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Clairaut invariant Riemannian maps with Kähler structure

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Abstract: In this paper, we study Clairaut invariant Riemannian maps from Kähler manifolds to Riemannian manifolds, and from Riemannian manifolds to Kähler manifolds. We find necessary and sufficient conditions for the curves on the total spaces and base spaces of invariant Riemannian maps to be geodesic. Further, we obtain necessary and sufficient conditions for invariant Riemannian maps from Kähler manifolds to Riemannian manifolds to be Clairaut invariant Riemannian maps. Moreover, we obtain a necessary and sufficient condition for invariant Riemannian maps from Riemannian manifolds to Kähler manifolds to be Clairaut invariant Riemannian maps. We also give nontrivial examples of Clairaut invariant Riemannian maps whose total manifolds or base manifolds are Kähler manifolds.

Key words: Riemannian manifold, Kähler manifold, Clairaut Riemannian submersion, invariant Riemannian map, Clairaut Riemannian map

1. Introduction

In 1992, Fischer introduced Riemannian map between Riemannian manifolds in [6] as a generalization of the notion of an isometric immersion and Riemannian submersion. The geometry of Riemannian submersions have been discussed in [5]. We note that a remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation $\|F_*\|^2 = rankF$, which is a bridge between geometric optics and physical optics [6]. The eikonal equation of geometrical optics was solved by using Cauchy's method of characteristics. In [6] Fischer also proposed an approach to build a quantum model and he pointed out the success of such a program of building a quantum model of nature using Riemannian maps would provide an interesting relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side, and Maxwell's equation, Shrödinger's equation and their proposed generalization on the physical side.

In [20], as an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases and also obtained some fundamental properties of this map. In [9], B. Şahin introduced holomorphic Riemannian maps as a generalization of holomorphic submersions and holomorphic submanifolds and in [12], B. Şahin studied Riemannian submersions from almost Hermitian manifolds. The notion of invariant Riemannian maps has been introduced by B. Şahin in [10] as a generalization of invariant immersion of almost Hermitian manifolds and holomorphic Riemannian submersions. In [10], B. Şahin introduced invariant and antiinvariant Riemannian maps to Kähler manifolds. The notion of invariant Riemannian maps to Kähler manifolds is a generalization of submanifolds of Kähler manifolds. It can be see

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that every holomorphic Riemannian map is an invariant Riemannian map but an invariant Riemannian map may not be holomorphic Riemannian map. Further, holomorphic, antiinvariant and semiinvariant Riemannian maps with Kähler structure were studied in [1–3, 10, 11, 15, 17–19].

In elementary differential geometry, if θ is the angle between the velocity vector of a geodesic and a meridian, and r is the distance to the axis of a surface of revolution, then Clairaut's relation states that $rsin\theta$ is constant. In 1972 ([5], p. 29), Bishop defined Clairaut Riemannian submersion with connected fibers and gave a necessary and sufficient condition for Riemannian submersion to be Clairaut Riemannian submersion, also in [7], B. Şahin studied Clairaut submersions. Further in [16], B. Şahin and H. M. Taştan studied Clariaut submersions from almost Hermitian manifolds. In [13], B. Şahin introduced Clairaut Riemannian maps, in which he obtained necessary and sufficient conditions for Riemannian map to be Clairaut Riemannian map. In [22], present authors studied Clairaut antiinvariant Riemannian maps from Kähler manifolds and investigated some geometric properties. In [21], we defined Clairaut Riemannian maps by using geodesic curves [4] on base spaces and obtained necessary and sufficient conditions for Riemannian maps to be Clairaut Riemannian maps. Also we obtained interesting result that such Clairaut Riemannian maps are umbilical maps.

In this paper we study Clairaut invariant Riemannian maps from Kähler manifolds, and to Kähler manifolds. In Section 2, we give some basic information about Riemannian map which is needed for this paper. In Section 3, we obtain necessary and sufficient conditions for invariant Riemannian maps to be Clairaut invariant Riemannian maps whose total manifolds are Kähler manifolds and investigate geometric properties of such maps with a nontrivial example. Finally, in Section 4, we obtain a necessary and sufficient condition for invariant Riemannian maps to be Clairaut invariant Riemannian maps whose base manifolds are Kähler manifolds and give a nontrivial example of such maps.

2. Preliminaries

In this section, we recall the notion of Riemannian maps between Riemannian manifolds and give a brief review of basic facts of Riemannian maps.

Let $F:(M^m,g_M)\to(N^n,g_N)$ be a smooth map between Riemannian manifolds such that $0<rankF\leq min\{m,n\}$, where $dim(M)=m$ and $dim(N)=n$. Then we denote the kernel space of F_* by $\nu_p=kerF_{*p}$ at $p\in M$ and consider the orthogonal complementary space $\mathcal{H}_p=(kerF_{*p})^\perp$ to $kerF_{*p}$ in T_pM. Then the tangent space T_pM of M at p has the decomposition $T_pM=(kerF_{*p})\oplus(kerF_{*p})^\perp=\nu_p\oplus\mathcal{H}_p$. We denote the range of F_* by $rangeF_*$ at $p\in M$ and consider the orthogonal complementary space $(rangeF_{*p})^\perp$ to $rangeF_{*p}$ in the tangent space $T_{F(p)}N$ of N at $F(p)\in N$. Since $rankF\leq min\{m,n\}$, we have $(rangeF_*)^\perp\neq\{0\}$. Thus the tangent space $T_{F(p)}N$ of N at $F(p)\in N$ has the decomposition $T_{F(p)}N=(rangeF_{*p})\oplus(rangeF_{*p})^\perp$. Now, the map $F:(M^m,g_M)\to(N^n,g_N)$ is called Riemannian map at $p\in M$ if the horizontal restriction $F^h_{*p}:(kerF_{*p})^\perp\to(rangeF_{*p})$ is a linear isometry between the inner product spaces $((kerF_{*p})^\perp,g_{M(p)}|_{(kerF_{*p})^\perp})$ and $(rangeF_{*p},g_{N(p_1)}|_{(rangeF_{*p})})$, where $F(p)=p_1$. In other words, F_* satisfies the equation

$$g_N(F_*X,F_*Y)=g_M(X,Y), \tag{2.1}$$

for all X,Y vector field tangent to $\Gamma(kerF_{*p})^\perp$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_*=\{0\}$ and $(rangeF_*)^\perp=\{0\}$.

The O'Neill tensors A and T defined in [5] as

$$A_E F = \mathcal{H}\nabla^M_{\mathcal{H}E}\nu F + \nu\nabla^M_{\mathcal{H}E}\mathcal{H}F, \tag{2.2}$$

$$T_E F = \mathcal{H}\nabla^M_{\nu E}\nu F + \nu\nabla^M_{\nu E}\mathcal{H}F, \tag{2.3}$$

for all vector fields E, F on M, where ∇^M is the Levi–Civita connection of g_M and ν, $\mathcal{H}$ denote the projections to vertical subbundle and horizontal subbundle, respectively. For any $E \in \Gamma(TM)$, T_E and A_E are skew-symmetric operators on $(\Gamma(TM), g_M)$ reversing the horizontal and the vertical distributions. It is also easy to see that T is vertical, $T_E = T_{\nu E}$ and A is horizontal, $A_E = A_{\mathcal{H}E}$. We note that the tensor field T satisfies $T_U W = T_W U$, for all $U, W \in \Gamma(kerF_*)$.
Now, from (2.2) and (2.3), we have

$$\nabla^M_V W = T_V W + \hat{\nabla}_V W, \tag{2.4}$$

$$\nabla^M_X V = A_X V + \nu\nabla^M_X V, \tag{2.5}$$

$$\nabla^M_X Y = A_X Y + \mathcal{H}\nabla^M_X Y, \tag{2.6}$$

for all $X, Y \in \Gamma(kerF_*)^\perp$ and $V, W \in \Gamma(kerF_*)$, where $\hat{\nabla}_V W = \nu\nabla^M_V W$.
Also, a Riemannian map is a Riemannian map with totally umbilical fibers if [11]

$$T_U V = g_M(U, V)H, \tag{2.7}$$

for all $U, V \in \Gamma(kerF_*)$, where H is the mean curvature vector field of fibers.

Let $F : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds (M, g_M) and (N, g_N). Then the differential F_* of F can be viewed as a section of bundle $Hom(TM, F^{-1}TN) \to M$, where $F^{-1}TN$ is the pullback bundle whose fibers at $p \in M$ is $(F^{-1}TN)_p = T_{F(p)}N$, $p \in M$. The bundle $Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi–Civita connection ∇^M and the pullback connection $\overset{N}{\nabla^F}$. Then the second fundamental form of F is given by [8]

$$(\nabla F_*)(X_1, Y_1) = \overset{N}{\nabla^F_{X_1}} F_* Y_1 - F_*(\nabla^M_{X_1} Y_1), \tag{2.8}$$

for all $X_1, Y_1 \in \Gamma(TM)$, where $\overset{N}{\nabla^F_{X_1}} F_* Y_1 \circ F = \nabla^N_{F_* X_1} F_* Y_1$. It is known that the second fundamental form is symmetric. In [10] B. Şahin proved that $(\nabla F_*)(X, Y)$ has no component in $rangeF_*$, for all $X, Y \in \Gamma(kerF_*)^\perp$. More precisely, we have

$$(\nabla F_*)(X, Y) \in \Gamma(rangeF_*)^\perp. \tag{2.9}$$

For any vector field X_1 on M and any section V of $(rangeF_*)^\perp$, we have $\nabla^{F\perp}_{X_1} V$, which is the orthogonal projection of $\nabla^N_{X_1} V$ on $(rangeF_*)^\perp$, where $\nabla^{F\perp}$ is linear connection on $(rangeF_*)^\perp$ such that $\nabla^{F\perp} g_N = 0$.
Now, for a Riemannian map F we define $\mathcal{S}_V$ as ([14], p. 188)

$$\nabla^N_{F_* X} V = -\mathcal{S}_V F_* X + \nabla^{F\perp}_X V, \tag{2.10}$$

where ∇^N is the Levi–Civita connection on N, $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla^N_{F_*X} V$. Thus at $p \in M$, we have $\nabla^N_{F_*X} V(p) \in T_{F(p)}N$, $\mathcal{S}_V F_* X \in F_{*p}(T_pM)$ and $\nabla^{F\perp}_X V(p) \in (F_{*p}(T_pM))^\perp$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V, and $F_* X$ at p depends only on V_p and $F_{*p} X_p$.

Lemma 2.1 *[15] Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then F is umbilical Riemannian map if and only if*

$$(\nabla F_*)(X, Y) = g_M(X, Y)H_2, \tag{2.11}$$

for $X, Y \in \Gamma(ker F_*)^\perp$ and H_2 is, nowhere zero, vector field on $(range F_*)^\perp$.

Let (M, g_M) be an almost Hermitian manifold [23], then M admits a tensor J of type (1, 1) on M such that $J^2 = -I$ and

$$g_M(JX_1, JY_1) = g_M(X_1, Y_1), \tag{2.12}$$

for all $X_1, Y_1 \in \Gamma(TM)$. An almost Hermitian manifold M is called Kähler manifold if

$$(\nabla^M_{X_1} J)Y_1 = 0, \tag{2.13}$$

for all $X_1, Y_1 \in \Gamma(TM)$, where ∇^M is a Levi–Civita connection on M.

3. Clairaut invariant Riemannian maps from Kähler manifolds

In this section, we introduce Clairaut invariant Riemannian maps from Kähler manifolds and investigate the geometry of such maps with a nontrivial example. For arbitrary Clairaut invariant Riemannian maps, we present the following definitions:

Definition 3.1 *[9] Let $F : (M, g_M, J) \to (N, g_N)$ be a Riemannian map from an almost Hermitian manifold M to a Riemannian manifold N with almost complex structure J. We say that F is an invariant Riemannian map at $p \in M$ if the $J(ker F_{*p}) = ker F_* p$. If F is an invariant Riemannian map for every $p \in M$ then F is called an invariant Riemannian map, i.e. the vertical distribution is invariant with respect to J, or $J(ker F_*) = ker F_*$. In this case, we have the horizontal distribution $(ker F_*)^\perp$ is also invariant with respect to J, i.e. $J(ker F_*)^\perp = (ker F_*)^\perp$.*

The notion of Clairaut Riemannian map was defined by B. Şahin in [13]. According to the definition, a Riemannian map $F : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $r : M \to \mathbb{R}^+$ such that for every geodesic α on M, the function $(r \circ \alpha) sin\theta$ is constant, where, for all t, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

Theorem 3.2 *[13] Let $F : (M, g_M) \to (N, g_N)$ be a Riemannian map with connected fibers, then F is Clairaut Riemannian map with $r = e^f$ if and only if each fiber is umbilical and has mean curvature vector field $H = -grad f$, where f is a smooth function on M and $grad f$ is the gradient of the function f with respect to g_M.*

Definition 3.3 *An invariant Riemannian map from Kähler manifold to a Riemannian manifold is called Clairaut invariant Riemannian map if it satisfies the condition of Clairaut Riemannian map.*

The origin of the notion of Clairaut Riemannian maps comes from geodesic curves on a surface. Therefore we are going to find necessary and sufficient conditions for the curves on the total spaces and the base spaces to be geodesic.

Lemma 3.4 *Let $F:(M,g_M,J)\to(N,g_N)$ be an invariant Riemannian map from a Kähler manifold M to a Riemannian manifold N. Let $\alpha:I\to M$ be a regular curve and $X(t),U(t)$ denote the horizontal and vertical components of its tangent vector field. Then α is a geodesic curve on M if and only if*

$$\nu\nabla^M_{\dot\alpha}JU + A_XJX + T_UJX = 0, \tag{3.1}$$

$$\mathcal{H}\nabla^M_{\dot\alpha}JX + A_XJU + T_UJU = 0. \tag{3.2}$$

Proof Let $\alpha:I\to M$ be a regular curve on M. Then from (2.13), we get $\nabla^M_{\dot\alpha}\dot\alpha = -J\nabla^M_{\dot\alpha}J\dot\alpha$. Since $\dot\alpha(t)=X(t)+U(t)$, for all $t\in I$, so we can write $\nabla^M_{\dot\alpha}\dot\alpha = -J\nabla^M_{X+U}J(X+U)$ and obtain

$$\nabla^M_{\dot\alpha}\dot\alpha = -J(\nabla^M_XJX + \nabla^M_UJX + \nabla^M_XJU + \nabla^M_UJU). \tag{3.3}$$

Using (2.4), (2.5), (2.6) in (3.3), we get

$$\nabla^M_{\dot\alpha}\dot\alpha = \ -J(A_XJX + T_UJX + \nu\nabla^M_XJU + \nu\nabla^M_UJU + \mathcal{H}\nabla^M_XJX + \mathcal{H}\nabla^M_UJX + A_XJU + T_UJU),$$

which implies

$$\nabla^M_{\dot\alpha}\dot\alpha = -J(A_XJX + T_UJX + \nu\nabla^M_{\dot\alpha}JU + \mathcal{H}\nabla^M_{\dot\alpha}JX + A_XJU + T_UJU). \tag{3.4}$$

Applying J both sides of (3.4) and take the vertical and horizontal components, we get

$$\nu J\nabla^M_{\dot\alpha}\dot\alpha = A_XJX + T_UJX + \nu\nabla^M_{\dot\alpha}JU,$$

$$\mathcal{H}J\nabla^M_{\dot\alpha}\dot\alpha = A_XJU + T_UJU + \mathcal{H}\nabla^M_{\dot\alpha}JX.$$

Now, α is a geodesic on M if and only if $\nu J\nabla^M_{\dot\alpha}\dot\alpha=0$ and $\mathcal{H}J\nabla^M_{\dot\alpha}\dot\alpha=0$, which completes the proof. □

Theorem 3.5 *Let $F:(M,g_M,J)\to(N,g_N)$ be an invariant Riemannian map from a Kähler manifold M to a Riemannian manifold N. Then F is a Clairaut invariant Riemannian map with $r=e^f$ if and only if $g_{M\alpha(t)}(A_XJU+T_UJU,JX)+g_{M\alpha(t)}(\nabla^Mf,X)g_{M\alpha(t)}(U(t),U(t))=0$, where $\alpha:I\to M$ is a geodesic curve on M with $X(t)$, $U(t)$ are horizontal and vertical components of $\dot\alpha(t)$ and f is a smooth function on M.*

Proof Let $\alpha:I\to M$ be a geodesic on M with $U(t)=\nu\dot\alpha(t)$ and $X(t)=\mathcal{H}\dot\alpha(t)$, $\theta(t)$ denote the angle in $[0,\pi]$ between $\dot\alpha(t)$ and $X(t)$. Assuming $a=\|\dot\alpha(t)\|^2$, then we get

$$g_{M\alpha(t)}(X(t),X(t)) = acos^2\theta(t), \tag{3.5}$$

$$g_{M\alpha(t)}(U(t),U(t)) = asin^2\theta(t). \tag{3.6}$$

Now, differentiating (3.5), we get

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = -2acos\theta(t)sin\theta(t)\frac{d\theta}{dt}. \tag{3.7}$$

On the other hand using (2.12), we get

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = \frac{d}{dt}g_{M\alpha(t)}(JX,JX). \tag{3.8}$$

Since F is Riemannian map, then using (2.1) in (3.8), we get

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = 2g_N(F_*(\nabla^M_{\dot\alpha}JX),F_*(JX)). \tag{3.9}$$

Using (2.8) in (3.9), we obtain

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = 2g_N(\overset{N}{\nabla}{}^F_{\dot\alpha}F_*(JX) - (\nabla F_*)(\dot\alpha,JX),F_*(JX)). \tag{3.10}$$

By putting $\dot\alpha = X + U$ in (3.10) and then using (2.9), we get

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = 2g_N(\overset{N}{\nabla}{}^F_X F_*(JX) - (\nabla F_*)(U,JX),F_*(JX)).$$

Using (2.8) and (2.9) in above equation, we get

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = 2g_N(F_*(\nabla^M_X JX) + F_*(\nabla^M_U JX),F_*(JX)),$$

which implies

$$\frac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = 2g_N(F_*(\mathcal{H}\nabla^M_{\dot\alpha}JX),F_*(JX)). \tag{3.11}$$

By using (3.2) in (3.11), we obtain

$$\tfrac{d}{dt}g_{M\alpha(t)}(X(t),X(t)) = -2g_M(A_XJU + T_UJU,JX). \tag{3.12}$$

Now from (3.7) and (3.12), we get

$$g_M(A_XJU + T_UJU,JX) = acos\theta(t)sin\theta(t)\frac{d\theta}{dt}. \tag{3.13}$$

Moreover, F is a Clairaut Riemannian map with $r = e^f$ if and only if $\frac{d}{dt}(e^{f\circ\alpha}sin\theta(t)) = 0$, that is, $e^{f\circ\alpha}(cos\theta(t)\frac{d\theta}{dt} + sin\theta(t)\frac{d(f\circ\alpha)}{dt}) = 0$. By multiplying this with nonzero factor $asin\theta(t)$, we get

$$-acos\theta(t)sin\theta(t)\frac{d\theta}{dt} = asin^2\theta(t)\frac{d(f\circ\alpha)}{dt}. \tag{3.14}$$

Using (3.6) in (3.14), we get

$$g_{M\alpha(t)}(U(t),U(t))\frac{d(f\circ\alpha)}{dt} = -acos\theta(t)sin\theta(t)\frac{d\theta}{dt}. \tag{3.15}$$

Thus from (3.13) and (3.15), we get

$$g_M(A_X JU + T_U JU, JX) + g_{M\alpha(t)}(U(t), U(t))\frac{d(f \circ \alpha)}{dt} = 0,$$

which implies

$$g_M(A_X JU + T_U JU, JX) + g_{M\alpha(t)}(U(t), U(t))g_{M\alpha(t)}(\nabla^M f, \dot{\alpha}) = 0.$$

Since $\nabla^M f \in \Gamma(kerF_*)^\perp$ therefore above equation can be written as

$$g_M(A_X JU + T_U JU, JX) + g_{M\alpha(t)}(U(t), U(t))g_{M\alpha(t)}(\nabla^M f, X) = 0,$$

which completes the proof. □

Theorem 3.6 *Let $F : (M, g_M, J) \to (N, g_N)$ be a Clairaut invariant Riemannian map with $r = e^f$ from a Kähler manifold M to a Riemannian manifold N. Then $T_U V = 0$, for $U \neq V \in \Gamma(kerF_*)$.*

Proof Let F be a Clairaut invariant Riemannian map from a Kähler manifold to a Riemannian manifold. Then by using (2.7) in Theorem 3.2, we get

$$T_U V = -g_M(U, V)gradf, \tag{3.16}$$

for $U, V \in \Gamma(kerF_*)$, which implies

$$g_M(T_U V, JX) = -g_M(U, V)g_M(gradf, JX), \tag{3.17}$$

for $X \in \Gamma(kerF_*)^\perp$. Using (2.4) and (2.12) in (3.17), we get

$$g_M(\nabla_U JV, X) = g_M(U, V)g_M(gradf, JX).$$

Using (2.4) and (3.16) in above equation, we get

$$g_M(U, JV)g_M(gradf, X) = -g_M(U, V)g_M(gradf, JX). \tag{3.18}$$

Interchanging the role of U and V $(U \neq V)$, we obtain

$$g_M(V, JU)g_M(gradf, X) = -g_M(U, V)g_M(gradf, JX). \tag{3.19}$$

From (3.19) and (3.18), we get

$$(g_M(U, JV) - g_M(V, JU))g_M(gradf, X) = 0.$$

Using (2.12) in above equation, we get

$$2g_M(U, JV)g_M(\nabla^M f, X) = 0.$$

Thus

$$g_M(\nabla^M f, X) = 0. \tag{3.20}$$

Also, by (3.16), we get

$$g_M(T_U V, X) = -g_M(U, V) g_M(\nabla^M f, X), \tag{3.21}$$

for $X \in \Gamma(kerF_*)^{\perp}$. By using (3.20) in (3.21), we get

$$g_M(T_U V, X) = 0.$$

Thus $T_U V = 0$ for $U \neq V \in \Gamma(kerF_*)$, which completes the proof. □

Proposition 3.7 *Let $F : (M, g_M, J) \to (N, g_N)$ be an invariant Riemannian map from Kähler manifold M to a Riemannian manifold N and $\alpha : I \to M$ is geodesic curve on M with $U(t) = \nu\dot{\alpha}(t)$ and $X(t) = \mathcal{H}\dot{\alpha}(t)$. Then the curve $\beta = F \circ \alpha$ is geodesic on N if and only if*

$$(\nabla F_*)(X, JX) = 0, F_*(\nabla^M_U JX + A_X JU + T_U JU) = 0. \tag{3.22}$$

Proof Let $\alpha : I \to M$ is geodesic on M. Then by (3.2), we have

$$\mathcal{H}\nabla^M_X JX = -(\mathcal{H}\nabla^M_U JX + A_X JU + T_U JU),$$

for $U \in \Gamma(kerF_*)$ and $X \in \Gamma(kerF_*)^{\perp}$. By applying definition of Riemannian map, we get

$$F_*(\nabla^M_X JX) = -F_*(\nabla^M_U JX + A_X JU + T_U JU). \tag{3.23}$$

Using (2.8) in (3.23), we get

$$\overset{N}{\nabla}{}^F_X F_*(JX) = (\nabla F_*)(X, JX) - F_*(\nabla^M_U JX + A_X JU + T_U JU).$$

Now β is geodesic on N if and only if $\overset{N}{\nabla}{}^F_X F_*(JX) = 0$, which completes the proof. □

Definition 3.8 *[21] A Riemannian map $F : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $s : N \to \mathbb{R}^+$ such that for every geodesic β on N, the function $(s \circ \beta) sin\omega(t)$ is constant, where, $F_*X \in \Gamma(rangeF_*)$ and $V \in \Gamma(rangeF_*)^{\perp}$ are components of $\dot{\beta}(t)$, and $\omega(t)$ is the angle between $\dot{\beta}(t)$ and V for all t.*

Remark 3.9 *As we know that Clairaut's relation comes from geodesic curve. In [13], B. Şahin considered geodesic curve on the total manifold of a Riemannian map F, then by using Clairaut relation fibers of F are totally umbilical. On the other hand, in Definition 3.8, we considered geodesic curve on base manifold of F, then by using Clairaut's relation F becomes totally umbilical.*

Theorem 3.10 *Let $F : (M, g_M, J) \to (N, g_N)$ be an invariant Riemannian map from a Kähler manifold M to a Riemannian manifold N and α, $\beta = F \circ \alpha$ are geodesic curves on M and N, respectively. Then F is Clairaut invariant Riemannian map with $s = e^g$ if and only if $g_N(\nabla^N_V F_*JX, F_*JX) + g_N(F_*JX, F_*JX)\frac{d(g\circ\beta)}{dt} = 0$, where g is a smooth function on N and $F_*X \in \Gamma(rangeF_*), V \in \Gamma(rangeF_*)^{\perp}$ are components of $\dot{\beta}(t)$.*

Proof Let $\alpha : I \to M$ be a geodesic on M with $U(t) = \nu\dot{\alpha}(t)$ and $X(t) = \mathcal{H}\dot{\alpha}(t)$. Let $\beta = F \circ \alpha$ be a geodesic on N with $F_*X \in \Gamma(rangeF_*)$ and $V \in \Gamma(rangeF_*)^{\perp}$ are components of $\dot{\beta}(t)$ and $\omega(t)$ denote the angle between $\dot{\beta}$ and V. Assuming $b = \|\dot{\beta}(t)\|^2$, then we get

$$g_{N\beta(t)}(V, V) = bcos^2\omega(t), \tag{3.24}$$

$$g_{N\beta(t)}(F_*X, F_*X) = bsin^2\omega(t). \tag{3.25}$$

Now, differentiating (3.25), we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2bsin\omega cos\omega\frac{d\omega}{dt}. \tag{3.26}$$

On the other hand, since M is almost Hermitian manifold and using (2.1), we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = \frac{d}{dt}g_{N\beta(t)}(F_*JX, F_*JX),$$

which implies

$$\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_{N\beta(t)}(\nabla^N_{\dot{\beta}}F_*JX, F_*JX). \tag{3.27}$$

By putting $\dot{\beta} = F_*X + V$ in (3.27), we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_{N\beta(t)}(\nabla^N_{F_*X}F_*JX + \nabla^N_V F_*JX, F_*JX),$$

which implies

$$\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_{N\beta(t)}(\overset{N}{\nabla}{}^F_X F_*(JX) \circ F + \nabla^N_V F_*(JX), F_*(JX)). \tag{3.28}$$

Using (2.8) in (3.28), we get

$$\tfrac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_{N\beta(t)}((\nabla F_*)(X, JX) + F_*(\nabla^M_X JX) + \nabla^N_V F_*(JX), F_*(JX)).$$

Using (3.23) in above equation, we get

$$\tfrac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = \quad 2g_{N\beta(t)}((\nabla F_*)(X, JX) - F_*(\nabla^M_U JX + A_X JU + T_U JU) + \nabla^N_V F_*(JX), F_*(JX)).$$

Since β is geodesic on N then using (3.22) in above equation, we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X, F_*X) = 2g_{N\beta(t)}(\nabla^N_V F_*(JX), F_*(JX)). \tag{3.29}$$

Thus from (3.26) and (3.29), we get

$$g_{N\beta(t)}(\nabla^N_V F_*(JX), F_*(JX)) = bsin\omega cos\omega\frac{d\omega}{dt}. \tag{3.30}$$

Moreover, F is a Clairaut Riemannian map with $s = e^g$ if and only if $\frac{d}{dt}(e^{g\circ\beta} sin\omega) = 0$, that is, $e^{g\circ\beta} sin\omega \frac{d(g\circ\beta)}{dt} + e^{g\circ\beta} cos\omega \frac{d\omega}{dt} = 0$. By multiplying this with nonzero factor $bsin\omega$, and using (3.25), we get

$$g_{N\beta(t)}(F_*X, F_*X)\frac{d(g\circ\beta)}{dt} = -bsin\omega cos\omega \frac{d\omega}{dt}. \tag{3.31}$$

Thus (3.30), (3.31), (2.1) and (2.12) completes the proof. □

Example 3.10 Let M be an Euclidean space given by $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \neq 0, x_2 \neq 0, x_3 \neq 0, x_4 \neq 0\}$. We define the Riemannian metric g_M on M given by $g_M = x_4^2 dx_1^2 + x_4^2 dx_2^2 + x_4^2 dx_3^2 + dx_4^2$ and the complex structure J on M defined as $J(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$. Let $N = \{(y_1, y_2, y_3) \in \mathbb{R}^3\}$ be a Riemannian manifold with Riemannian metric g_N on N given by $g_N = dy_1^2 + y_3^2 dy_2^2 + dy_3^2$. Consider a map $F : (M, g_M, J) \to (N, g_N)$ defined by

$$F(x_1, x_2, x_3, x_4) = (0, x_3, x_4).$$

Then, we get

$$kerF_* = Span\{U_1 = e_1, U_2 = e_2\},$$

and

$$(kerF_*)^\perp = Span\{X_1 = e_3, X_2 = e_4\},$$

where $\left\{e_1 = \frac{1}{x_4}\frac{\partial}{\partial x_1}, e_2 = \frac{1}{x_4}\frac{\partial}{\partial x_2}, e_3 = \frac{1}{x_4}\frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial x_4}\right\}$, $\left\{e_1^* = \frac{\partial}{\partial y_1}, e_2^* = \frac{1}{y_3}\frac{\partial}{\partial y_2}, e_3^* = \frac{\partial}{\partial y_3}\right\}$ are bases on T_pM and $T_{F(p)}N$ respectively, for all $p \in M$. By direct computations, we can see that $F_*(X_1) = e_2^*, F_*(X_2) = e_3^*$ and $g_M(X_i, X_j) = g_N(F_*X_i, F_*X_j)$ for all $X_i, X_j \in \Gamma(kerF_*)^\perp$. Thus F is Riemannian map with $(rangeF_*)^\perp = Span\{e_1^*\}$. Moreover it is easy to see that $JU_1 = -U_2, JU_2 = U_1$ and $(kerF_*)^\perp$ is also invariant with respect to J. Thus F is an invariant Riemannian map.

Now we will find a smooth function f on M satisfying $T_UU = -g_M(U, U)\nabla^M f$ for all $U \in \Gamma(kerF_*)$. Here any $U \in \Gamma(kerF_*)$ can be written as $U = \lambda_1 U_1 + \lambda_2 U_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Since covariant derivative for vector fields $E = E_i\frac{\partial}{\partial x_i}, F = F_j\frac{\partial}{\partial x_j}$ on M is defined as

$$\nabla_E^M F = E_i F_j \nabla^M_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} + E_i\frac{\partial F_j}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{3.32}$$

where the covariant derivatives of basis vector fields $\frac{\partial}{\partial x_j}$ along $\frac{\partial}{\partial x_i}$ is defined by

$$\nabla^M_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \Gamma_{ij}^k\frac{\partial}{\partial x_k}, \tag{3.33}$$

and Christoffel symbols are defined by

$$\Gamma_{ij}^k = \frac{1}{2}g_M^{kl}\Big(\frac{\partial g_{Mjl}}{\partial x_i} + \frac{\partial g_{Mil}}{\partial x_j} - \frac{\partial g_{Mij}}{\partial x_l}\Big). \tag{3.34}$$

Now, we have

$$g_{Mij} = \begin{pmatrix} x_4^2 & 0 & 0 & 0 \\ 0 & x_4^2 & 0 & 0 \\ 0 & 0 & x_4^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g_M^{ij} = \begin{pmatrix} \frac{1}{x_4^2} & 0 & 0 & 0 \\ 0 & \frac{1}{x_4^2} & 0 & 0 \\ 0 & 0 & \frac{1}{x_4^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.35}$$

By using (3.34) and (3.35), we obtain

$$\begin{array}{l} \Gamma_{11}^{1}=0,\Gamma_{11}^{2}=0,\Gamma_{11}^{3}=0,\Gamma_{11}^{4}=-x_{4}, \\ \Gamma_{22}^{1}=0,\Gamma_{22}^{2}=0,\Gamma_{22}^{3}=0,\Gamma_{22}^{4}=-x_{4}, \\ \Gamma_{12}^{1}=0=\Gamma_{21}^{1},\Gamma_{12}^{2}=0=\Gamma_{21}^{2},\Gamma_{12}^{3}=0=\Gamma_{21}^{3},\Gamma_{12}^{4}=0=\Gamma_{21}^{4}. \end{array} \tag{3.36}$$

By using (3.32), (3.33) and (3.36), we obtain

$$\nabla^M_{e_1}e_1=\nabla^M_{e_2}e_2=-\frac{1}{x_4}\frac{\partial}{\partial x_4},\nabla^M_{e_2}e_1=\nabla^M_{e_1}e_2=0.$$

Therefore

$$\nabla^M_U U=\nabla^M_{\lambda_1e_1+\lambda_2e_2}\lambda_1e_1+\lambda_2e_2=\lambda_1^2\nabla^M_{e_1}e_1+\lambda_1\lambda_2\nabla^M_{e_1}e_2+\lambda_1\lambda_2\nabla^M_{e_2}e_1+\lambda_2^2\nabla^M_{e_2}e_2$$

$$=-(\lambda_1^2+\lambda_2^2)\Big(\frac{1}{x_4}\frac{\partial}{\partial x_4}\Big).$$

Then by (2.4), we get

$$T_UU=-(\lambda_1^2+\lambda_2^2)\Big(\frac{1}{x_4}\frac{\partial}{\partial x_4}\Big).$$

For any smooth function f on M, the gradient of f with respect to the metric g_M is given by $\nabla^M f = \sum_{i,j=1}^{4} g_M^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$. Therefore

$$\nabla^M f=\frac{1}{x_4^2}\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1}+\frac{1}{x_4^2}\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2}+\frac{1}{x_4^2}\frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_3}+\frac{\partial f}{\partial x_4}\frac{\partial}{\partial x_4}.$$

Hence $\nabla^M f=\frac{1}{x_4}\frac{\partial}{\partial x_4}$ for the function $f=log(x_4)$. Then it is easy to verify that $T_UU=-g_M(U,U)\nabla^M f$, where $g_M(U,U)=(\lambda_1^2+\lambda_2^2)$, for any vertical vector field U. Thus by Theorem 3.2, we see that F is Clairaut invariant Riemannian map.

4. Clairaut invariant Riemannian maps to Kähler manifolds

In this section, we introduce Clairaut invariant Riemannian maps to Kähler manifolds and find a necessary and sufficient condition for invariant Riemannian maps from Riemannian maps to Kähler manifolds to be Clairaut invariant Riemannian maps and give a non-trivial example of such maps.

Definition 4.1 *[10] Let $F:(M,g_M)\to(N,g_N)$ be a proper Riemannian map from a Riemannian manifold M to a Hermitian manifold N with almost complex structure J_N. We say that F is an invariant Riemannian map at $p\in M$ if $J_N(rangeF_{*p})=rangeF_{*p}$. If F is an invariant Riemannian map for every $p\in M$ then F is called an invariant Riemannian map.*

Definition 4.2 *An invariant Riemannian map from Riemannian manifold to Kähler manifold is called Clairaut invariant Riemannian map if it satisfies the condition of Definition 3.8.*

Theorem 4.3 *[21] Let* $F:(M^m,g_M)\to(N^n,g_N)$ *be a Riemannian map between Riemannian manifolds such that* $(rangeF_*)^\perp$ *is totally geodesic and* α, $\beta=F\circ\alpha$ *are geodesic curves on* M *and* N*, respectively. Then* F *is Clairaut Riemannian map with* $s=e^g$ *if and only if* F *is umbilical, and has* $H_2=-\nabla^N g$*, where* H_2 *is mean curvature vector field of* $rangeF_*$ *and* g *is a smooth function on* N*.*

Proposition 4.4 *Let* $F:(M,g_M)\to(N,g_N,J_N)$ *be an invariant Riemannian map from a Riemannian manifold* M *to a Kähler manifold* N *such that* $(rangeF_*)^\perp$ *is totally geodesic and* $\alpha:I\to M$ *be a geodesic curve on* M*. Then the curve* $\beta=F\circ\alpha$ *is geodesic curve on* N *if and only if*

$$J_N(\nabla F_*)(X,X)+\nabla_X^{F\perp}J_NV+\nabla_V^{F\perp}J_NV=0, \tag{4.1}$$

$$-\mathcal{S}_{J_NV}F_*X+J_NF_*(\nabla_X^MX)+\nabla_V^NJ_NF_*X=0, \tag{4.2}$$

where $F_*X\in\Gamma(rangeF_*), V\in\Gamma(rangeF_*)^\perp$ *are components of* $\dot\beta(t)$ *and* ∇^N *is the Levi–Civita connection on* N *and* $\nabla^{F\perp}$ *is a linear connection on* $(rangeF_*)^\perp$*.*

Proof Let F be an invariant Riemannian map from a Riemannian manifold M to a Kähler manifold N such that $(rangeF_*)^\perp$ is totally geodesic, i.e. $\nabla_U^NV=\nabla_U^{F\perp}V$ for all $U,V\in\Gamma(rangeF_*)^\perp$. Let $\alpha:I\to M$ be a geodesic on M with $U(t)=\nu\dot\alpha(t)$ and $X(t)=\mathcal{H}\dot\alpha(t)$. Let $\beta=F\circ\alpha$ be a geodesic on N with $F_*X\in\Gamma(rangeF_*)$ and $V\in\Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$. Since N is Kähler manifold, we have $\nabla_{\dot\beta}^N\dot\beta=-J_N\nabla_{\dot\beta}^NJ_N\dot\beta$.

Now,

$$\nabla_{\dot\beta}^NJ_N\dot\beta=\nabla_{F_*X+V}^NJ_N(F_*X+V),$$

which is equal to

$$\nabla_{\dot\beta}^NJ_N\dot\beta=J_N\nabla_{F_*X}^NF_*X+\nabla_{F_*X}^NJ_NV+\nabla_V^NJ_NF_*X+\nabla_V^NJ_NV.$$

Using (2.10) in above equation, we get

$$\nabla_{\dot\beta}^NJ_N\dot\beta=J_N\overset{N}{\nabla}{}_X^FF_*X\circ F-\mathcal{S}_{J_NV}F_*X+\nabla_X^{F\perp}J_NV+\nabla_V^NJ_NF_*X+\nabla_V^NJ_NV.$$

Since $g_N(\nabla_V^NJ_NF_*X,U)=-g_N(J_NF_*X,\nabla_V^NU)=-g_N(J_NF_*X,\nabla_V^{F\perp}U)=0$ for $U\in\Gamma(rangeF_*)^\perp$, which implies $\nabla_V^NJ_NF_*X\in\Gamma(rangeF_*)$ and using (2.8) in above equation, we get

$$\nabla_{\dot\beta}^NJ_N\dot\beta=\quad J_N((\nabla F_*)(X,X)+F_*(\nabla_X^MX))-\mathcal{S}_{J_NV}F_*X+\nabla_X^{F\perp}J_NV+\nabla_V^NJ_NF_*X+\nabla_V^NJ_NV. \tag{4.3}$$

Since $(rangeF_*)^\perp$ is totally geodesic therefore putting $\nabla_V^NJ_NV=\nabla_V^{F\perp}J_NV$ in (4.3), we get

$$\nabla_{\dot\beta}^NJ_N\dot\beta=\quad J_N(\nabla F_*)(X,X)+J_NF_*(\nabla_X^MX)-\mathcal{S}_{J_NV}F_*X+\nabla_X^{F\perp}J_NV+\nabla_V^NJ_NF_*X+\nabla_V^{F\perp}J_NV. \tag{4.4}$$

Now β is geodesic on N if and only if $\nabla_{\dot\beta}^N\dot\beta=0$ if and only if $-J_N(\nabla_{\dot\beta}^NJ_N\dot\beta)=0$ if and only if $\nabla_{\dot\beta}^NJ_N\dot\beta=0$. Then (4.4) implies $J_N(\nabla F_*)(X,X)+J_NF_*(\nabla_X^MX)-\mathcal{S}_{J_NV}F_*X+\nabla_X^{F\perp}J_NV+\nabla_V^NJ_NF_*X+\nabla_V^{F\perp}J_NV=0$, which completes the proof. □

Theorem 4.5 *Let $F:(M,g_M)\to(N,g_N,J_N)$ be an invariant Riemannian map from Riemannian manifold M to Kähler manifold N such that $(rangeF_*)^\perp$ is totally geodesic and α, $\beta=F\circ\alpha$ are geodesic curves on M and N, respectively. Then F is Clairaut invariant Riemannian map with $s=e^g$ if and only if $\mathcal{S}_{J_NV}F_*X=-V(g)J_NF_*X$, where g is a smooth function on N and $F_*X\in\Gamma(rangeF_*), V\in\Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$.*

Proof Let $\alpha:I\to M$ be a geodesic on M with $U(t)=\nu\dot\alpha(t)$ and $X(t)=\mathcal{H}\dot\alpha(t)$. Let $\beta=F\circ\alpha$ be a geodesic on N with $F_*X\in\Gamma(rangeF_*)$ and $V\in\Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$ and $\omega(t)$ denote the angle between $\dot\beta$ and V. Assuming $b=\|\dot\beta(t)\|^2$, then we get

$$g_{N\beta(t)}(V,V)=bcos^2\omega(t), \tag{4.5}$$

$$g_{N\beta(t)}(F_*X,F_*X)=bsin^2\omega(t). \tag{4.6}$$

Now differentiating (4.6), we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X,F_*X)=2bsin\omega cos\omega\frac{d\omega}{dt}. \tag{4.7}$$

On the other hand

$$\frac{d}{dt}g_{N\beta(t)}(F_*X,F_*X)=2g_N(\nabla^N_{\dot\beta}J_NF_*X,J_NF_*X).$$

By putting $\dot\beta=F_*X+V$ in above equation, we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X,F_*X)=2g_N(\nabla^N_{F_*X}J_NF_*X+\nabla^N_VJ_NF_*X,J_NF_*X),$$

which implies

$$\frac{d}{dt}g_{N\beta(t)}(F_*X,F_*X)=2g_N(J_N\overset{N}{\nabla}{}^F_XF_*X\circ F+\nabla^N_VJ_NF_*X,J_NF_*X). \tag{4.8}$$

Using (2.8) and (4.2) in (4.8), we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X,F_*X)=2g_N(J_N(\nabla F_*)(X,X)+\mathcal{S}_{J_NV}F_*X,J_NF_*X).$$

Using (2.9) in above equation, we get

$$\frac{d}{dt}g_{N\beta(t)}(F_*X,F_*X)=2g_N(\mathcal{S}_{J_NV}F_*X,J_NF_*X). \tag{4.9}$$

Now from (4.7) and (4.9), we get

$$g_N(\mathcal{S}_{J_NV}F_*X,J_NF_*X)=bsin\omega cos\omega\frac{d\omega}{dt}. \tag{4.10}$$

Moreover, F is a Clairaut Riemannian map with $s=e^g$ if and only if $\frac{d}{dt}(e^{g\circ\beta}sin\omega)=0$, that is, $e^{g\circ\beta}sin\omega\frac{d(g\circ\beta)}{dt}+e^{g\circ\beta}cos\omega\frac{d\omega}{dt}=0$. By multiplying this with nonzero factor $bsin\omega$ and using (4.6), we get

$$g_{N\beta(t)}(F_*X,F_*X)\frac{d(g\circ\beta)}{dt}=-bsin\omega cos\omega\frac{d\omega}{dt}. \tag{4.11}$$

Now from (4.10) and (4.11), we get

$$g_N(\mathcal{S}_{J_N V}F_*X, J_N F_*X) = -g_{N\beta(t)}(F_*X, F_*X)\frac{d(g\circ\beta)}{dt}.$$

Since N is Kähler manifold then above equation can be written as

$$g_N(\mathcal{S}_{J_N V}F_*X, J_N F_*X) = -g_{N\beta(t)}(J_N F_*X, J_N F_*X)\frac{d(g\circ\beta)}{dt},$$

which means

$$g_N(\mathcal{S}_{J_N V}F_*X, J_N F_*X) = -g_{N\beta(t)}(J_N F_*X, J_N F_*X)g_N(\nabla^N g, \dot{\beta}). \tag{4.12}$$

Since $\nabla^N g \in \Gamma(rangeF_*)^\perp$ and $\dot{\beta} = F_*X + V$, then from (4.12), we get

$$g_N(\mathcal{S}_{J_N V}F_*X, J_N F_*X) = -g_{N\beta(t)}(J_N F_*X, J_N F_*X)g_N(\nabla^N g, V). \tag{4.13}$$

Thus $\mathcal{S}_{J_N V}F_*X = -V(g)J_N F_*X$, where $V(g)$ is a smooth function on N, which completes the proof. □

Example 4.6 Let M be an Euclidean space given by $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 \neq 0, x_2 \neq 0, x_3 \neq 0, x_4 \neq 0\}$. We define the Riemannian metric g_M on M given by $g_M = e^{2x_4}dx_1^2 + e^{2x_4}dx_2^2 + e^{2x_4}dx_3^2 + dx_4^2$. Let $N = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4\}$ be a Riemannian manifold with Riemannian metric g_N on N given by $g_N = e^{2x_4}dy_1^2 + e^{2x_4}dy_2^2 + dy_3^2 + dy_4^2$ and the complex structure J_N on N defined as $J_N(y_1, y_2, y_3, y_4) = (-y_2, y_1, -y_4, y_3)$. Consider a map $F : (M, g_M) \to (N, g_N, J_N)$ defined by

$$F(x_1, x_2, x_3, x_4) = \Big(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, 0, 0\Big).$$

Then, we get

$$kerF_* = Span\{U_1 = e_3, U_2 = e_4\},$$

and

$$(kerF_*)^\perp = Span\{X_1 = e_1, X_2 = e_2\},$$

where $\Big\{e_1 = e^{-x_4}\frac{\partial}{\partial x_1}, e_2 = e^{-x_4}\frac{\partial}{\partial x_2}, e_3 = e^{-x_4}\frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial x_4}\Big\}$, $\Big\{e_1^* = e^{-x_4}\frac{\partial}{\partial y_1}, e_2^* = e^{-x_4}\frac{\partial}{\partial y_2}, e_3^* = \frac{\partial}{\partial y_3}, e_4^* = \frac{\partial}{\partial y_4}\Big\}$ are bases on T_pM and $T_{F(p)}N$ respectively, for all $p \in M$. By easy computations, we see that $F_*(X_1) = \frac{1}{\sqrt{2}}(e_1^* + e_2^*), F_*(X_2) = \frac{1}{\sqrt{2}}(e_1^* - e_2^*)$ and $g_M(X_i, X_j) = g_N(F_*X_i, F_*X_j)$ for all $X_i, X_j \in \Gamma(kerF_*)^\perp$. Thus F is Riemannian map with $rangeF_* = Span\Big\{F_*(X_1) = \frac{1}{\sqrt{2}}(e_1^* + e_2^*), F_*(X_2) = \frac{1}{\sqrt{2}}(e_1^* - e_2^*)\Big\}$ and $(rangeF_*)^\perp = Span\{e_3^*, e_4^*\}$. Moreover it is easy to see that $J_N F_*X_1 = F_*X_2$, $J_N F_*X_2 = -F_*X_1$ and $(rangeF_*)^\perp$ is also invariant with respect to J_N. Thus F is an invariant Riemannian map.

Now to show that F is Clairaut Riemannian map we will find a smooth function g on N satisfying $(\nabla F_*)(X, X) = -g_M(X, X)\nabla^N g$ for any $X \in \Gamma(kerF_*)^\perp$. Since by (2.9) $(\nabla F_*)(X, X) \in \Gamma(rangeF_*)^\perp$ for any $X \in \Gamma(kerF_*)^\perp$. So, here we can write $(\nabla F_*)(X, X) = \lambda_1 e_3^* + \lambda_2 e_4^*$, and $g_M(X, X) = g_M(\alpha_1 X_1 + \alpha_2 X_2, \alpha_1 X_1 + \alpha_2 X_2) = \alpha_1^2 + \alpha_2^2$ for some $\lambda_1, \lambda_2, \alpha_1, \alpha_2 \in \mathbb{R}$.

Since, we have

$$g_{Nij}=\begin{pmatrix} e^{2x_4} & 0 & 0 & 0\\ 0 & e^{2x_4} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, g_N^{ij}=\begin{pmatrix} e^{-2x_4} & 0 & 0 & 0\\ 0 & e^{-2x_4} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

For any smooth function g on N, the gradient of g with respect to the metric g_N is given by $\nabla^N g = \sum_{i,j=1}^{4} g_N^{ij}\frac{\partial g}{\partial y_i}\frac{\partial}{\partial y_j}$. Thus

$$\nabla^N g = e^{-2x_4}\frac{\partial g}{\partial y_1}\frac{\partial}{\partial y_1}+e^{-2x_4}\frac{\partial g}{\partial y_2}\frac{\partial}{\partial y_2}+\frac{\partial g}{\partial y_3}\frac{\partial}{\partial y_3}+\frac{\partial g}{\partial y_4}\frac{\partial}{\partial y_4}.$$

Hence $\nabla^N g = -\Big(\frac{\lambda_1}{\alpha_1^2+\alpha_2^2}\frac{\partial}{\partial y_3}+\frac{\lambda_2}{\alpha_1^2+\alpha_2^2}\frac{\partial}{\partial y_4}\Big)$ for the function $g=-\frac{\lambda_1}{\alpha_1^2+\alpha_2^2}y_3-\frac{\lambda_2}{\alpha_1^2+\alpha_2^2}y_4$. Then it is easy to verify that $(\nabla F_*)(X,X) = -g_M(X,X)\nabla^N g$, for any vector field $X\in\Gamma(kerF_*)^{\perp}$. Thus by Theorem 4.3 and Lemma 2.1, we see that F is Clairaut invariant Riemannian map.

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