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Some properties of the matrix Wiener transform with related topics on Hilbert space

Hyun Soo CHUNG*
Department of Mathematics, Dankook University, Cheonan, South Korea

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Abstract: Main purpose of this paper is to obtain fundamental relationships for the integrals and the matrix Wiener transforms on Hilbert space. Using some technics and properties of matrices of real numbers, we state some algebraic structure of matrices. We then establish evaluation formulas with examples. Furthermore, we define the matrix Wiener transform, and investigate some properties of the matrix Wiener transform. Finally, we establish relationships for the matrix Wiener transform.

Key words: Evaluation formulas, matrix Wiener transform, bounded linear operators, eigenvalue, composition formula

1. Introduction

With the development of scientific techniques, some integration theories and relationships transforms on Hilbert space have appeared in engineering and humanities, health sciences to economics, finance and machine learning. According to these applications, the concept of Wiener transform on a Hilbert space was introduced and studied by Segal, Hida, Negrin, Hayek et al. They then developed various theories and structures for the Wiener transform via the second quantization by use of an integral operator and some concepts of matrix on Hilbert space, see [7, 12, 13, 15–20].

Research results of various integral transforms on function space have been published in many papers [1–6, 9–11]. Recently studies on integral transform using matrices, it is called the matrix transform, have been conducted, and are being used in various fields. All formulas and results for the matrix transform are more generalized versions in previous papers [7, 14, 17].

In this paper, we first obtain two evaluation formula via the some concepts and properties of matrices of real numbers, and give some examples. We then define the matrix Wiener transform, and establish some relationships. Our results and formulas take a different form than the previous ones. Various research results can be obtained by using the properties of matrices.

2. Definitions and preliminaries

In this section, we state some definitions and notations to understand the paper.

Let $\mathbf{H}$ be a real Hilbert space with inner product $\langle x, x' \rangle_{\mathbf{H}}$ for $x, x' \in \mathbf{H}$ and let $\mathcal{L} \equiv \mathcal{L}(\mathbf{H} : \mathbf{H})$ be the set of all bounded linear operators on $\mathbf{H}$ [7, 8]. Let A and B be elements of $\mathcal{L}$ such that there exists an

*Correspondence: hschung@dankook.ac.kr
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orthonormal basis $\mathcal{B} = \{e_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathbf{H}$ ($\mathcal{A}$ being some index set) consisting of elements of $\mathbf{H}$ with

$$Ae_\alpha = \mu_\alpha e_\alpha, \qquad Be_\alpha = \lambda_\alpha e_\alpha \tag{2.1}$$

for some real numbers μ_α and λ_α. Then we note that for each $x \in \mathbf{H}$,

$$x = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle_{\mathbf{H}} e_\alpha$$

and hence,

$$Ax = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle_{\mathbf{H}} \mu_\alpha e_\alpha \quad \text{and} \quad Bx = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle_{\mathbf{H}} \lambda_\alpha e_\alpha.$$

For any arbitrary set $\mathcal{F}$, natural numbers k and n, let $\mathcal{M}^{\mathcal{F}}_{k\times n}$ be the set of all matrices whose components are in $\mathcal{F}$, namely, let

$$\mathcal{M}^{\mathcal{F}}_{k\times n} = \{A = (a_{ij})_{k\times n} | a_{ij} \in \mathcal{F}, 1 \le i \le k, 1 \le j \le n\}. \tag{2.2}$$

Then the space $(\mathcal{M}^{\mathcal{L}}_{k\times n}, \|\cdot\|_0)$ is a Banach space with the norm

$$\|A\|_0 = \max\{\|T_{ij}\|_{op} : 1 \le i \le k, 1 \le j \le n\},$$

where $A = (T_{ij}) \in \mathcal{M}^{\mathcal{L}}_{k\times n}$ and $\|T\|_{op}$ is the operator norm of $T \in \mathcal{L}$, see [7].

We give simple examples of $\mathcal{M}^{\mathcal{F}}_{k\times n}$.

Example 2.1 *When $k = 2, n = 1$ and $\mathcal{F} = \mathbf{H}$,*

$$\mathcal{M}^{K}_{2\times 1} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbf{H} \right\}.$$

If $k = n = 2$ and $\mathcal{F} = \mathcal{L}$, then

$$\mathcal{M}^{\mathcal{L}}_{2\times 2} = \left\{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : T_{11}, T_{12}, T_{21}, T_{22} \in \mathcal{L} \right\}.$$

We shall explain algebraic calculations on $\mathcal{M}^{\mathcal{F}}_{k\times n}$ to develop our theories and results. According to the various algebraic calculations for the matrices with constant coefficients, we shall introduce some algebraic calculations on $\mathcal{M}^{\mathcal{F}}_{k\times n}$.

Remark 2.2 *We adopt some notations and properties of matrices of real numbers as follows:*

(i) For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}^{\mathbf{H}}_{n\times 1}$ *and* $T = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ & \cdots & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \equiv (T_{ij}) \in \mathcal{M}^{\mathcal{L}}_{n\times n}$, *let*

$$Tx = \begin{pmatrix} \sum_{r=1}^{n} T_{1r}x_r \\ \vdots \\ \sum_{r=1}^{n} T_{nr}x_r \end{pmatrix} \in \mathcal{M}^{\mathbf{H}}_{n\times 1}.$$

(ii) For $T_1 = (T_{ij}^{(1)})$ *and* $T_2 = (T_{ij}^{(2)})$ *in* $\mathcal{M}_{n\times n}^{\mathcal{L}}$*, let*

$$A = \big(T_1 | T_2\big) = \begin{pmatrix} T_{11}^{(1)} & \cdots & T_{1n}^{(1)} & T_{11}^{(2)} & \cdots & T_{1n}^{(2)} \\ & & \cdots & & & \\ T_{n1}^{(1)} & \cdots & T_{nn}^{(1)} & T_{n1}^{(2)} & \cdots & T_{nn}^{(2)} \end{pmatrix} \in \mathcal{M}_{n\times 2n}^{\mathcal{L}}$$

and

$$B = \begin{pmatrix} T_1 \\ - \\ T_2 \end{pmatrix} = \begin{pmatrix} T_{11}^{(1)} & \cdots & T_{1n}^{(1)} \\ & \cdots & \\ T_{n1}^{(1)} & \cdots & T_{nn}^{(1)} \\ T_{11}^{(2)} & \cdots & T_{1n}^{(2)} \\ & \cdots & \\ T_{11}^{(2)} & \cdots & T_{1n}^{(2)} \end{pmatrix} \in \mathcal{M}_{2n\times n}^{\mathcal{L}}.$$

(iii) Let $T_1 = (T_{ij}^{(1)}), T_2 = (T_{ij}^{(2)}), S_1 = (S_{ij}^{(1)})$ *and* $S_2 = (S_{ij}^{(2)})$ *be elements of* $\mathcal{M}_{n\times n}^{\mathcal{L}}$ *with*

$$T_{ij}^{(1)} e_\alpha = \mu_{ij}^{(1),\alpha} e_\alpha, \qquad T_{ij}^{(2)} e_\alpha = \mu_{ij}^{(2),\alpha} e_\alpha, \qquad S_{ij}^{(1)} e_\alpha = \lambda_{ij}^{(1),\alpha} e_\alpha$$

and

$$S_{ij}^{(2)} e_\alpha = \lambda_{ij}^{(2),\alpha} e_\alpha$$

for $e_\alpha \in \mathcal{B}$*. Let* $(A)_0$ *be the matrix of the eigenvalues of* T_1, T_2, S_1 *and* S_2 *as below*

$$(A)_0 = \begin{pmatrix} (\mu_{ij}^{(1),\alpha}) \mid (\lambda_{ij}^{(1),\alpha}) \\ --- \\ (\mu_{ij}^{(2),\alpha}) \mid (\lambda_{ij}^{(2),\alpha}) \end{pmatrix} \in \mathcal{M}_{2n\times 2n}^{\mathbb{R}}.$$

For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n\times 1}^{\mathbf{H}}$ *and* $\alpha \in \mathcal{A}$*, let* $(x)_\alpha$ *be the matrix of real numbers as below*

$$(x)_\alpha = \begin{pmatrix} \langle x_1, e_\alpha \rangle_{\mathbf{H}} \\ \vdots \\ \langle x_n, e_\alpha \rangle_{\mathbf{H}} \end{pmatrix} \in \mathcal{M}_{n\times 1}^{\mathbb{R}}.$$

(iv) For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ *and* $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{M}_{n\times 1}^{\mathbf{H}}$*, let*

$$x_y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \equiv \begin{pmatrix} x \\ -- \\ y \end{pmatrix} \in \mathcal{M}_{2n\times 1}^{\mathbf{H}}.$$

(v) For x_y is in (iv), let $\mathcal{H}$ be a transform from $\mathcal{M}^{\mathbf{H}}_{2n\times 1}$ to $\mathcal{M}^{\mathbf{H}}_{n\times 1}\equiv \mathbf{H}^n$ defined by the formula

$$\mathcal{H}(x_y) = \big(x+y\big)\,. \tag{2.3}$$

Then $\mathcal{H}$ is a linear transform.

(vi) For $T\in\mathcal{M}^{\mathcal{L}}_{k\times n}$, let T^t be the transpose matrix of T. Then T^t is an element of $\mathcal{M}^{\mathcal{L}}_{n\times k}$. For example, if $T=\begin{pmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}$, then $T^t=\begin{pmatrix} T_{11} & T_{21}\\ T_{12} & T_{22}\end{pmatrix}$.

We now state a class of functions used in this paper.

Definition 2.3 *For a natural number n, let $\mathcal{S}^{(n)}$ be the class of functions on $\mathbb{R}^n$ which satisfies the condition*

$$|f(u)|\le M_f\exp\bigg\{N_f\sum_{r=1}^{n}|u_j|\bigg\} \tag{2.4}$$

where $u=(u_1,\cdots,u_n)\in\mathbb{R}^n$, $f\in C^\infty(\mathbb{R}^n)$, (the set of infinitely many differential functions), M_f and N_f are nonnegative real numbers. In this case, f is called an exponential type function on $\mathbb{R}^n$. Let $\mathbb{E}_0^{(n)}$ be the class of all functions F_β on $\mathbf{H}^n$ of the form

$$F_\beta(x) = f((x)_\beta) = f\begin{pmatrix}\langle x_1, e_\alpha\rangle_{\mathbf{H}}\\ \vdots \\ \langle x_n, e_\alpha\rangle_{\mathbf{H}}\end{pmatrix} \tag{2.5}$$

where $f\in\mathcal{S}^{(n)}$ and $e_\beta\in\mathcal{B}$. We say that $F_\beta(x)$ is a exponential type function on $\mathbf{H}^n$.

For a function F_β on $\mathbf{H}^n$, we denote the $\mathbf{H}^n$-integral by

$$\int_{\mathbf{H}^n} F(x)dg_c(x)$$

where the integration on $\mathbf{H}^n$ is performed with respect to the normalized distribution g_c of the variance parameter $c>0$ if it exists. Then we have

$$\int_{\mathbf{H}^n} F_\beta(x)dg_c(x) = \bigg(\frac{1}{\sqrt{2\pi c}}\bigg)^n\int_{\mathbb{R}^n} f(v)\exp\bigg\{-\frac{v^tv}{2c}\bigg\}dv \tag{2.6}$$

if it exists, where $v=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\in\mathcal{M}^{\mathbb{R}}_{n\times 1}$.

3. Evaluation formulas

In this section, we obtain two evaluation formulas for the $\mathbf{H}^n$-integral, and give some examples to explain the usefulness of our evaluation formulas.

When calculating integrals, there are many functions that are actually difficult to calculate integrals. In the following Theorem 3.1 below, it can be seen that these difficulties can be solved under appropriate conditions.

Theorem 3.1 *Let* $T = (T_{ij})$ *be an element of* $\mathcal{M}^{\mathcal{L}}_{n\times n}$ *with real valued eigenvalues,* $T_{ij}e_\alpha = \mu^\alpha_{ij}e_\alpha$ *for* $i,j = 1,\cdots,n$*, where* $e_\alpha \in \mathcal{B}$*. Then*

$$\int_{\mathbf{H}^n} F_\beta(x)dg_c(x) = \int_{\mathbf{H}^n} F_\beta(Tx)dg_c(x) \tag{3.1}$$

for all $F_\beta \in \mathbb{E}_0^{(n)}$ *if following conditions hold:*

(a) The determinant of $(T)_0$ *is* 1 *and hence* $(T)_0$ *has the inverse matrix* $(T)_0^{-1}$*.*

(b) $((T)_0^{-1})^t((T)_0^{-1}) = I_n$*, where* I_n *denotes the identity matrix in* $\mathcal{M}^{\mathbb{R}}_{n\times n}$*.*

Proof First, using Equations (2.6) and (2.4), one can see that

$$\begin{aligned}
&\left|\int_{\mathbf{H}^n} F_\beta(x)dg_c(x)\right| \\
&\le \left(\frac{1}{\sqrt{2\pi c}}\right)^n \int_{\mathbb{R}^n} |f(v^t)| \exp\left\{-\frac{v^t v}{2c}\right\}dv \\
&\le \left(\frac{1}{\sqrt{2\pi c}}\right)^n M_f \int_{\mathbb{R}^n} \exp\left\{N_f \sum_{r=1}^n |v_j| - \frac{v^t v}{2c}\right\}dv < \infty
\end{aligned}$$

and so F_β is integrable on $\mathbf{H}^n$. We left to show that Equation (3.1) holds. In order to do this, using Equations (2.6) and (2.4), we obtain that

$$\begin{aligned}
\int_{\mathbf{H}^n} F_\beta(Tx)dg_c(x) &= \int_{\mathbf{H}^n} f\Big((T)_0(x)_\beta\Big)dg_c(x) \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^n \int_{\mathbb{R}^n} f\Big((T)_0 v\Big) \exp\left\{-\frac{v^t v}{2c}\right\}dv.
\end{aligned}$$

Now, let $w = (T)_0 v$. Then using conditions (a) and (b), we can obtain that

$$\begin{aligned}
&\int_{\mathbf{H}^n} F_\beta(Tx)dg_c(x) \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^n \int_{\mathbb{R}^n} f(w) \exp\left\{-\frac{w^t((T)_0^{-1})^t(T)_0^{-1}w}{2c}\right\}dw \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^n \int_{\mathbb{R}^n} f(w) \exp\left\{-\frac{w^t w}{2c}\right\}dw,
\end{aligned}$$

which completes the proof of Theorem 3.1 as desired. □

We shall give two examples to illustrate the usefulness of Theorem 3.1 above.

Example 3.2 *Let* T *be as in Theorem 3.1 and let* $f(u_1,u_2) = \sec^4(u_1+u_2)$*. Then* $F_\beta(x) = \sec^4(\langle x_1, e_\beta\rangle_{\mathbf{H}} + \langle x_2, e_\beta\rangle_{\mathbf{H}})$*. One can see that the calculation of the following integral*

$$\int_{\mathbf{H}^2} F_\beta(x)dg_c(x)$$

is not easy. However, using Equation (3.1) we have

$$\begin{aligned}
&\int_{\mathbf{H}^2} F_\beta(x)dg_c(x) \\
&= \int_{\mathbf{H}^2} F_\beta(Tx)dg_c(x) \\
&= \int_{\mathbf{H}^2} \sec^4((\mu_{11}^\beta + \mu_{21}^\beta)\langle x_1, e_\beta\rangle_{\mathbf{H}} + (\mu_{12}^\beta + \mu_{22}^\beta)\langle x_2, e_\beta\rangle_{\mathbf{H}})dg_c(x) \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^2 \int_{\mathbb{R}^2} \sec^4((\mu_{11}^\beta + \mu_{21}^\beta)v_1 + (\mu_{12}^\beta + \mu_{22}^\beta)v_2) \exp\left\{-\frac{v_1^2 + v_2^2}{2c}\right\}dv \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^2 \int_{\mathbb{R}^2} \exp\left\{-\frac{v_1^2 + v_2^2}{2c}\right\}dv = 1
\end{aligned}$$

provided $\mu_{11}^\beta + \mu_{21}^\beta = 0$ *and* $\mu_{12}^\beta + \mu_{22}^\beta = 0$.

In our next example, we give an example for the 3-dimensional version of Theorem 3.1.

Example 3.3 *Let* T *be as in Theorem 3.1 and let* $f(u_1, u_2, u_3) = \exp\{-\frac{u_2+u_3}{u_1}\}$. *Then*

$$F_\beta(x) = \exp\left\{-\frac{\langle x_2, e_\beta\rangle_{\mathbf{H}} + \langle x_3, e_\beta\rangle_{\mathbf{H}}}{\langle x_1, e_\beta\rangle_{\mathbf{H}}}\right\}.$$

One can see that the calculation of the following integral

$$\int_{\mathbf{H}^3} F_\beta(x)dg_c(x)$$

is also not easy. However, using Equation (3.1) we have

$$\begin{aligned}
&\int_{\mathbf{H}^3} F_\beta(x)dg_c(x) \\
&= \int_{\mathbf{H}^3} F_\beta(Tx)dg_c(x) \\
&= \int_{\mathbf{H}^3} \exp\left\{-\frac{(\mu_{12}^\beta + \mu_{22}^\beta + \mu_{32}^\beta)\langle x_2, e_\beta\rangle_{\mathbf{H}} + (\mu_{13}^\beta + \mu_{23}^\beta + \mu_{33}^\beta)\langle x_3, e_\beta\rangle_{\mathbf{H}}}{(\mu_{11}^\beta + \mu_{21}^\beta + \mu_{31}^\beta)\langle x_1, e_\beta\rangle_{\mathbf{H}}}\right\}dg_c(x) \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^3 \int_{\mathbb{R}^3} \exp\left\{-\frac{(\mu_{12}^\beta + \mu_{22}^\beta + \mu_{32}^\beta)v_2 + (\mu_{13}^\beta + \mu_{23}^\beta + \mu_{33}^\beta)v_3}{(\mu_{11}^\beta + \mu_{21}^\beta + \mu_{31}^\beta)v_1}\right\} \\
&\qquad\qquad \times \exp\left\{-\frac{v_1^2 + v_2^2 + v_3^2}{2c}\right\}dv \\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^3 \int_{\mathbb{R}^3} \exp\left\{-K - \frac{v_1^2 + v_2^2 + v_3^2}{2c}\right\}dv = e^{-K}
\end{aligned}$$

provided $\mu_{11}^\beta = K\mu_{21}^\beta, \mu_{12}^\beta = K\mu_{22}^\beta$ *and* $\mu_{13}^\beta = K\mu_{23}^\beta$.

When $n = 2$, we have the following evaluation formula.

Corollary 3.4 *Let* $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ *be an element of* $\mathcal{M}^{\mathcal{L}}_{2\times 2}$ *with real valued eigenvalues,* $T_{11}e_\alpha = \mu^\alpha_{11}e_\alpha, T_{12}e_\alpha = \mu^\alpha_{12}e_\alpha, T_{21}e_\alpha = \mu^\alpha_{21}e_\alpha$ *and* $T_{22}e_\alpha = \mu^\alpha_{22}e_\alpha$ *for* $e_\alpha \in \mathcal{B}$*. Then*

$$\int_{\mathbf{H}^2} F_\beta(x)dg_c(x) = \int_{\mathbf{H}^2} F_\beta(Tx)dg_c(x) \tag{3.2}$$

for all $F_\beta \in \mathbb{E}^{(2)}_0$ *if following conditions hold:*

(a') $(\mu^\beta_{11})^2 + (\mu^\beta_{21})^2 = 1, (\mu^\beta_{12})^2 + (\mu^\beta_{22})^2 = 1$.

(b') $\mu^\beta_{11}\mu^\beta_{21} + \mu^\beta_{12}\mu^\beta_{22} = 0$.

(c') $\mu^\beta_{11}\mu^\beta_{22} - \mu^\beta_{12}\mu^\beta_{21} = 1$.

In fact, the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *satisfies conditions (a'), (b'), and (c').*

The following lemma plays a key role to obtain the second evaluation formula for $\mathbf{H}^n$-integrals.

Lemma 3.5 *Let* φ *be a measurable function on* $\mathbb{R}$*. Then for all nonzero complex numbers* p *and* q*,*

$$\int_{\mathbf{H}}\int_{\mathbf{H}} \varphi(p\langle x, e_\alpha\rangle + q\langle y, e_\alpha\rangle)dg_c(x)dg_c(y) \overset{*}{=} \int_{\mathbf{H}} \varphi(\sqrt{p^2+q^2}\langle z, e_\alpha\rangle)dg_c(z) \tag{3.3}$$

where by $\overset{*}{=}$ *we means that if either side exists, both sides exist and equality holds.*

Proof In order to prove the Lemma 3.5, we define a transform R_θ on $\mathbf{H}\times\mathbf{H}$. For each real number θ, let $R_\theta : \mathbf{H}\times\mathbf{H} \to \mathbf{H}\times\mathbf{H}$ be the transform defined by $R_\theta(x) = (X)$, where

$$X \equiv \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let h be a measurable function on $\mathbb{R}\times\mathbb{R}$ and let

$$H(x) = h\begin{pmatrix} \langle x_1, e_\alpha\rangle_{\mathbf{H}} \\ \langle x_2, e_\alpha\rangle_{\mathbf{H}} \end{pmatrix} = h((x)_\alpha).$$

Then using Equation (2.6) it follows that

$$\begin{aligned}&\int_{\mathbf{H}^2} H(R_\theta(x))dg_c(x)\\ &= \left(\frac{1}{\sqrt{2\pi c}}\right)^2 \int_{\mathbb{R}^2} h(Xu)\exp\left\{-\frac{u_1^2+u_2^2}{2c}\right\}du,\end{aligned}$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Now, let $u_1 \cos\theta - u_2 \sin\theta = u_1'$ and $u_1 \sin\theta + u_2 \cos\theta = u_2'$. Then the last expression above becomes

$$\left(\frac{1}{\sqrt{2\pi c}}\right)^2 \int_{\mathbb{R}^2} h(u') \exp\left\{-\frac{(u_1')^2 + (u_2')^2}{2c}\right\} du',$$

where $u' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$, and hence by using Equation (2.3) again, we obtain Equation (3.4) as below

$$\int_{\mathbf{H}^2} H(x) dg_c(x) \overset{*}{=} \int_{\mathbf{H}^2} H(R_\theta(x)) dg_c(x). \tag{3.4}$$

Now, let $P_2 : \mathbf{H}^2 \to \mathbf{H}$ be the projection map given by $P_2(x) = x_2$ and let $H(x) = K(P_2(x))$ where K is a measurable function on $\mathbf{H}$. Then

$$\int_{\mathbf{H}^2} K(x_1 \sin\theta + x_2 \cos\theta) dg_c(x) \overset{*}{=} \int_{\mathbf{H}} K(z) dg_c(z). \tag{3.5}$$

For given real numbers p and q, letting $\sin\theta = \frac{p}{\sqrt{p^2+q^2}}$, $\cos\theta = \frac{q}{\sqrt{p^2+q^2}}$. Replacing K with $F(x) = K(\sqrt{p^2+q^2}x)$ where $F(x) = \varphi(\langle x, e_\alpha \rangle)$, we can complete the proof of Lemma 3.5. □

In Theorem 3.6, we establish the second evaluation formula. Equation (3.6) is called the composition formula for $\mathbf{H}^n$-integrals with matrices.

Theorem 3.6 *Let F_β be an element of $\mathbb{E}_0^{(n)}$ and let $T_1, T_2 \in \mathcal{M}_{n\times n}^{\mathcal{L}}$ with*

$$T_k = (T_{ij}^{(k)}),$$

where $T_{ij}^{(k)} e_\alpha = \mu_{ij}^{(k),\alpha} e_\alpha$ for $k = 1, 2$. Then we have

$$\int_{\mathbf{H}^n} \int_{\mathbf{H}^n} F_\beta(T_1 x + T_2 y) dg_c(x) dg_c(y) = \int_{\mathbf{H}^n} F_\beta(T_3 z) dg_c(z) \tag{3.6}$$

where $(T_3)_0 = (\lambda_{ij}^\beta) \in \mathcal{M}_{n\times n}^{\mathbb{R}}$ with

$$(\lambda_{ij}^\beta)^2 = (\mu_{ij}^{(1),\beta})^2 + (\mu_{ij}^{(2),\beta})^2$$

for all $i, j = 1, \cdots, n$.

Proof We first note that

$$T_1 x + T_2 y = \begin{pmatrix} \sum_{r=1}^n (T_{1r}^{(1)} x_r + T_{1r}^{(2)} y_r) \\ \vdots \\ \sum_{r=1}^n (T_{nr}^{(1)} x_r + T_{nr}^{(2)} y_r) \end{pmatrix}$$

and so

$$F_\beta(T_1x+T_2y)=f\begin{pmatrix}\sum\limits_{r=1}^{n}(\mu_{1r}^{(1),\beta}\langle x_r,e_\beta\rangle+\mu_{1r}^{(2),\beta}\langle y_r,e_\beta\rangle)\\ \vdots\\ \sum\limits_{r=1}^{n}(\mu_{nr}^{(1),\beta}\langle x_r,e_\beta\rangle+\mu_{nr}^{(2),\beta}\langle y_r,e_\beta\rangle)\end{pmatrix}.$$

Also we note that

$$F_\beta(T_3z)=f\begin{pmatrix}\sum\limits_{r=1}^{n}\lambda_{1r}^{\beta}\langle z_r,e_\beta\rangle\\ \vdots\\ \sum\limits_{r=1}^{n}\lambda_{nr}^{\beta}\langle z_r,e_\beta\rangle\end{pmatrix}.$$

Let Pro_s be a s-th projection map on $\mathbb{R}^n$ defined by the formula $Pro_s(u)=u_s, s=1,2,\cdots,n$ and let $f=\psi\circ p_s$ for a measurable function ψ. Then using these facts and (3.3) in Lemma 3.5 with φ replaced by ψ repeatedly, we have

$$\begin{aligned}&\int_{\mathbf{H}^n}\int_{\mathbf{H}^n}F_\beta(T_1x+T_2y)dg_c(x)dg_c(y)\\ &=\int_{\mathbf{H}^n}\int_{\mathbf{H}^n}f\Big((T_1)_0(x)_\beta+(T_2)_0(y)_\beta\Big)dg_c(x)dg_c(y)\\ &=\int_{\mathbf{H}^n}\int_{\mathbf{H}^n}\psi\Big(\sum_{r=1}^{n}(\mu_{sr}^{(1),\beta}\langle x_r,e_\beta\rangle+\mu_{sr}^{(2),\beta}\langle y_r,e_\beta\rangle)\Big)dg_c(x)dg_c(y)\\ &=\int_{\mathbf{H}^n}\int_{\mathbf{H}^n}\psi\Big(\sum_{r=1}^{n}\lambda_{sr}^{\beta}\langle z_r,e_\beta\rangle\Big)dg_c(x)dg_c(y)\\ &=\int_{\mathbf{H}^n}f\Big((T_3)_0(z)_\beta\Big)dg_c(z)\\ &=\int_{\mathbf{H}^n}F_\beta(T_3z)dg_c(z).\end{aligned}$$

Hence we have the desired results. □

4. Matrix Wiener transform

In this section we first define the matrix Wiener transform, and establish the existence of matrix Wiener transform for functions in $\mathbb{E}_0^{(n)}$. We then obtain some relationships for the matrix Wiener transform. In order to do this, we need following items.

For T_1, T_2, S_1 and S_2 in $\mathcal{M}_{n\times n}^{\mathcal{L}}$, let

$$A=\begin{pmatrix}T_1 & | & S_1\\ - & - & -\\ T_2 & | & S_2\end{pmatrix}\in\mathcal{M}_{2n\times 2n}^{\mathcal{L}}. \tag{4.1}$$

Then we have

$$Ax_y = \begin{pmatrix} T_1 \mid S_1 \\ ---\\ T_2 \mid S_2 \end{pmatrix} \begin{pmatrix} x \\ - \\ y \end{pmatrix} = \begin{pmatrix} T_1x + S_1y \\ --- \\ T_2x + S_2y \end{pmatrix} \in \mathcal{M}^{\mathbf{H}}_{2n\times 1}.$$

Hence we have

$$\mathcal{H}(Ax_y) = \big((T_1 + T_2)x + (S_1 + S_2)y\big) \in \mathcal{M}^{\mathbf{H}}_{n\times 1}.$$

Furthermore, one can see that

$$\big((T_1 + T_2)x + (S_1 + S_2)y\big) = \begin{pmatrix} \sum_{r=1}^{n}(T_{1r}^{(1)} + T_{1r}^{(2)})x_r + \sum_{r=1}^{n}(S_{1r}^{(1)} + S_{1r}^{(2)})y_r \\ \vdots \\ \sum_{r=1}^{n}(T_{nr}^{(1)} + T_{nr}^{(2)})x_r + \sum_{r=1}^{n}(S_{nr}^{(1)} + S_{nr}^{(2)})y_r \end{pmatrix} \in \mathcal{M}^{\mathbf{H}}_{n\times 1}$$

and so

$$F_\beta(\mathcal{H}(Ax_y)) = f\bigg((A)_0(x_y)_\beta\bigg).$$

We are ready to define a matrix Wiener transform via the bounded linear operators on the product space of Hilbert spaces.

Definition 4.1 *Let F be a exponential type function on $\mathbf{H}^n$ and let $\mathcal{H}$ be as in Equation (2.3). Let A be as in Equation (4.1) above. The ($\mathcal{H}$) matrix Wiener transform $\mathbb{T}^{\mathcal{H}}_A(F)$ of F is defined by the formula*

$$\mathbb{T}^{\mathcal{H}}_A(F)(y) = \int_{\mathbf{H}^n} (F \circ \mathcal{H})(Ax_y)dg_c(x), \tag{4.2}$$

if it exists.

In our next theorem, we establish the existence of the matrix Wiener transform for functions in $\mathbb{E}_0^{(n)}$.

Theorem 4.2 (Existence theorem) *Let F_β be as in Equation (2.5) above. Then the matrix Wiener transform $\mathbb{T}^{\mathcal{H}}_A(F_\beta)$ of F_β exists, belongs to $\mathbb{E}_0^{(n)}$ and is given by the formula*

$$\mathbb{T}^{\mathcal{H}}_A(F_\beta)(y) = \Gamma_{\mathbb{T}^{\mathcal{H}}_A(F_\beta)}((y)_\beta), \tag{4.3}$$

where

$$\Gamma_{\mathbb{T}^{\mathcal{H}}_A(F_\beta)}(u) = \bigg(\frac{1}{\sqrt{2\pi c}}\bigg)^n \int_{\mathbb{R}^n} f\bigg(((T_1)_0 + (T_2)_0)v + ((S_1)_0 + (S_2)_0)u\bigg) \times \exp\bigg\{-\frac{v^tv}{2c}\bigg\}dv.$$

Proof Using Equations (4.2) and (2.6), we have

$$\begin{aligned}
&\mathbb{T}_A^{\mathcal{H}}(F_\beta)(y)\\
&=\int_{\mathbf{H}^n} F_\beta((T_1+T_2)x+(S_1+S_2)y)dg_c(x)\\
&=\int_{\mathbf{H}^n} f\Big(((T_1)_0+(T_2)_0)(x)_0+((S_1)_0+(S_2)_0)(y)_\beta\Big)dg_c(x)\\
&=\left(\frac{1}{\sqrt{2\pi c}}\right)^n\int_{\mathbb{R}^n} f\Big(((T_1)_0+(T_2)_0)v+((S_1)_0+(S_2)_0)(y)_\beta\Big)\exp\Big\{-\frac{v^tv}{2c}\Big\}dv.
\end{aligned}$$

We next use Equation (2.4) to obtain the existence as below:

$$\begin{aligned}
&|\Gamma_{\mathbb{T}_A^{\mathcal{H}}(F_\beta)}(u)|\\
&\le\left(\frac{1}{\sqrt{2\pi c}}\right)^n M_f\int_{\mathbb{R}^n}\exp\left\{N_fM_1\sum_{r=1}^n|v_r|\right\}\exp\Big\{-\frac{v^tv}{2c}\Big\}dv\\
&\qquad\qquad\times\exp\left\{N_fM_2\sum_{r=1}^n|u_r|\right\}\\
&\le M_{\mathbb{T}_A^{\mathcal{H}}(F_\beta)}\exp\left\{N_{\mathbb{T}_A^{\mathcal{H}}(F_\beta)}\sum_{r=1}^n|u_r|\right\}<\infty,
\end{aligned}$$

where $M_1=Max\{|\mu_{ij}^{(1),\beta}|,|\mu_{ij}^{(2),\beta}|\}$, $M_2=Max\{|\lambda_{ij}^{(1),\beta}|,|\lambda_{ij}^{(2),\beta}|\}$ for $i,j=1,\cdots,n$, $N_{\Gamma_{\mathbb{T}_A^{\mathcal{H}}(F_\beta)}}=N_fM_2$ and

$$M_{\Gamma_{\mathbb{T}_A^{\mathcal{H}}(F_\beta)}}=\left(\frac{1}{\sqrt{2\pi c}}\right)^n M_f\int_{\mathbb{R}^n}\exp\left\{N_fM_1\sum_{r=1}^n|v_r|\right\}\exp\Big\{-\frac{v^tv}{2c}\Big\}dv<\infty.$$

Hence we have the desired result. □

Theorem 4.3 tells us that the matrix Wiener transforms are commutative.

Theorem 4.3 (Relationship 1: commutation of the matrix Wiener transform) *Let* F_β *be an element of* $\mathbb{E}_0^{(n)}$ *and let*

$$A_k=\begin{pmatrix}T_{k1} & | & S_{k1}\\ - & - & -\\ T_{k2} & | & S_{k2}\end{pmatrix}\in\mathcal{M}_{2n\times 2n}^{\mathcal{L}}$$

where T_{kq} *and* S_{kq} *satisfy the condition (iii) in Remark 2.2 respectively for* $k,q=1,2$. *Then*

$$\mathbb{T}_{A_2}^{\mathcal{H}}(\mathbb{T}_{A_1}^{\mathcal{H}}(F_\beta))(y)=\mathbb{T}_{A_1}^{\mathcal{H}}(\mathbb{T}_{A_2}^{\mathcal{H}}(F_\beta))(y)$$

if

$$(T_{11})_0+(T_{12})_0=(T_{21})_0+(T_{22})_0,\quad (S_{11})_0+(S_{12})_0=(S_{21})_0+(S_{22})_0,$$

and $((S_{21})_0+(S_{22})_0)$ *and* $((S_{11})_0+(S_{12})_0)$ *are commutative.*

Proof Using Equations (4.2) and (2.6), we have

$$
\begin{aligned}
&\mathbb{T}^{\mathcal{H}}_{A_2}(\mathbb{T}^{\mathcal{H}}_{A_1}(F_\beta)(y)\\
&= \int_{\mathbf{H}^n}\int_{\mathbf{H}^n} F_\beta((T_{11}+T_{12})z+(S_{11}+S_{12})(T_{21}+T_{22})x\\
&\qquad\qquad +(S_{11}+S_{12})(S_{21}+S_{22})y)dg_c(z)dg_c(x)\\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^{2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\Big(((T_{11})_0+(T_{12})_0)w+((S_{11})_0+(S_{12})_0)((T_{21})_0+(T_{22})_0)v\\
&\qquad\qquad +((S_{11})_0+(S_{12})_0)((S_{21})_0+(S_{22})_0)(y)_\beta\Big)\\
&\qquad\qquad \times \exp\left\{-\frac{w^tw+v^tv}{2c}\right\}dwdv.
\end{aligned} \tag{4.4}
$$

Using the similar methods, we have

$$
\begin{aligned}
&\mathbb{T}^{\mathcal{H}}_{A_1}(\mathbb{T}^{\mathcal{H}}_{A_2}(F_\beta)(y)\\
&= \left(\frac{1}{\sqrt{2\pi c}}\right)^{2n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\Big(((T_{21})_0+(T_{22})_0)w+((S_{21})_0+(S_{22})_0)((T_{11})_0+(T_{12})_0)v\\
&\qquad\qquad +((S_{21})_0+(S_{22})_0)((S_{11})_0+(S_{12})_0)(y)_\beta\Big)\\
&\qquad\qquad \times \exp\left\{-\frac{w^tw+v^tv}{2c}\right\}dwdv.
\end{aligned} \tag{4.5}
$$

Comparing two equations (4.4) and (4.5), we can obtain the desired results. □

We give a composition formula and a inverse transform for the matrix Wiener transform.

Theorem 4.4 (Relationship 2: composition formula and inverse transform) *Under the hypothesis of Theorem 4.3, we have*

$$\mathbb{T}^{\mathcal{H}}_{A_2}(\mathbb{T}^{\mathcal{H}}_{A_1}(F_\beta))(y) = \mathbb{T}^{\mathcal{H}}_{A_3}(F_\beta)(y)$$

if

$$(A_3)_0 = (\lambda^\beta_{ij}) \in \mathcal{M}^{\mathcal{L}}_{n\times n}$$

where $(\lambda^\beta_{ij})^2 = [(\mu^{(11),\beta}_{ij}) + (\mu^{(12),\beta}_{ij})]^2 + [((\mu^{(21),\beta}_{ij}) + (\mu^{(22),\beta}_{ij}))((\lambda^{(11),\beta}_{ij}) + (\lambda^{(12),\beta}_{ij}))]^2$. *Furthermore, we have*

$$\mathbb{T}^{\mathcal{H}}_{A_2}(\mathbb{T}^{\mathcal{H}}_{A_1}(F_\beta))(y) = F_\beta(y)$$

if

$$(T_{11})_0 + (T_{12})_0 \quad \text{or} \quad (T_{21})_0 + (T_{22})_0$$

is the zero matrix, and $((S_{21})_0+(S_{22})_0)((S_{11})_0+(S_{12})_0) = I_n$ *where* I_n *denotes the identity matrix in* $\mathcal{M}^{\mathbb{R}}_{n\times n}$. *In this case, we write*

$$(\mathbb{T}^{\mathcal{H}}_{A_1})^{-1} = \mathbb{T}^{\mathcal{H}}_{A_2}.$$

We finish this paper by a stating some differences the matrix Wiener transform $\mathbb{T}_{A_1}^{\mathcal{H}}$ used in this paper and the matrix Wiener transform $\mathcal{F}_{c,A,B}$ used in [13].

Remark 4.5 *In [13], the authors defined the matrix Wiener transform $\mathcal{F}_{c,A,B}$ on a Hilbert space $\mathbb{R}^n$, and then they analyzed some properties of the matrix Wiener transform $\mathcal{F}_{c,A,B}$ involving the composition, inverse and Parseval's formulas corresponding to Theorem 4.4 in this paper. These formulas are meaningful. While, our matrix Wiener transform $\mathbb{T}_{A}^{\mathcal{H}}$ is defined on general Hilbert space $\mathbf{H}$. This tells that all formulas and results in [13] are corollaries of our formulas and results. More specifically, when $\mathbf{H} = \mathbb{R}$*

$$\mathbb{T}_{A}^{\mathcal{H}}(F_\beta)(y) = \mathcal{F}_{c,T_0,S_0}(f)(u)$$

where $F_\beta(x) = f((x)_\beta)$, $T_0 = (T_1)_0 + (T_2)_0$ and $S_0 = (S_1)_0 + (S_2)_0$ for a matrix A given by Equation (4.1) above. Furthermore, we obtained an evaluation formula (3.1) with some examples. This version was not established in [13]. In addition to this formula, it is expected that other formulas that have not been dealt with in [13] can be obtained.

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