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Second main theorem for meromorphic mappings intersecting moving targets on parabolic manifolds

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Abstract: In this paper, we establish a new second main theorem for meromorphic mappings from M into $\mathbb{P}(V)$ intersecting moving targets $g_j : M \to \mathbb{P}(V^*)$, $1 \le j \le q$, where M is a parabolic manifold and V is a Hermitian vector space. As an application, we prove the algebraic dependence problem for meromorphic mappings with moving targets in general position.

Key words: Second main theorem, meromorphic mappings, algebraic dependence problem, parabolic manifolds, moving targets

1. Introduction and main results

In 1933, Cartan [2] established the second main theorem for linearly nondegenerate holomorphic curves into complex projective spaces intersecting hyperplanes in general position which extended Nevanlinna's results to the high dimensional case. Later on, value distribution theory for meromorphic mappings of $\mathbb{C}^m$ into the complex projective space $\mathbb{P}^n(\mathbb{C})$ intersecting fixed or moving targets had been studied and many interesting results had been established [10, 14, 16, 19]. For the case of degenerate meromorphic mappings, M. Ru and J. Wang [15] obtained a second main theorem for moving hyperplanes with truncated counting function. In 2008, D. D. Thai and S. D. Quang [22] improved the result of Ru-Wang [15].

In 2019, S. D. Quang [11] proved the second main theorems for degenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting moving hyperplanes with counting function truncated to level k as follows.

Theorem A ([11, Theorem 1.1]) *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q (q \ge 2n-k+2)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})^*$ in general position such that $(f, a_i) \not\equiv 0 (1 \le i \le q)$, where $k+1 = \operatorname{rank}_{\mathcal{R}}(f)$. Then the following assertions hold:*

$$\| \frac{q-(n-k)}{n+2} T_f(r) \le \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\le i\le q} T_{a_i}(r,s)).$$

Theorem B ([11, Theorem 1.3]) *With the assumption of Theorem A, we assume further that $q \ge (n-k)(k+1)+n+2$. Then we have*

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$$\| \ \frac{q}{k+2}T_f(r) \leq \sum_{i=1}^{q} N^{[k]}_{(f,a_i)}(r) + o(T_f(r)) + O(\max_{1\leq i\leq q} T_{a_i}(r,s)).$$

In this paper, by the notation " $\|$ " we mean that the inequality holds for all $r \in [0,\infty)$ outside a Borel subset E of the interval $[0,\infty)$ with $\int_E dr < \infty$.

W. Stoll [18] obtained the second main theorem for linearly nondegenerate meromorphic maps from an m-dimensional parabolic manifold to $\mathbb{P}^n(\mathbb{C})$ with fixed hyperplanes. In the case of meromorphic mappings interesting moving targets, Ashline [1] obtained the logarithmic derivative lemma on parabolic manifolds with an additional assumption and proved the corresponding second main theorem. In 1997, M. Ru [12] established the second main theorem with moving targets on parabolic manifold and the logarithmic derivative lemma could be avoided. In 2015, Q. Yan [25] proved the following truncated second main theorem for meromorphic maps with moving targets in general position on parabolic manifolds.

Theorem C ([25, Theorem 1.1]) *Let (M,τ) be an admissible parabolic manifold with dimension m. Let V be a Hermitian vector space with* $\dim V = n+1 > 1$. *Let $g_1,\ldots,g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position. Let $f : M \to \mathbb{P}(V)$ be a nonconstant meromorphic map such that* $\mathrm{Ric}_\tau(r,s) = o(T_f(r,s))$ *and* $\log Y(r) = o(T_f(r,s))$ *for $r \to \infty$. Assume that (f,g_j) is free for $1 \leq j \leq q$ and* $\dim(\mathrm{supp}\mu_{(f,g_i)} \cap \mathrm{supp}\mu_{(f,g_j)}) \leq m-2$ *for $1 \leq i < j \leq q$. If $q \geq 2n+1$, then for $s > 0$,*

$$\| \ \frac{q}{2n+1}T_f(r,s) \leq \sum_{j=1}^{q} N^{[n]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1\leq j\leq q} T_{g_j}(r,s)),$$

where $\mathrm{Ric}_\tau(r,s)$ is the Ricci function, which depends only on the geometry (topology) of the manifold M.

The main purpose of this paper is to give a new second main theorem for degenerate meromorphic mappings with moving targets on parabolic manifolds. Throughout this paper, we shall use the standard notations in the value distribution theory of meromorphic mappings on parabolic manifolds ([18, 23]). To establish the value distribution theory, we shall work on admissible parabolic manifolds, which satisfy the following assumptions:

(i) M is a connected complex manifold of dimension m.

(ii) There exists a parabolic exhaustion function τ on M.

(iii) For any positive integer n, let $\Psi : M \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map. Then there is a holomorphic differential form B of degree $(m-1,0)$ on M such that Ψ is general for B and

$$m i_{m-1} B \wedge \bar{B} \leq Y(r) v^{m-1}$$

on $M[r]$ for some real position valued function $Y(r)$ on M, which is independent of Ψ ($Y(r)$ is called a majorant for B). Here, for any positive integer m,

$$i_m := \left(\frac{\sqrt{-1}}{2\pi}\right)^m (-1)^{\frac{m(m-1)}{2}} m!.$$

We can prove the following second main theorem.

Theorem 1.1 *Let* (M,τ) *be an admissible parabolic manifold with dimension* m. *Let* V *be a Hermitian vector space with* $\dim V = n+1 > 1$. *Let* $g_1,\dots,g_q : M \to \mathbb{P}(V^*)$ *be meromorphic maps located in general position. Let* $f : M \to \mathbb{P}(V)$ *be a nonconstant meromorphic map with* $\mathrm{rank}_{\mathcal{R}_{\mathcal{G}}}(f) = k+1$ *such that* $\mathrm{Ric}_\tau(r,s) = o(T_f(r,s))$ *and* $\log Y(r) = o(T_f(r,s))$ *for* $r \to \infty$. *Assume that* (f,g_j) *is free for* $1 \le j \le q$ *and* $\dim(\mathrm{supp}\mu_{(f,g_i)} \cap \mathrm{supp}\mu_{(f,g_j)}) \le m-2$ *for* $1 \le i < j \le q$. *If* $q \ge 2n-k+2$, *then for* $s > 0$,

$$\| \ \frac{q-(n-k)}{n+2} T_f(r,s) \le \sum_{j=1}^{q} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1\le j\le q} T_{g_j}(r,s)).$$

We see that the Theorem 1.1 is an improvement of Theorem B.

We will prove a better second main theorem as follows if the number of moving targets is large enough.

Theorem 1.2 *With the assumptions of Theorem 1.1, we assume further that* $q \ge (n-k+1)(k+2)$. *Then for* $s > 0$, *we have*

$$\| \ \frac{q}{k+2} T_f(r,s) \le \sum_{j=1}^{q} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1\le j\le q} T_{g_j}(r,s)).$$

Remark 1.3 *In the case where the mappings are assumed to be nondegenerate, we see that the coefficients in front of the characteristic function in Theorem 1.1 are the same as in Theorem 1.2.*

As the application of the second main theorem, the uniqueness problem have been given over the last few decades. In 1926, Nevanlinna [8] proved the well-known five-value theorem and four-value theorem for meromorphic functions on the complex plane $\mathbb{C}$. Fujimoto [6] generalized the Nevanlinna's five value theorem to the case of meromorphic mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with counting multiplicities. After that, many scholars had interest in the uniqueness problem of meromorphic mappings with counting multiplicities or ignoring multiplicities and obtained a lot of significant results [3–5, 7, 9, 21, 24, 26]. W. Stoll [20] proved algebraic dependence problem for meromorphic mappings with fixed hyperplanes on parabolic covering space which extended the results given by Smiley [17]. Ru [13] extended Stoll's result to moving targets, and he gave the algebraic dependence problem for the case of meromorphic mappings with moving targets. To state our results, we need the following notations [20, 25].

Let M be a connected complex manifold of dimension m with surjective, proper holomorphic map $\pi : M \to \mathbb{C}^m$. Then $\tau = \|\pi\|^2$ is parabolic exhaustion of M and (M,τ) is called a parabolic covering space of $\mathbb{C}^m$. Let V be a Hermitian vector space of dimension $n+1 > 1$. Let $f_t : M \to \mathbb{P}(V)$ be a meromorphic mapping for $t = 1,\dots,\lambda$ and A be a nonempty subset of M. $f_1,\dots,f_\lambda$ are said to be in p-special position on A if, for any $x \in A$, there exists an open, connected neighborhood U_x of x and a reduced representation $F_t : U_x \to V$ (of f_t) such that, for any $1 \le t_1 < t_2 < \ldots < t_p \le \lambda$,

$$F_{t_1}(x) \wedge \ldots \wedge F_{t_p}(x) = 0.$$

If $A = M$ omit "on A". Also "special position" means "λ-special position". Then f_1 and f_2 are in 2-special position if and only if $f_1 \equiv f_2$. If $f_1,\dots,f_\lambda$ are in λ-special position, then $f_1,\dots,f_\lambda$ are algebraically dependent, where $f_1,\dots,f_\lambda$ are said to be algebraically dependent if and only if there is a proper analytic subset S of $\mathbb{P}(V)^\lambda$ such that $(f_1,\dots,f_\lambda) \subset S$.

Using Theorem B, Q. Yan [25] gave an algebraic dependence theorem for meromorphic mappings with moving targets as following.

Theorem D ([25, Theorem 1.2]) *Let $f_1, \ldots, f_\lambda : M \to \mathbb{P}(V)$ be nonconstant meromorphic maps such that $\mathrm{Ric}_\tau(r,s) = o(T_{f_t}(r,s))$ and $\log Y(r) = o(T_{f_t}(r,s))$ for $r \to \infty$, $1 \le t \le \lambda$. Let $g_1, \ldots, g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position and $T_{g_j}(r,s) = o(\max_{1\le t\le\lambda} T_{f_t}(r,s))$ for $r \to \infty, 1 \le j \le q$. Assume that (f_t, g_j) is free for $1 \le t \le \lambda, 1 \le j \le q$. Assume that $A_j = \mathrm{supp}\mu_{(f_1,g_j)} = \mathrm{supp}\mu_{(f_2,g_j)} = \cdots = \mathrm{supp}\mu_{(f_\lambda,g_j)}$ for each $1 \le j \le q$, and $\dim(A_i \cap A_j) \le m-2$ for $1 \le i < j \le q$. Define $A = \bigcup_{j=1}^q A_j$. Assume that $f_1, \ldots, f_\lambda$ are in l-special position on A, where l is an integer with $2 \le l \le \lambda$. If $q > \frac{n(2n+1)\lambda}{\lambda-l+1}$, then $f_1, \ldots, f_\lambda$ are in special position.*

We will apply Theorem 1.1 and show the following result.

Theorem 1.4 *Let $f_1, \ldots, f_\lambda : M \to \mathbb{P}(V)$ be nonconstant meromorphic maps such that $\mathrm{Ric}_\tau(r,s) = o(T_{f_t}(r,s))$ and $\log Y(r) = o(T_{f_t}(r,s))$ for $r \to \infty$, $1 \le t \le \lambda$. Let $g_1, \ldots, g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position and $T_{g_j}(r,s) = o(\max_{1\le t\le\lambda} T_{f_t}(r,s))$ for $r \to \infty, 1 \le j \le q$. Let $m_j (j = 1, \ldots, q)$ be positive integers or ∞ with $m_1 \ge m_2 \ge \ldots \ge m_q$. Assume that (f_t, g_j) is free for $1 \le t \le \lambda, 1 \le j \le q$. Assume that $A_j = \mathrm{supp}\mu_{(f_1,g_j),\le m_j} = \mathrm{supp}\mu_{(f_2,g_j),\le m_j} = \cdots = \mathrm{supp}\mu_{(f_\lambda,g_j),\le m_j}$ for each $1 \le j \le q$, and $\dim(A_i \cap A_j) \le m-2$ for $1 \le i < j \le q$. Define $A = \cup_{j=1}^q A_j$. Assume that $f_1, \ldots, f_\lambda$ are in l-special position on A, where l is an integer with $2 \le l \le \lambda$. If $k+1 = \max_{1\le t\le\lambda}\{\mathrm{rank}_{\mathcal{R}_\mathcal{G}}(f_t)\}$, $k'+1 = \min_{1\le t\le\lambda}\{\mathrm{rank}_{\mathcal{R}_\mathcal{G}}(f_t)\}$,*

$$\sum_{j=3}^{q} \frac{m_j}{m_j+1} > q - 2 - \frac{q-(n-k)}{(n+2)k} + \frac{2}{m_2+1} + (1 - \frac{k'}{m_2+1})\frac{\lambda}{\lambda-l+1},$$

then $f_1, \ldots, f_\lambda$ are in special position.

Similar to the proof of Theorem 1.4, we have the following result by using Theorem 1.2.

Theorem 1.5 *With the assumptions of Theorem 1.4, if $k+1 = \max_{1\le t\le\lambda}\{\mathrm{rank}_{\mathcal{R}_\mathcal{G}}(f_t)\}$, $k'+1 = \min_{1\le t\le\lambda}\{\mathrm{rank}_{\mathcal{R}_\mathcal{G}}(f_t)\}$,*

$$\sum_{j=3}^{q} \frac{m_j}{m_j+1} > q - 2 - \frac{q}{(k+2)k} + \frac{2}{m_2+1} + (1 - \frac{k'}{m_2+1})\frac{\lambda}{\lambda-l+1},$$

then $f_1, \ldots, f_\lambda$ are in special position.

Remark 1.6 (1) *Letting $m_j = +\infty$, $(1 \le j \le q)$, we have the following results.*
(a) *By Theorem 1.4, we have*

$$q > \frac{\lambda(n+2)k}{\lambda-l+1} + (n-k).$$

(b) *By Theorem 1.5, we have $q > \frac{\lambda(k+2)k}{\lambda-l+1}$, which is improvement of the Theorem D.*
(2) *Letting $\lambda = l = 2$ and $m_j = +\infty$, we get the following uniqueness theorems.*
(a) *By Theorem 1.4, if $q > 2nk + n + 3k$, then $f_1 = f_2$.*
(b) *By Theorem 1.5, if $q > 2k(k+2)$, then $f_1 = f_2$.*

We organize our paper as follows: In next section, we recall some basic notations and definitions in the value distribution theory on parabolic manifolds. We give the proofs of Theorem 1.1 and Theorem 1.2 in Section 3 and prove Theorem 1.4 in Section 4. In this paper, we adopt the methods and techniques by Quang [11] to prove the main theorems.

2. Preliminaries

We recall some fundamental notations and results of meromorphic maps on parabolic manifolds, for references, see [18, 23].

2.1. Parabolic manifolds and Hermitian vector space

Let M be a connected, complex manifold of dimension m. Let τ be a nonnegative function of class C^∞ on M. For $r \geq 0$ and $A \subseteq M$, define

$$A[r] = \{x \in A | \tau(x) \leq r^2\},\ A(r) = \{x \in A | \tau(x) < r^2\},$$

$$A\langle r\rangle = \{x \in A | \tau(x) = r^2\},\ A_* = \{x \in A | \tau(x) > 0\},$$

$$v = dd^c\tau,\ \omega = dd^c \log \tau,\ \sigma = d^c \log \tau \wedge \omega^{m-1}.$$

If $M[r]$ is compact for each $r > 0$, then the function τ is said to be an exhaustion of M. The function τ is said to be parabolic if

$$\omega \geq 0, \quad d\sigma = \omega^m \equiv 0, \quad v^m \not\equiv 0 \text{ on } M_*.$$

Note that $v \geq 0$ on M. If τ is a parabolic exhaustion, (M, τ) is said to be a parabolic manifold. Define $\hat{\mathbb{R}}_\tau = \{r \in \mathbb{R}^+ | d\tau(x) \neq 0 \text{ for all } x \in M\langle r\rangle\}$. Then $\mathbb{R}^+ \setminus \hat{\mathbb{R}}_\tau$ has measure zero. If $r \in \hat{\mathbb{R}}_\tau$, then $M\langle r\rangle$ is the boundary of $M(r)$ and $M\langle r\rangle$ is a differentiable, $(2m-1)$-dimensional submanifold of class C^∞ which we orient to the exterior of $M(r)$. By Stoll ([18]), for all $r \in \hat{\mathbb{R}}_\tau$, $\int_{M\langle r\rangle} \sigma$ is a positive constant, independent of r. Let $\kappa = \int_{M\langle r\rangle} \sigma$. In addition,

$$\int_{M[r]} v^m = \int_{M(r)} v^m = \kappa r^{2m}.$$

We will introduce the definition of Hermitian vector space as follows.

Let V be a complex vector space with dimension $n+1 > 1$. A function $\varphi : V \times V \to \mathbb{C}$ is called a Hermitian metric on V if and only if

(i) $\varphi(\alpha, \alpha) > 0$ for all $\alpha \in V - \{0\}$.

(ii) $\varphi(a\alpha + b\beta, \gamma) = a\varphi(\alpha, \gamma) + b\varphi(\beta, \gamma)$ for all a and b in $\mathbb{C}$ and all α, β and γ in V.

(iii) $\varphi(\alpha, \beta) = \overline{\varphi(\beta, \alpha)}$ for all $\alpha,\ \beta \in V$.

A norm $\|\|_\varphi : V \to \mathbb{R}_+$ called the Hermitian norm defined by φ is given by

$$\|\alpha\|_\varphi = \sqrt{\varphi(\alpha, \alpha)}.$$

Then

$$|\varphi(\alpha, \beta)| \leq \|\alpha\|_\varphi \|\beta\|_\varphi.$$

A complex vector space (V, φ) with a Hermitian metric is called a Hermitian vector spaces.

2.2. Meromorphic maps, reduced representation

Let M be a complex manifold with $\dim M = m$. Let $A \neq \emptyset$ be an open subset of M such that $S = M - A$ is analytic. Then A is dense in M. Let V be a complex vector space with dimension $n+1 > 1$. Let $f : A \to \mathbb{P}(V)$ be a holomorphic map on A. The closure Γ of the graph $\{(x, f(x))|x \in A\}$ in $M \times \mathbb{P}(V)$ is called the closed graph of f. The map f is said to be meromorphic on M if (i) $\Gamma(f)$ is analytic in $M \times \mathbb{P}(V)$ and (ii) $\Gamma(f) \cap (K \times \mathbb{P}(V))$ is compact for each compact subset $K \subseteq M$, i.e. the project $\rho : \Gamma(f) \to M$ is proper. If f is meromorphic, then the set of *indeterminacy* $I_f = \{x \in M | \sharp\rho^{-1}(x) > 1\}$ is analytic with $\dim I_f \leq m-2$ and is contained in S. The holomorphic map $f : A \to \mathbb{P}(V)$ continues to a holomorphic map $f : M - I_f \to \mathbb{P}(V)$ such that we can assume that $S = I_f$.

Suppose that $f : A \to \mathbb{P}(V)$ is a holomorphic map as above. Also, suppose that U is a nonempty, open, connected subset of M. A holomorphic map $F : U \to V$ is called a representation of f on U if $F \not\equiv 0$ and if $f(x) = \mathbb{P}(F(x))$ for all $x \in U \cap A$ such that $F(x) \neq 0$. The representation is called reduced if $\dim F^{-1}(0) \leq m-2$. If $F : U \to V$ is reduced representation, then $U \cap I_f = F^{-1}(0)$. Also, f is meromorphic if and only if for every point $x \in M$, there is a representation $F : U \to V$ of f with $x \in U$.

2.3. Divisor

Let ν be a divisor on M with $S = \mathrm{supp}\nu$, and let k, L be positive integers or ∞. We define the following counting function of ν by:

$$\nu^{[L]}(z) = \min\{\nu(z), L\}, \quad \nu^{[L]}_{\leq k}(z) = \begin{cases} 0, & \text{if } \nu(z) > k; \\ \nu^{[L]}(z), & \text{if } \nu(z) \leq k, \end{cases}$$

$$n_\nu(t) = \begin{cases} t^{2-2m} \int_{S[t]} \nu v^{m-1}, & \text{if } m \geq 2; \\ \sum_{z \in S[t]} \nu(z), & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^{[L]}_\nu(t)$, $n^{[L]}_{\nu,>k}(t)$ and $n^{[L]}_{\nu,\leq k}(t)$. The counting function of ν is defined to be

$$N_\nu(r,s) = \int_s^r n_\nu(t) \frac{dt}{t},$$

Similarly, we define $N^{[L]}_\nu(r,s)$, $N^{[L]}_{\nu,>k}(r,s)$ and $N^{[L]}_{\nu,\leq k}(r,s)$.

In addition, if $L = \infty$, we will omit the L for brevity.

2.4. Projective distance

Suppose that $f : M \to \mathbb{P}(V)$ and $g : M \to \mathbb{P}(V^*)$ are meromorphic maps. Let U be an open, connected subset of M. Let $F : U \to V$ be a reduced representation of f and $G : U \to V^*$ be a reduced representation of g. Let $\{v_0, \ldots, v_n\}$ be an orthonormal basis of V, and let $\{v_0^*, \ldots, v_n^*\}$ be the dual basis.

Take $a \in V^*$ and $b \in V^* \backslash \{0\}$, there is a unique meromorphic function $f_{a,b}|_U$ called a coordinate function on M such that

$$f_{a,b}|_U = \frac{\langle F, a \rangle}{\langle F, b \rangle},$$

if $\langle F, b\rangle \not\equiv 0$.

Define

$$F \llcorner G = \sum_{i=0}^{n} \langle F, v_i^* \rangle \langle G, v_i \rangle,$$

$$\|F\| = \max_{0\le i\le n} |\langle F, v_i^*\rangle|, \quad \|G\| = \max_{0\le i\le n} |\langle G, v_i\rangle|.$$

Then the project distance between f and g is defined by

$$\|f; g\||_U = \frac{|F \llcorner G|}{\|F\|\|G\|}.$$

Note that $\|f; g\|$ is global function on M.

(f, g) is called *free* if and only if there exist representations $F : U \to V$ of f and $G :\to V^*$ of g such that $F \llcorner G \not\equiv 0$. Suppose that (f, g) is free. Define the intersection divisor of f and g by

$$\mu_{(f,g)}|_U = \mu_{F \llcorner G},$$

which is well-defined. For the intersection divisor of f and g, we denote the counting function by

$$N_{(f,g)}(r, s) = N_{\mu_{(f,g)}}(r, s), \quad N^{[L]}_{(f,g)}(r, s) = N_{\mu^{[L]}_{(f,g)}}(r, s),$$

$$N^{[L]}_{(f,g),>k}(r, s) = N_{\mu^{[L]}_{(f,g),>k}}(r, s), \quad N^{[L]}_{(f,g),\le k}(r, s) = N_{\mu^{[L]}_{(f,g),\le k}}(r, s).$$

For $r \in \hat{\mathbb{R}}_\tau$, define

$$m_{(f,g)}(r) = \int_{M\langle r\rangle} \log \frac{1}{\|f; g\|} \sigma.$$

2.5. Family of moving targets

Let g be a meromorphic map from M into $\mathbb{P}(V^*)$, and G be a reduced representation of g on U and $\bar{j}$ with $0 \le \bar{j} \le n$ such that $\langle G, v_{\bar{j}}\rangle \not\equiv 0$. Then $\tilde{G} := \sum_{i=0}^{n} \frac{\langle G, v_i\rangle}{\langle G, v_{\bar{j}}\rangle} v_i^*$ is a global meromorphic representation of g. Let $\mathcal{G} = \{g_1, \dots, g_q\}$ be a family of meromorphic mappings from M into $\mathbb{P}(V^*)$. Denote $\tilde{\mathcal{G}} = \{\tilde{G}_1, \dots, \tilde{G}_q\}$. We note that $\frac{\langle G_j, v_i\rangle}{\langle G_j, v_{\bar{j}}\rangle}$ is meromorphic function on M for $i = 0, \dots, n$ and $j = 1, \dots, q$. Denote by $\mathcal{R}_{\mathcal{G}}$ the smallest subfield containing $\mathbb{C}$ and all meromorphic functions $\frac{\langle G_j, v_i\rangle}{\langle G_j, v_{\bar{j}}\rangle}$ for all i, j.

2.6. The first main theorem

The characteristic function of a meromorphic map $f : M \to \mathbb{P}(V)$ is defined by

$$T_f(r, s) = \int_s^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^*(\Omega) \wedge v^{m-1} \quad \text{for } 0 < s < r,$$

where Ω is the Fubini-Study form on $\mathbb{P}(V)$.

The First Main Theorem *For r, s in $\hat{\mathbb{R}}_\tau$ with $0 < s < r$, we have*

$$T_f(r,s) + T_g(r,s) = N_{(f,g)}(r,s) + m_{(f,g)}(r) - m_{(f,g)}(s).$$

3. Proof of Theorem 1.1 and Theorem 1.2

In order to prove Theorem 1.1 and according to the proof the Theorem 3.1 in [25], we can obtain the following lemma.

Lemma 3.1 *Let (M,τ) be an admissible parabolic manifold with dimension m. Let V be a Hermitian vector space with $\dim V = n+1 > 1$. Let $g_1, \dots, g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position. Let $f : M \to \mathbb{P}(V)$ be a nonconstant meromorphic map such that $\mathrm{Ric}_\tau(r,s) = o(T_f(r,s))$ and $\log Y(r) = o(T_f(r,s))$ for $r \to \infty$. Assume that (f, g_j) is free for $1 \le j \le q$ and $\dim(\mathrm{supp}\mu_{(f,g_i)} \cap \mathrm{supp}\mu_{(f,g_j)}) \le m-2$ for $1 \le i < j \le q$. Assume that there exists a partition $\{1,\dots,q\} = I_1 \cup I_2 \cup \ldots \cup I_l$ satisfying:*

(i) $\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_1}$ *is minimal over $\mathcal{R}_\mathcal{G}$, $\sharp I_t \ge 2$, and $\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_t}$ is linearly independent over $\mathcal{R}_\mathcal{G} (2 \le t \le l)$.*

(ii) *For any $2 \le t \le l$, $j \in I_t$, there exist meromorphic functions $c_j \in \mathcal{R}_\mathcal{G} \setminus \{0\}$ such that*

$$\sum_{j\in I_t} c_j F \llcorner \tilde{G}_{i_j} \in \left(\bigcup_{s=1}^{t-1} \bigcup_{j\in I_s} F \llcorner \tilde{G}_{i_j} \right)_{\mathcal{R}_\mathcal{G}}.$$

Then for $s > 0$, we have

$$\| \; T_f(r,s) \le \sum_{i=1}^{l} \sum_{j\in I_i} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1\le j\le q} T_{g_j}(r,s)),$$

where $k+1 = \mathrm{rank}_{\mathcal{R}_\mathcal{G}}(f)$, $k+1 \le n+1$.

Proof of Theorem 1.1. We denote by $\mathcal{I}$ the set of all permutations of q-tuple $(1,\dots,q)$. For each element $I = (i_1,\dots,i_q) \in \mathcal{I}$, we set

$$N_I = \{r \in \mathbb{R}^+; N^{[k]}_{(f,g_{i_1})}(r,s) \le \cdots \le N^{[k]}_{(f,g_{i_q})}(r,s)\}.$$

We consider an element I of $\mathcal{I}$. Without loss of generality, we set $I = (1,\dots,q)$. We will construct subsets I_t of the set $A_1 = \{1,\dots,2n-k+2\}$ as follows.

We choose a subset I_1 of A_1 which is the minimal subset of A_1 satisfying that $\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_1}$ is minimal over $\mathcal{R}_\mathcal{G}$. If $\mathrm{rank}_{\mathcal{R}_\mathcal{G}}\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_1} = k+1$, then we stop the process.

Otherwise, set $I_1' = \{i : F \llcorner \tilde{G}_i \in (\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_1})_{\mathcal{R}_\mathcal{G}}\}$, $A_2 = A_1 \setminus (I_1 \cup I_1')$ and see that $\sharp I_1 \cup I_1' < n+1$. We consider the following two case:

Case 1. Suppose that $\sharp A_2 \ge n+1$. Since $\{\tilde{g}_{i_j}\}_{j\in A_2}$ is in general position, we have

$$(\{F \llcorner \tilde{G}_{i_j}\}_{j\in A_2})_{\mathcal{R}_\mathcal{G}} \supset (\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_1})_{\mathcal{R}_\mathcal{G}} \not\equiv \emptyset.$$

Case 2. Suppose that $\sharp A_2 < n+1$. Then we have the following:

$$\dim_{\mathcal{R}_\mathcal{G}}(\{F \llcorner \tilde{G}_{i_j}\}_{j\in I_1})_{\mathcal{R}_\mathcal{G}} \ge k+1-(n+1-\sharp I_1 \cup I_1') = k-n+\sharp I_1 \cup I_1',$$

$$\dim_{\mathcal{R}_{\mathcal{G}}}(\{F\llcorner\tilde{G}_{i_j}\}_{j\in A_2})_{\mathcal{R}_{\mathcal{G}}} \geq k+1-(n+1-\sharp A_2) = k-n+\sharp A_2.$$

We note that $\sharp I_1 \cup I_1' + \sharp A_2 = 2n-k+2$. Hence, the above inequalities imply that

$$\begin{aligned}
& \dim_{\mathcal{R}_{\mathcal{G}}}\Big((\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1})_{\mathcal{R}_{\mathcal{G}}} \cap (\{F\llcorner\tilde{G}_{i_j}\}_{j\in A_2})_{\mathcal{R}_{\mathcal{G}}}\Big) \\
= \ & \dim_{\mathcal{R}_{\mathcal{G}}}(\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1\cup I_1'})_{\mathcal{R}_{\mathcal{G}}} + \dim_{\mathcal{R}_{\mathcal{G}}}(\{F\llcorner\tilde{G}_{i_j}\}_{j\in A_2})_{\mathcal{R}_{\mathcal{G}}} - (k+1) \\
\geq \ & k-n+\sharp I_1\cup I_1' + k-n+\sharp A_2 - (k+1) = 1.
\end{aligned}$$

Therefore, from the above two cases, we have

$$(\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1})_{\mathcal{R}_{\mathcal{G}}} \cap (\{F\llcorner\tilde{G}_{i_j}\}_{j\in A_2})_{\mathcal{R}_{\mathcal{G}}} \neq \emptyset.$$

Thus, we may choose a subset $I_2 \subset A_2$ which is the minimal subset of A_2 satisfying that there exist nonzero meromorphic functions $c_i \in \mathcal{R}_{\mathcal{G}}(i \in I_2)$,

$$\sum_{i\in I_2} c_i F\llcorner\tilde{G}_i \in \left(\bigcup_{i\in I_1} F\llcorner\tilde{G}_i\right)_{\mathcal{R}_{\mathcal{G}}}.$$

We see that $\sharp I_2 \geq 2$, and the family $\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_2}$ is linearly independent over $\mathcal{R}_{\mathcal{G}}$. Hence, $\sharp I_2 \leq k+1$ and

$$\sharp(I_1\cup I_2) \leq \min\{2n-k+2, n+k+1\}.$$

If $\operatorname{rank}_{\mathcal{R}_{\mathcal{G}}}\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1\cup I_2} = k+1$, then we stop the process. Otherwise, by repeating the above argument, we have a subset $I_2' = \{i : F\llcorner\tilde{G}_i \in (\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1\cup I_2})_{\mathcal{R}_{\mathcal{G}}}\}$, and a subset I_3 of $A_3 = A_1 \setminus (I_1\cup I_2\cup I_2')$, which satisfy the following:

- there exist nonzero meromorphic functions $c_i \in \mathcal{R}_{\mathcal{G}}(i \in I_3)$ such that

$$\sum_{i\in I_3} c_i F\llcorner\tilde{G}_i \in \left(\bigcup_{i\in I_1\cup I_2} F\llcorner\tilde{G}_i\right)_{\mathcal{R}_{\mathcal{G}}},$$

- $\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_3}$ is linearly independent over $\mathcal{R}_{\mathcal{G}}$.

Continuing this process, we get a sequence of subsets $I_1,\ldots,I_l$ which satisfy:

- $\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1}$ is minimal over $\mathcal{R}_{\mathcal{G}}$, $\sharp I_t \geq 2$, and $\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_t}$ is linearly independent over $\mathcal{R}_{\mathcal{G}}(2\leq t\leq l)$,
- for any $2\leq t\leq l$, $j\in I_t$, there exist meromorphic functions $c_j \in \mathcal{R}_{\mathcal{G}}\setminus\{0\}$ such that

$$\sum_{j\in I_t} c_j F\llcorner\tilde{G}_{i_j} \in \left(\bigcup_{s=1}^{t-1}\bigcup_{j\in I_s} F\llcorner\tilde{G}_{i_j}\right)_{\mathcal{R}_{\mathcal{G}}},$$

- $\operatorname{rank}_{\mathcal{R}_{\mathcal{G}}}\{F\llcorner\tilde{G}_{i_j}\}_{j\in I_1\cup\cdots\cup I_l} = k+1.$

If $\sharp I_1 = 2$, we will remove one element from I_1 and combine the remaining element with I_2 to become a new set I_1. Therefore, we will get a sequence $I_1, \ldots, I_l$ which satisfies the above three properties and $\sharp I_1 \geq 3$, $\sharp I_t \geq 2 (2 \leq t \leq l)$. We set $n_1 = \sharp I_1 - 2,\ n_t = \sharp I_t - 1 (2 \leq t \leq l),\ n_0 = \max_{2 \leq t \leq l} n_t,\ J = I_1 \cup \cdots \cup I_l$, and $d + 2 = \sharp J$. Then we can estimate

$$(n_1 + 2) + (n_2 + 1) + \cdots + (n_l + 1) = d + 2, \quad \sharp J = d + 2 \leq n + 2.$$

Now the family of subsets $I_1, \ldots, I_l$ satisfy the assumptions of Lemma 3.1. Therefore, by Lemma 3.1, we have

$$T_f(r,s) \leq \sum_{t=1}^{l} \sum_{j \in I_t} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)). \tag{3.1}$$

For all $r \in N_I$, from (3.1) we obtain

$$\begin{aligned} \| \ T_f(r,s) &\leq \sum_{j \in J} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\ &\leq \frac{\sharp J}{q - (2n - k + 2) + \sharp J} (\sum_{j \in J} N^{[k]}_{(f,g_j)}(r,s) + \sum_{j=2n-k+3}^{q} N^{[k]}_{(f,g_j)}(r,s)) \\ &\quad + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)). \end{aligned}$$

Since $\sharp J = d + 2 \leq n + 2$, then one can imply that

$$\| \ T_f(r,s) \leq \frac{n+2}{q-(n-k)} \sum_{j=1}^{q} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)). \tag{3.2}$$

We see that $\bigcup_{I \in \mathcal{I}} N_I = \mathbb{R}^+$ and the above inequality holds for every $r \in N_I,\ I \in \mathcal{I}$ outside a finite Borel measure subset. Thus,

$$\| \ \frac{q-(n-k)}{n+2} T_f(r,s) \leq \sum_{j=1}^{q} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)).$$

Thus, Theorem 1.1 is proved. □

Proof of Theorem 1.2. We denote by $\mathcal{I}$ the set of all permutations of q-tuple $(1, \ldots, q)$. For each element

$I = (i_1, \ldots, i_q) \in \mathcal{I}$, we set

$$N_I = \{r \in \mathbb{R}^+; N^{[k]}_{(f,g_{i_1})}(r,s) \leq \cdots \leq N^{[k]}_{(f,g_{i_q})}(r,s)\}.$$

We now consider an element I_0 of $\mathcal{I}$. Without loss of generality, we set $I_0 = (1, \ldots, q)$. $\{g_j\}_{j=1}^{q}$ are located in general position and $\operatorname{rank}_{\mathcal{R}_{\mathcal{G}}}(f) = k + 1$, then there is a maximal linearly independent subset of the set $\{F \llcorner \tilde{G}_j : 1 \leq j \leq n + 1\}$ which is of exactly $k + 1$ elements and contains $F \llcorner \tilde{G}_1$. We assume that they are $\{F \llcorner \tilde{G}_{i_j} : 1 = i_1 < \ldots < i_{k+1} \leq n + 1\}$. For each $j \in \{1, \ldots, k + 1\}$, we set

$$\Omega(j) = \{i \in (1, \ldots, q); F \llcorner \tilde{G}_i \in (F \llcorner \tilde{G}_{i_t}; 1 \leq t \leq k + 1, t \neq j)_{\mathcal{R}_{\mathcal{G}}}\}.$$

Note that the space $(F \llcorner \tilde{G}_{i_t}; 1 \leq t \leq k+1, t \neq j)_{\mathcal{R}_{\mathcal{G}}}$ is of dimension k. Then $\Omega(j)$ has at most n elements. Therefore,

$$\sharp \bigcup_{j=1}^{k+1} \Omega(j) = \sharp \bigcup_{j=1}^{k+1} (\Omega(j) \setminus \{i_j\}_{j=1}^{k+1}) + k + 1 \leq (n-k)(k+1) + k + 1 = (n-k+1)(k+1).$$

Hence, there exists an index $i_0 \leq (n-k+1)(k+1)+1$ such that $i_0 \notin \bigcup_{j=1}^{k+1} \Omega(j)$. This yields that the set $\{F \llcorner \tilde{G}_{i_j}; 0 \leq j \leq k+1\}$ is minimal over $\mathcal{R}_{\mathcal{G}}$. Then by Lemma 3.1, for all $r \in N_I$ we have

$$\begin{aligned}
\| \; T_f(r,s) &\leq \sum_{j=0}^{k+1} N^{[k]}_{(f,g_{i_j})}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\
&\leq N^{[k]}_{(f,g_1)}(r,s) + \sum_{j=2}^{k+1} N^{[k]}_{(f,g_{i_j})}(r,s) + N^{[k]}_{(f,g_\lambda)}(r,s) \\
&\quad + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\
&\leq N^{[k]}_{(f,g_1)}(r,s) + \sum_{i=n-k+2}^{n+1} N^{[k]}_{(f,g_i)}(r,s) + N^{[k]}_{(f,g_\lambda)}(r,s) \\
&\quad + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\
&\leq \frac{1}{n-k+1} \Big(\sum_{i=1}^{n-k+1} N^{[k]}_{(f,g_i)}(r,s) + \sum_{i=n-k+2}^{\lambda-1} N^{[k]}_{(f,g_i)}(r,s) \\
&\quad + \sum_{i=\lambda}^{(n-k+1)(k+2)} N^{[k]}_{(f,g_i)}(r,s) \Big) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\
&= \frac{1}{n-k+1} \sum_{i=1}^{(n-k+1)(k+2)} N^{[k]}_{(f,g_i)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\
&\leq \frac{1}{n-k+1} \frac{(n-k+1)(k+2)}{q} \sum_{i=1}^{q} N^{[k]}_{(f,g_i)}(r,s) \\
&\quad + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)) \\
&= \frac{k+2}{q} \sum_{i=1}^{q} N^{[k]}_{(f,g_i)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)),
\end{aligned}$$

where $\lambda = (n-k+1)(k+1)+1$.

We see that $\bigcup_{I \in \mathcal{I}} N_I = \mathbb{R}^+$ and the above inequality holds for every $r \in N_I$, $I \in \mathcal{I}$ outside a finite Borel measure subset. Thus,

$$\| \; \frac{q}{k+2} T_f(r,s) \leq \sum_{j=1}^{q} N^{[k]}_{(f,g_j)}(r,s) + o(T_f(r,s)) + O(\max_{1 \leq j \leq q} T_{g_j}(r,s)).$$

Thus, Theorem 1.2 is proved. □

4. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemma.

Lemma 4.1 [20, 25] *Let M be a connected complex manifold of dimension m. Let A be a pure $(m-1)$-dimensional analytic subset of M and V be a complex vector space of dimension $n+1>1$. Let λ and l be integers with $1 \le l \le \lambda \le n+1$. Let $f_j : M \to \mathbb{P}(V)$, $1 \le j \le \lambda$ be meromorphic mappings. Assume that $f_1, \dots, f_\lambda$ are in general position (not in special position), and $f_1, \dots, f_\lambda$ are in l-special position on A. Then, we have*

$$\mu_{f_1 \wedge \dots \wedge f_\lambda} \ge (\lambda - l + 1)\nu_A.$$

Proof of Theorem 1.5. Assume that $f_0, \dots, f_\lambda$ are not in special position, and $k_t + 1 = \operatorname{rank}_{\mathcal{R}_\mathcal{G}}(f_t)$, $k = \max\{k_t : 1 \le t \le \lambda\}$, $k' = \min\{k_t : 1 \le t \le \lambda\}$. Let $F_t : U \to V$ be a representation of f_t on U for $t = 1, \dots, \lambda$. Then $F_1 \wedge \cdots \wedge F_\lambda : U \to \bigwedge_\lambda V$ is not identically zero, there exists one and only one divisor defined by

$$\mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda}|_U = \mu_{F_1 \wedge \cdots \wedge F_\lambda}.$$

By Lemma 4.1, we have, for every $1 \le t \le \lambda$,

$$\sum_{j=1}^{q} N^{[k_t]}_{(f_t, g_j)}(r,s) \le \frac{k_t}{\lambda - l + 1} N_{\mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda}}(r,s). \tag{4.1}$$

By the first main theorem of the exterior product (cf. (3.28) of [20]),

$$N_{\mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda}}(r,s) \le \sum_{i=1}^{\lambda} T_{f_i}(r,s) + O(1). \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\sum_{j=1}^{q} N^{[k_t]}_{(f_t, g_j)}(r,s) \le \frac{k_t}{\lambda - l + 1} \sum_{i=1}^{\lambda} T_{f_i}(r,s) + O(1). \tag{4.3}$$

By Theorem 1.1, for each $1 \le t \le \lambda$, we have

$$\begin{aligned}
\frac{q-(n-k_t)}{n+2}T_{f_t}(r,s) &\le \sum_{j=1}^{q} N^{[k_t]}_{(f_t,g_j)}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\le \sum_{j=1}^{q}(N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + N^{[k_t]}_{(f_t,g_j),>m_j}(r,s)) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\le \sum_{j=1}^{q}\left(N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + \frac{k_t}{m_j+1}N_{(f_t,g_j),>m_j}(r,s)\right) \\
&\quad + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\le \sum_{j=1}^{q} N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\quad + \sum_{j=1}^{q}\frac{k_t}{m_j+1}(N_{(f_t,g_j)}(r,s) - N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s)).
\end{aligned}$$

We note that $N_{(f_t,g_j)}(r,s) \le T_{f_t}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s))$. Thus, we have

$$\begin{aligned}
\frac{q-(n-k_t)}{n+2}T_{f_t}(r,s) &\le \sum_{j=1}^{q}(1-\frac{k_t}{m_j+1})N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + \sum_{j=1}^{q}\frac{k_t}{m_j+1}T_{f_t}(r,s) \\
&\quad + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\le (1-\frac{k_t}{m_1+1})N^{[k_t]}_{(f_t,g_1),\le m_1}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\quad + \sum_{j=2}^{q}(1-\frac{k_t}{m_2+1})N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + \sum_{j=1}^{q}\frac{k_t}{m_j+1}T_{f_t}(r,s) \\
&\le (\frac{k_t}{m_2+1} - \frac{k_t}{m_1+1})N^{[k_t]}_{(f_t,g_1),\le m_1}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\quad + \sum_{j=1}^{q}(1-\frac{k_t}{m_2+1})N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + \sum_{j=1}^{q}\frac{k_t}{m_j+1}T_{f_t}(r,s).
\end{aligned}$$

Hence, together with $N^{[k_t]}_{(f_t,g_1),\le m_1}(r,s) \le T_{f_t}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s))$, the above inequality implies that

$$\begin{aligned}
\frac{q-(n-k_t)}{n+2}T_{f_t}(r,s) &\le (\frac{k_t}{m_2+1} - \frac{k_t}{m_1+1})T_{f_t}(r,s) + o(\max_{1\le t\le\lambda} T_{f_t}(r,s)) \\
&\quad + \sum_{j=1}^{q}(1-\frac{k_t}{m_2+1})N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) + \sum_{j=1}^{q}\frac{k_t}{m_j+1}T_{f_t}(r,s).
\end{aligned}$$

Noting that $N^{[k_t]}_{(f_t,g_j),\le m_j}(r,s) \le N^{[k_t]}_{(f_t,g_j)}(r,s)$, then, combining with (4.3), one can imply that

$$\begin{aligned}\frac{q-(n-k_t)}{n+2}T_{f_t}(r,s) &\le (\frac{k_t}{m_2+1}-\frac{k_t}{m_1+1})T_{f_t}(r,s)+\sum_{j=1}^{q}(1-\frac{k_t}{m_2+1})N^{[k_t]}_{(f_t,g_j)}(r,s)\\ &\quad +\sum_{j=1}^{q}\frac{k_t}{m_j+1}T_{f_t}(r,s)+o(\max_{1\le t\le\lambda}T_{f_t}(r,s))\\ &\le (\frac{k_t}{m_2+1}-\frac{k_t}{m_1+1})T_{f_t}(r,s)+\sum_{j=1}^{q}\frac{k_t}{m_j+1}T_{f_t}(r,s)\\ &\quad +(1-\frac{k_t}{m_2+1})\frac{k_t}{\lambda-l+1}\sum_{i=1}^{\lambda}T_{f_i}(r,s)+o(\max_{1\le t\le\lambda}T_{f_t}(r,s)).\end{aligned}$$

Then, for each $1\le t\le\lambda$, we have

$$\begin{aligned}-\sum_{j=3}^{q}\frac{1}{m_j+1}T_{f_t}(r,s) &\le \frac{2}{m_2+1}T_{f_t}(r,s)-\frac{q-(n-k_t)}{(n+2)k_t}T_{f_t}(r,s)\\ &\quad +(1-\frac{k_t}{m_2+1})\frac{1}{\lambda-l+1}\sum_{i=1}^{\lambda}T_{f_i}(r,s)+o(\max_{1\le t\le\lambda}T_{f_t}(r,s))\\ &\le \frac{2}{m_2+1}T_{f_t}(r,s)-\frac{q-(n-k)}{(n+2)k}T_{f_t}(r,s)\\ &\quad +(1-\frac{k'}{m_2+1})\frac{1}{\lambda-l+1}\sum_{i=1}^{\lambda}T_{f_i}(r,s)+o(\max_{1\le t\le\lambda}T_{f_t}(r,s)).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{j=3}^{q}\frac{m_j}{m_j+1}\sum_{t=1}^{\lambda}T_{f_t}(r,s) &\le (q-2)\sum_{t=1}^{\lambda}T_{f_t}(r,s)+\frac{2}{m_2+1}\sum_{t=1}^{\lambda}T_{f_t}(r,s)\\ &\quad -\frac{q-(n-k)}{(n+2)k}\sum_{t=1}^{\lambda}T_{f_t}(r,s)\\ &\quad +(1-\frac{k'}{m_2+1})\frac{\lambda}{\lambda-l+1}\sum_{i=1}^{\lambda}T_{f_i}(r,s)+o(\sum_{t=1}^{\lambda}T_{f_t}(r,s)).\end{aligned}$$

Let $r\to\infty$,

$$\sum_{j=3}^{q}\frac{m_j}{m_j+1}\le q-2-\frac{q-(n-k)}{(n+2)k}+\frac{2}{m_2+1}+(1-\frac{k'}{m_2+1})\frac{\lambda}{\lambda-l+1}.$$

This is a contradiction. Hence, $f_1,\ldots,f_\lambda$ are in special position. □

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