(U, V) \text{grad} f, \tag{3.16} \]
for \( U, V \in \Gamma(\ker F_*) \), which implies
\[ g_M(T_U V, JX) = -g_M(U, V) g_M(\text{grad} f, JX), \tag{3.17} \]
for \( X \in \Gamma(\ker F_*)^\perp \). Using (2.4) and (2.12) in (3.17), we get
\[ g_M(\nabla_U JV, X) = g_M(U, V) g_M(\text{grad} f, JX). \]
Using (2.4) and (3.16) in above equation, we get
\[ g_M(U, JV) g_M(\text{grad} f, X) = -g_M(U, V) g_M(\text{grad} f, JX). \tag{3.18} \]
Interchanging the role of \( U \) and \( V \) \( (U \neq V) \), we obtain
\[ g_M(V, JU) g_M(\text{grad} f, X) = -g_M(U, V) g_M(\text{grad} f, JX). \tag{3.19} \]
From (3.19) and (3.18), we get
\[ (g_M(U, JV) - g_M(V, JU)) g_M(\text{grad} f, X) = 0. \]
Using (2.12) in above equation, we get
\[ 2g_M(U, JV) g_M(\nabla^M f, X) = 0. \]
Thus
\[ g_M(\nabla^M f, X) = 0. \tag{3.20} \]
Also, by (3.16), we get
\[ g_M(T_U V, X) = -g_M(U, V) g_M(\nabla^M f, X), \] (3.21)
for \( X \in \Gamma(\ker F_\ast) \). By using (3.20) in (3.21), we get
\[ g_M(T_U V, X) = 0. \]

Thus \( T_U V = 0 \) for \( U \neq V \in \Gamma(\ker F_\ast) \), which completes the proof. \( \square \)

**Proposition 3.7** Let \( F : (M, g_M, J) \to (N, g_N) \) be an invariant Riemannian map from Kähler manifold \( M \) to a Riemannian manifold \( N \) and \( \alpha : I \to M \) is geodesic curve on \( M \) with \( U(t) = \nu \alpha(t) \) and \( X(t) = H\alpha(t) \).

Then the curve \( \beta = F \circ \alpha \) is geodesic on \( N \) if and only if
\[ (\nabla F_\ast)(X, JX) = 0, F_\ast(\nabla^N_X JX + A_X JU + T_U JU) = 0. \] (3.22)

**Proof** Let \( \alpha : I \to M \) is geodesic on \( M \). Then by (3.2), we have
\[ H\nabla^N_X JX = -(H\nabla^N_U JX + A_X JU + T_U JU), \]
for \( U \in \Gamma(\ker F_\ast) \) and \( X \in \Gamma(\ker F_\ast) \). By applying definition of Riemannian map, we get
\[ F_\ast(\nabla^N_X JX) = -F_\ast(\nabla^N_U JX + A_X JU + T_U JU). \] (3.23)

Using (2.8) in (3.23), we get
\[ \nabla^N_X F_\ast(JX) = (\nabla F_\ast)(X, JX) - F_\ast(\nabla^N_U JX + A_X JU + T_U JU). \]

Now \( \beta \) is geodesic on \( N \) if and only if \( \nabla^N_X F_\ast(JX) = 0 \), which completes the proof. \( \square \)

**Definition 3.8** [21] A Riemannian map \( F : (M, g_M) \to (N, g_N) \) between Riemannian manifolds is called Clairaut Riemannian map if there is a function \( s : N \to \mathbb{R}^+ \) such that for every geodesic \( \beta \) on \( N \), the function \( (s \circ \beta)\sin\omega(t) \) is constant, where, \( F_\ast X \in \Gamma(\text{range}F_\ast) \) and \( V \in \Gamma(\text{range}F_\ast) \) are components of \( \dot{\beta}(t) \), and \( \omega(t) \) is the angle between \( \dot{\beta}(t) \) and \( V \) for all \( t \).

**Remark 3.9** As we know that Clairaut’s relation comes from geodesic curve. In [13], B. Şahin considered geodesic curve on the total manifold of a Riemannian map \( F \), then by using Clairaut relation fibers of \( F \) are totally umbilical. On the other hand, in Definition 3.8, we considered geodesic curve on base manifold of \( F \), then by using Clairaut’s relation \( F \) becomes totally umbilical.

**Theorem 3.10** Let \( F : (M, g_M, J) \to (N, g_N) \) be an invariant Riemannian map from a Kähler manifold \( M \) to a Riemannian manifold \( N \) and \( \alpha, \beta = F \circ \alpha \) are geodesic curves on \( M \) and \( N \), respectively. Then \( F \) is Clairaut invariant Riemannian map with \( s = e^9 \) if and only if \( g_N(\nabla^N_U F_\ast JX, F_\ast JX) = g_N(F_\ast JX, F_\ast JX) \frac{dg(\beta)}{dt} = 0 \), where \( g \) is a smooth function on \( N \) and \( F_\ast X \in \Gamma(\text{range}F_\ast), V \in \Gamma(\text{range}F_\ast) \) are components of \( \dot{\beta}(t) \).
Proof Let $\alpha : I \to M$ be a geodesic on $M$ with $U(t) = \nu \dot{\alpha}(t)$ and $X(t) = \mathcal{H} \dot{\alpha}(t)$. Let $\beta = F \circ \alpha$ be a geodesic on $N$ with $F_*X \in \Gamma(\text{range} F_*)$ and $V \in \Gamma(\text{range} F_*)^\perp$ are components of $\dot{\beta}(t)$ and $\omega(t)$ denote the angle between $\dot{\beta}$ and $V$. Assuming $b = \|\dot{\beta}(t)\|^2$, then we get
\[ g_{N\beta(t)}(V, V) = b \cos^2 \omega(t), \] (3.24)
\[ g_{N\beta(t)}(F_*X, F_*X) = b \sin^2 \omega(t). \] (3.25)

Now, differentiating (3.25), we get
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 b \sin \omega \cos \omega \frac{d\omega}{dt}. \] (3.26)

On the other hand, since $M$ is almost Hermitian manifold and using (2.1), we get
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = \frac{d}{dt} g_{N\beta(t)}(F_*JX, F_*JX), \]
which implies
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 g_{N\beta(t)}(\nabla^N_{F_*X} F_*JX, F_*JX). \] (3.27)

By putting $\dot{\beta} = F_*X + V$ in (3.27), we get
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 g_{N\beta(t)}(\nabla^N_{F_*X} F_*JX + \nabla^N_{V} F_*JX, F_*JX), \]
which implies
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 g_{N\beta(t)}(\nabla^N_{F_*X} F_*JX + \nabla^N_{V} F_*JX, F_*JX). \] (3.28)

Using (2.8) in (3.28), we get
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 g_{N\beta(t)}((\nabla F_*)(X, JX) + F_*(\nabla^M_JX) + \nabla^N_{V} F_*(JX), F_(JX)). \]

Using (3.23) in above equation, we get
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 g_{N\beta(t)}((\nabla F_*)(X, JX) - F_*(\nabla^M_JX + A_* X JU + T_U JU) + \nabla^N_{V} F_*(JX), F_(JX)). \]

Since $\beta$ is geodesic on $N$ then using (3.22) in above equation, we get
\[ \frac{d}{dt} g_{N\beta(t)}(F_*X, F_*X) = 2 g_{N\beta(t)}(\nabla^N_{V} F_*(JX), F_*(JX)). \] (3.29)

Thus from (3.26) and (3.29), we get
\[ g_{N\beta(t)}(\nabla^N_{V} F_*(JX), F_*(JX)) = b \sin \omega \cos \omega \frac{d\omega}{dt}. \] (3.30)
Moreover, $F$ is a Clairaut Riemannian map with $s = e^g$ if and only if $\frac{d}{dt}(e^{g\beta} \sin \omega) = 0$, that is, $e^{g\beta} \sin \omega \frac{d(g\beta)}{dt} + e^{g\beta} \cos \omega \frac{d\omega}{dt} = 0$. By multiplying this with nonzero factor $bsin\omega$, and using (3.25), we get

$$g_{N\beta(t)}(F_*X, F_*X) \frac{d(g \circ \beta)}{dt} = -bsin\omega \cos\omega \frac{d\omega}{dt}. \tag{3.31}$$

Thus (3.30), (3.31), (2.1) and (2.12) completes the proof. \hfill \Box

**Example 3.10** Let $M$ be an Euclidean space given by $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \neq 0, x_2 \neq 0, x_3 \neq 0, x_4 \neq 0\}$. We define the Riemannian metric $g_M$ on $M$ given by $g_M = x_1^2 dx_1^2 + x_2^2 dx_2^2 + x_3^2 dx_3^2 + dx_4^2$ and the complex structure $J$ on $M$ defined as $J(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$. Let $N = \{(y_1, y_2, y_3) \in \mathbb{R}^3\}$ be a Riemannian manifold with Riemannian metric $g_N$ on $N$ given by $g_N = dy_1^2 + y_3^2 dy_2^2 + dy_3^2$. Consider a map $F : (M, g_M, J) \to (N, g_N)$ defined by

$$F(x_1, x_2, x_3, x_4) = (0, x_3, x_4).$$

Then, we get

$$\ker F_* = \text{Span}\{U_1 = e_1, U_2 = e_2\},$$

and

$$(kerF_*)^\perp = \text{Span}\{X_1 = e_3, X_2 = e_4\},$$

where $\begin{cases} e_1 = \frac{1}{x_4} \frac{\partial}{\partial x_1}, e_2 = \frac{1}{x_4} \frac{\partial}{\partial x_2}, e_3 = \frac{1}{x_4} \frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial x_4} \end{cases}$, $\begin{cases} e_1^* = \frac{\partial}{\partial y_1}, e_2^* = \frac{1}{y_3} \frac{\partial}{\partial y_2}, e_3^* = \frac{\partial}{\partial y_3} \end{cases}$ are bases on $T_p M$ and $T_F(p)N$ respectively, for all $p \in M$. By direct computations, we can see that $F_*(X_1) = e_2^*, F_*(X_2) = e_3^*$ and $g_M(X_i, X_j) = g_N(F_*X_i, F_*X_j)$ for all $X_i, X_j \in \Gamma(\ker F_*)^\perp$. Thus $F$ is Riemannian map with $(\text{range } F_*)^\perp = \text{Span}\{e_1^*\}$. Moreover it is easy to see that $JU_1 = -U_2, JU_2 = U_1$ and $(\ker F_*)^\perp$ is also invariant with respect to $J$. Thus $F$ is an invariant Riemannian map.

Now we will find a smooth function $f$ on $M$ satisfying $T_U f = -g_M(U, U) \nabla^M f$ for all $U \in \Gamma(\ker F_*)$. Here any $U \in \Gamma(\ker F_*)$ can be written as $U = \lambda_1 U_1 + \lambda_2 U_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Since covariant derivative for vector fields $E = E_i \frac{\partial}{\partial x_i}, F = F_j \frac{\partial}{\partial x_j}$ on $M$ is defined as

$$\nabla^M_E F = E_i F_j \nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + E_i \frac{\partial F_j}{\partial x_i} \frac{\partial}{\partial x_j}, \tag{3.32}$$

where the covariant derivatives of basis vector fields $\frac{\partial}{\partial x_j}$ along $\frac{\partial}{\partial x_i}$ is defined by

$$\nabla^M_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \tag{3.33}$$

and Christoffel symbols are defined by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{Mji}}{\partial x_l} + \frac{\partial g_{Mkj}}{\partial x_i} - \frac{\partial g_{Mik}}{\partial x_j} \right). \tag{3.34}$$

Now, we have

$$g_{Mij} = \begin{pmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & x_2^2 & 0 & 0 \\ 0 & 0 & x_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g^M_{ij} = \begin{pmatrix} \frac{1}{x_1^2} & 0 & 0 & 0 \\ 0 & \frac{1}{x_2^2} & 0 & 0 \\ 0 & 0 & \frac{1}{x_3^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.35}$$
By using (3.34) and (3.35), we obtain

\[
\begin{align*}
\Gamma_{11}^1 &= 0, \Gamma_{11}^2 = 0, \Gamma_{11}^3 = 0, \Gamma_{11}^4 = -x_4, \\
\Gamma_{22}^1 &= 0, \Gamma_{22}^2 = 0, \Gamma_{22}^3 = 0, \Gamma_{22}^4 = -x_4, \\
\Gamma_{12}^1 &= 0 = \Gamma_{12}^2, \Gamma_{12}^3 = 0 = \Gamma_{12}^4. 
\end{align*}
\]  

(3.36)

By using (3.32), (3.33) and (3.36), we obtain

\[
\nabla^M_{e_1} e_1 = \nabla^M_{e_2} e_2 = -\frac{1}{x_4} \frac{\partial}{\partial x_4}, \nabla^M_{e_1} e_2 = \nabla^M_{e_2} e_1 = 0.
\]

Therefore

\[
\nabla^M_U = \nabla^M_{\lambda_1 e_1 + \lambda_2 e_2} = \lambda_1^2 \nabla^M_{e_1} e_1 + \lambda_1 \lambda_2 \nabla^M_{e_2} e_2 + \lambda_1 \lambda_2 \nabla^M_{e_1} e_1 + \lambda_2^2 \nabla^M_{e_2} e_2
\]

\[=-(\lambda_1^2 + \lambda_2^2) \left(\frac{1}{x_4} \frac{\partial}{\partial x_4}\right).
\]

Then by (2.4), we get

\[
T^U U = -(\lambda_1^2 + \lambda_2^2) \left(\frac{1}{x_4} \frac{\partial}{\partial x_4}\right).
\]

For any smooth function \( f \) on \( M \), the gradient of \( f \) with respect to the metric \( g_M \) is given by \( \nabla^M f = \sum_{i,j=1}^4 g^{ij}_M \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \). Therefore

\[
\nabla^M f = \frac{1}{x_4^2} \frac{\partial f}{\partial x_1} \frac{\partial}{\partial x_1} + \frac{1}{x_4^2} \frac{\partial f}{\partial x_2} \frac{\partial}{\partial x_2} + \frac{1}{x_4^2} \frac{\partial f}{\partial x_3} \frac{\partial}{\partial x_3} + \frac{\partial f}{\partial x_4} \frac{\partial}{\partial x_4}
\]

Hence \( \nabla^M f = \frac{1}{x_4} \frac{\partial f}{\partial x_4} \) for the function \( f = \log(x_4) \). Then it is easy to verify that \( T^U U = -g_M(U,U)\nabla^M f \), where \( g_M(U,U) = (\lambda_1^2 + \lambda_2^2) \), for any vertical vector field \( U \). Thus by Theorem 3.2, we see that \( F \) is Clairaut invariant Riemannian map.

4. Clairaut invariant Riemannian maps to Kähler manifolds

In this section, we introduce Clairaut invariant Riemannian maps to Kähler manifolds and find a necessary and sufficient condition for invariant Riemannian maps from Riemannian maps to Kähler manifolds to be Clairaut invariant Riemannian maps and give a non-trivial example of such maps.

**Definition 4.1** [10] Let \( F : (M, g_M) \to (N, g_N) \) be a proper Riemannian map from a Riemannian manifold \( M \) to a Hermitian manifold \( N \) with almost complex structure \( J_N \). We say that \( F \) is an invariant Riemannian map at \( p \in M \) if \( J_N(\text{range} F_p) = \text{range} F_p \). If \( F \) is an invariant Riemannian map for every \( p \in M \) then \( F \) is called an invariant Riemannian map.

**Definition 4.2** An invariant Riemannian map from Riemannian manifold to Kähler manifold is called Clairaut invariant Riemannian map if it satisfies the condition of Definition 3.8.
Theorem 4.3 [21] Let $F : (M^m, g_M) \to (N^n, g_N)$ be a Riemannian map between Riemannian manifolds such that $(\text{range}F_s)^\perp$ is totally geodesic and $\alpha, \beta = F \circ \alpha$ are geodesic curves on $M$ and $N$, respectively. Then $F$ is Clairaut Riemannian map with $s = e^g$ if and only if $F$ is umbilical, and has $H_2 = -\nabla^N g$, where $H_2$ is mean curvature vector field of $\text{range}F_s$ and $g$ is a smooth function on $N$.

Proposition 4.4 Let $F : (M, g_M) \to (N, g_N, J_N)$ be an invariant Riemannian map from a Riemannian manifold $M$ to a Kähler manifold $N$ such that $(\text{range}F_s)^\perp$ is totally geodesic and $\alpha : I \to M$ be a geodesic curve on $M$. Then the curve $\beta = F \circ \alpha$ is geodesic curve on $N$ if and only if

\begin{equation}
J_N(\nabla F_s)(X, X) + \nabla^F_X J_N V + \nabla^F_V J_N V = 0, \tag{4.1}
\end{equation}

\begin{equation}
-\mathcal{S}_{J_N} F_s X + J_N F_s (\nabla^M_X X) + \nabla^V_J N F_s X = 0, \tag{4.2}
\end{equation}

where $F_s X \in \Gamma(\text{range}F_s), V \in \Gamma(\text{range}F_s)^\perp$ are components of $\beta(t)$ and $\nabla^N$ is the Levi–Civita connection on $N$ and $\nabla^F$ is a linear connection on $(\text{range}F_s)^\perp$.

Proof Let $F$ be an invariant Riemannian map from a Riemannian manifold $M$ to a Kähler manifold $N$ such that $(\text{range}F_s)^\perp$ is totally geodesic, i.e. $\nabla^F_X V = \nabla^V_X V$ for all $U, V \in \Gamma(\text{range}F_s)^\perp$. Let $\alpha : I \to M$ be a geodesic on $M$ with $U(t) = \nu \alpha(t)$ and $X(t) = \mathcal{H} \alpha(t)$. Let $\beta = F \circ \alpha$ be a geodesic on $N$ with $F_s X \in \Gamma(\text{range}F_s)$ and $V \in \Gamma(\text{range}F_s)$. Then $(\text{range}F_s)^\perp$ are components of $\beta(t)$. Since $N$ is Kähler manifold, we have $\nabla^N \beta = -J_N \nabla^N J_N \beta$.

Now,

$$\nabla^N J_N \beta = \nabla^N_{F_s X + V} J_N (F_s X + V),$$

which is equal to

$$\nabla^N J_N \beta = J_N \nabla^N_{F_s X} F_s X + \nabla^N_{F_s X} J_N V + \nabla^N_{F_s} J_N F_s X + \nabla^N_{\nabla V} J_N V. \tag{4.3}$$

Using (2.10) in above equation, we get

$$\nabla^N J_N \beta = J_N \nabla^N_{F_s} F_s X \circ F - \mathcal{S}_{J_N} F_s X + \nabla^F_X J_N V + \nabla^N J_N F_s X + \nabla^V_{\nabla^N J_N V}.$$

Since $g_N(\nabla^N J_N F_s X, U) = -g_N(J_N F_s X, \nabla^N U) = -g_N(J_N F_s X, \nabla^F_X U) = 0$ for $U \in \Gamma(\text{range}F_s)^\perp$, which implies $\nabla^N J_N F_s X \in \Gamma(\text{range}F_s)$ and using (2.8) in above equation, we get

$$\nabla^N J_N \beta = J_N((\nabla^N F_s)(X, X) + F_s(\nabla^M_X X)) - \mathcal{S}_{J_N} F_s X + \nabla^F_X J_N V + \nabla^N J_N F_s X + \nabla^N_{\nabla^N J_N V}. \tag{4.3}$$

Since $(\text{range}F_s)^\perp$ is totally geodesic therefore putting $\nabla^N_{\nabla^N J_N V} = \nabla^F_X J_N V$ in (4.3), we get

$$\nabla^N J_N \beta = J_N(\nabla^N F_s)(X, X) + J_N F_s(\nabla^M_X X) - \mathcal{S}_{J_N} F_s X + \nabla^F_X J_N V + \nabla^N J_N F_s X + \nabla^F_X J_N V. \tag{4.4}$$

Now $\beta$ is geodesic on $N$ if and only if $\nabla^N \beta = 0$ if and only if $-J_N(\nabla^N J_N \beta) = 0$ if and only if $\nabla^N J_N \beta = 0$.

Then (4.4) implies $J_N(\nabla F_s)(X, X) + J_N F_s(\nabla^M_X X) - \mathcal{S}_{J_N} F_s X + \nabla^F_X J_N V + \nabla^N J_N F_s X + \nabla^F_X J_N V = 0,$ which completes the proof.

\[1031\]
Theorem 4.5 Let \( F : (M, g_M) \to (N, g_N, J_N) \) be an invariant Riemannian map from Riemannian manifold \( M \) to Kähler manifold \( N \) such that \((\text{range}F_*)^\perp\) is totally geodesic and \( \alpha, \beta = F \circ \alpha \) are geodesic curves on \( M \) and \( N \), respectively. Then \( F \) is Clairaut invariant Riemannian map with \( s = e^g \) if and only if \( S_{J_NV}F_*X = -V(g)J_NF_*X \), where \( g \) is a smooth function on \( N \) and \( F_*X \in \Gamma(\text{range}F_*) \), \( V \in \Gamma(\text{range}F_*)^\perp \) are components of \( \dot{\beta}(t) \).

**Proof** Let \( \alpha : I \to M \) be a geodesic on \( M \) with \( U(t) = \nu \dot{\alpha}(t) \) and \( X(t) = \mathcal{H} \dot{\alpha}(t) \). Let \( \beta = F \circ \alpha \) be a geodesic on \( N \) with \( F_*X \in \Gamma(\text{range}F_*) \) and \( \nu \in \Gamma(\text{range}F_*)^\perp \) are components of \( \dot{\beta}(t) \) and \( \omega(t) \) denote the angle between \( \dot{\beta} \) and \( V \). Assuming \( b = \| \dot{\beta}(t) \|^2 \), then we get
\[
g_{N\beta(t)}(V, V) = b \cos^2 \omega(t), \tag{4.5}
\]
\[
g_{N\beta(t)}(F_*X, F_*X) = b \sin^2 \omega(t). \tag{4.6}
\]
Now differentiating (4.6), we get
\[
\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2b\sin \omega \cos \omega \frac{d\omega}{dt}. \tag{4.7}
\]
On the other hand
\[
\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_N(\nabla^N_{F_*X}J_NF_*X, J_NF_*X).
\]
By putting \( \dot{\beta} = F_*X + V \) in above equation, we get
\[
\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_N(\nabla^N_{F_*X}J_NF_*X + \nabla^N_VJ_NF_*X, J_NF_*X),
\]
which implies
\[
\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_N(J_N\nabla^N_{F_*X}F_*X \circ F + \nabla^N_VJ_NF_*X, J_NF_*X). \tag{4.8}
\]
Using (2.8) and (4.2) in (4.8), we get
\[
\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_N(J_N(\nabla F_*)(X, X) + S_{J_NV}F_*X, J_NF_*X).
\]
Using (2.9) in above equation, we get
\[
\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_N(S_{J_NV}F_*X, J_NF_*X). \tag{4.9}
\]
Now from (4.7) and (4.9), we get
\[
g_N(S_{J_NV}F_*X, J_NF_*X) = b \sin \omega \cos \omega \frac{d\omega}{dt}. \tag{4.10}
\]
Moreover, \( F \) is a Clairaut Riemannian map with \( s = e^g \) if and only if \( \frac{d}{dt}(e^{g \circ \beta} \sin \omega) = 0 \), that is, \( e^{g \circ \beta} \sin \omega \frac{d(g \circ \beta)}{dt} + e^{g \circ \beta} \cos \omega \frac{d\omega}{dt} = 0 \). By multiplying this with nonzero factor \( b \sin \omega \) and using (4.6), we get
\[
g_{N\beta(t)}(F_*X, F_*X) \frac{d(g \circ \beta)}{dt} = -b \sin \omega \cos \omega \frac{d\omega}{dt}. \tag{4.11}
\]
Now from (4.10) and (4.11), we get
\[ g_N(S_{J_N}VF_*X, J_NF_*X) = -g_{N\beta(t)}(F_*X, F_*X) \frac{d(g \circ \beta)}{dt}. \]
Since \( N \) is Kähler manifold then above equation can be written as
\[ g_N(S_{J_N}VF_*X, J_NF_*X) = -g_{N\beta(t)}(J_NF_*X, J_NF_*X)g_N(\nabla^N g, \dot{\beta}). \tag{4.12} \]
Since \( \nabla^N g \in \Gamma(range F_\ast) \perp \) and \( \dot{\beta} = F_*X + V \), then from (4.12), we get
\[ g_N(S_{J_N}VF_*X, J_NF_*X) = -g_{N\beta(t)}(J_NF_*X, J_NF_*X)g_N(\nabla^N g, V). \tag{4.13} \]
Thus \( S_{J_N}VF_*X = -V(g)J_NF_*X \), where \( V(g) \) is a smooth function on \( N \), which completes the proof. \( \square \)

**Example 4.6** Let \( M \) be an Euclidean space given by \( M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \neq 0, x_2 \neq 0, x_3 \neq 0, x_4 \neq 0\} \). We define the Riemannian metric \( g_M \) on \( M \) given by \( g_M = e^{2x_1}dx_1^2 + e^{2x_2}dx_2^2 + e^{2x_3}dx_3^2 + dx_4^2 \). Let \( N = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4\} \) be a Riemannian manifold with Riemannian metric \( g_N \) on \( N \) given by \( g_N = e^{2y_1}dy_1^2 + e^{2y_2}dy_2^2 + dy_3^2 + dy_4^2 \) and the complex structure \( J_N \) on \( N \) defined as \( J_N(y_1, y_2, y_3, y_4) = (-y_2, y_1, -y_4, y_3) \). Consider a map \( F : (M, g_M) \to (N, g_N, J_N) \) defined by
\[ F(x_1, x_2, x_3, x_4) = \left( \frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, 0, 0 \right). \]
Then, we get
\[ \ker F_* = \text{Span}\{U_1 = e_3, U_2 = e_4\}, \]
and
\[ (\ker F_*)^\perp = \text{Span}\{X_1 = e_1, X_2 = e_2\}, \]
where \( \{e_1 = e^{-x_4} \frac{\partial}{\partial x_1}, e_2 = e^{-x_4} \frac{\partial}{\partial x_2}, e_3 = e^{-x_4} \frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial x_4}\} \) are bases on \( T_pM \) and \( T_{F(p)}N \) respectively, for all \( p \in M \). By easy computations, we see that \( F_*X_1 = \frac{1}{\sqrt{2}}(e_1^* + e_2^*), F_*X_2 = \frac{1}{\sqrt{2}}(e_1^* - e_2^*) \) and \( g_M(X_1, X_2) = g_N(F_*X_1, F_*X_2) \) for all \( X_1, X_2 \in \Gamma(\ker F_*)^\perp \).
Thus \( F \) is Riemannian map with \( \text{range} F_* = \text{Span}\{F_*X_1 = \frac{1}{\sqrt{2}}(e_1^* + e_2^*), F_*X_2 = \frac{1}{\sqrt{2}}(e_1^* - e_2^*)\} \) and \( (\text{range} F_*)^\perp = \text{Span}\{e_3^*, e_4^*\} \). Moreover it is easy to see that \( J_NF_*X_1 = F_*X_2, J_NF_*X_2 = -F_*X_1 \) and \( (\text{range} F_*)^\perp \) is also invariant with respect to \( J_N \). Thus \( F \) is an invariant Riemannian map.

Now to show that \( F \) is Clairaut Riemannian map we will find a smooth function \( g \) on \( N \) satisfying \( (\nabla F_*)(X, X) = -g_M(X, X)\nabla^N g \) for any \( X \in \Gamma(\ker F_*)^\perp \). Since by (2.9) \( (\nabla F_*)(X, X) \in \Gamma(\text{range} F_*)^\perp \) for any \( X \in \Gamma(\ker F_*)^\perp \). So, here we can write \( (\nabla F_*)(X, X) = \lambda_1 e_3^* + \lambda_2 e_4^* \), and \( g_M(X, X) = g_M(\alpha_1 X_1 + \alpha_2 X_2, \alpha_1 X_1 + \alpha_2 X_2) = \alpha_1^2 + \alpha_2^2 \) for some \( \lambda_1, \lambda_2, \alpha_1, \alpha_2 \in \mathbb{R} \).
Since, we have
\[ g_{Nij} = \begin{pmatrix} e^{2x_4} & 0 & 0 & 0 \\ 0 & e^{2x_4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ g^{ij}_{N} = \begin{pmatrix} e^{-2x_4} & 0 & 0 & 0 \\ 0 & e^{-2x_4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

For any smooth function \( g \) on \( N \), the gradient of \( g \) with respect to the metric \( g_N \) is given by
\[ \nabla^N g = \sum_{i,j=1}^{4} g^{ij}_{N} \frac{\partial g}{\partial y_i} \frac{\partial}{\partial y_j}. \]

Thus
\[ \nabla^N g = e^{-2x_4} \frac{\partial g}{\partial y_1} \frac{\partial}{\partial y_1} + e^{-2x_4} \frac{\partial g}{\partial y_2} \frac{\partial}{\partial y_2} + \frac{\partial g}{\partial y_3} \frac{\partial}{\partial y_3} + \frac{\partial g}{\partial y_4} \frac{\partial}{\partial y_4}. \]

Hence \( \nabla^N g = -\left(\frac{\lambda_1}{\alpha_1 + \alpha_2} \frac{\partial}{\partial y_3} + \frac{\lambda_2}{\alpha_1 + \alpha_2} \frac{\partial}{\partial y_4}\right) \) for the function \( g = -\frac{\lambda_1}{\alpha_1 + \alpha_2} y_3 - \frac{\lambda_2}{\alpha_1 + \alpha_2} y_4 \). Then it is easy to verify that \( (\nabla F_*)(X, X) = -g_M(X, X)\nabla^N g \), for any vector field \( X \in \Gamma(kerF_*)^\perp \). Thus by Theorem 4.3 and Lemma 2.1, we see that \( F \) is Clairaut invariant Riemannian map.

Acknowledgment
We are thankful to the anonymous referees for his/her valuable comments/suggestions towards improvement of the paper. Also, Kiran Meena gratefully acknowledges the financial support provided by the Council of Scientific and Industrial Research (C.S.I.R.), New Delhi, India [ File No.: 09/013(0887)/2019-EMR-I ].

References


