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# Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds

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**Abstract:** In this paper, we classify 3-dimensional Riemannian manifolds endowed with a special type of vector field if the Riemannian metrices are Ricci-Yamabe solitons and gradient Ricci-Yamabe solitons, respectively. Finally, we construct an example to illustrate our result.

**Key words:** Ricci-Yamabe soliton, gradient Ricci-Yamabe soliton, 3-dimensional Riemannian manifolds

## 1. Introduction

In the last twenty years, the hypothesis of geometric flows are the most fascinating mathematical tools to describe the geometric structures in Riemannian geometry. A specific section of solutions on which the metric evolves by diffeomorphisms has a significant impact in the investigation of singularities of the flows as they show up as possible singularity models. They are frequently called soliton solutions.

Yamabe flow was presented by Hamilton [8], simultaneously with Ricci flow. The Ricci soliton and Yamabe soliton are special solutions of the Ricci flow and Yamabe flow, respectively. The Ricci soliton and the Yamabe soliton are equivalent for dimension $m = 2$. Although, in dimension $m > 2$, they are not identical.

In the course of recent years, the theoretical concept of geometric flows, for example, Ricci flow and Yamabe flow have been the focal point of fascination of numerous geometers. As of late, in 2019, Guler and Crasmareanu [7] presented the investigation of another geometric flow under the name Ricci-Yamabe map. This map is nothing but a scalar combination of Ricci and Yamabe flow. This is additionally named $(\alpha, \beta)$ type Ricci-Yamabe flow. This type of flow is an advancement for the metrics on $N$, the Riemannian manifold defined by [7]

$$\frac{\partial}{\partial t} g(t) = \beta r(t) g(t) - 2\alpha \mathcal{S}(t), \quad g_0 = g(0), \tag{1.1}$$

where $\mathcal{S}$ is the Ricci tensor, $r$ denotes the scalar curvature, and $\lambda, \alpha, \beta \in \mathbb{R}$.

One can think Ricci-Yamabe flow as Riemannian or singular Riemannian or semi-Riemannian flow because of the signs of $\alpha$ and $\beta$, the involved scalars. This sort of different choices can be valuable in some mathematical or physical models such as relativistic theories. Therefore, normally Ricci-Yamabe soliton arises as the constraint of the soliton of Ricci-Yamabe flow. Another strong inspiration that started the investigation of Ricci-Yamabe

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solitons is that, in spite of the fact that Ricci solitons and Yamabe solitons are identical in dimensional 2, in higher dimension they are basically different.

A Ricci-Yamabe soliton on $(N, g)$ is a data $(g, X, \lambda, \alpha, \beta)$ fulfilling

$$\pounds_X g = -2\alpha\mathcal{S} - (2\lambda - \beta r)g, \tag{1.2}$$

where $\pounds$ being the Lie-derivative, $\mathcal{S}$ indicates the Ricci tensor, $r$ denotes the scalar curvature and $\lambda, \alpha, \beta \in \mathbb{R}$.

If $f$ is a smooth function and $X$ is the gradient of $f$ on $N$, then the foregoing notion is named gradient Ricci-Yamabe soliton and the equation (1.2) transforms to

$$\nabla^2 f = -\alpha\mathcal{S} - (\lambda - \frac{1}{2}\beta r)g, \tag{1.3}$$

where the Hessian of $f$ is denoted by $\nabla^2 f$.

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$. If $\lambda$, $\beta$ and $\alpha$ are smooth functions on $N$, then a Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is called an almost Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton). If $\beta = 0$, $\alpha = 1$, then Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) turns into Ricci soliton (or gradient Ricci soliton) [8]. Similarly, it turns into Yamabe soliton (or gradient Yamabe soliton) [9] if $\beta = 1$ , $\alpha = 0$ . Also, if $\beta = -1$, $\alpha = 1$, it reduces to an Einstein soliton (or gradient Einstein soliton) [3].

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be proper if $\alpha \neq 0, 1$.

Ricci solitons and Yamabe solitons have been investigated by several authors. A few of them are ( [1, 2, 15–18]) and ([4–6, 13, 19, 20]) respectively.

We present the paper as follows:

At first, in Section 2, we provide the basic results of $N^3$, a 3-dimensional Riemannian manifold. Then in Section 3, we consider Ricci-Yamabe solitons on $N^3$. In Section 4, we investigate gradient Ricci-Yamabe solitons on $N^3$. Finally, we construct an example to illustrate our result.

## 2. Preliminaries

Let $(N^m, g)$ be a Riemannian manifold of dimension $m$. $\mathcal{R}$, the Riemannian curvature tensor of $(N^3, g)$ is written by

$$\begin{aligned} \mathcal{R}(P,Q)G &= g(Q,G)\mathcal{Q}P - g(P,G)\mathcal{Q}Q + \mathcal{S}(Q,G)P - \mathcal{S}(P,G)Q \\ &\quad -\frac{r}{2}\{g(Q,G)P - g(P,G)Q\} \end{aligned} \tag{2.1}$$

for any $P, Q, G \in \chi(N)$ and $\mathcal{Q}$ indicates the Ricci operator defined by $\mathcal{S}(P,Q) = g(\mathcal{Q}P, Q)$. Many authors have investigated $(N^3, g)$ with various structures such as contact structure and complex structure in ([10, 11]) and others.

It may be mentioned that every compact, orientable $N^3$ has a contact structure [14], that is, there exists a global one-form $B$ such that $B \wedge dB \neq 0$ everywhere.

Throughout the paper, we assume that the 3-dimensional Riemannian manifold $N^3$ admits a unit vector field $\zeta$ such that

$$\nabla_P \zeta = B(P)\zeta - P, \tag{2.2}$$

where the 1-form $B$ is associated with the vector field $\zeta$ such that $g(P,\zeta) = B(P)$.

Equation (2.2) implies

$$\mathcal{R}(P,Q)\zeta = B(P)Q - B(Q)P, \tag{2.3}$$

$$\mathcal{R}(\zeta,P)Q = B(Q)P - g(P,Q)\zeta, \tag{2.4}$$

$$(\nabla_P B)Q = B(P)B(Q) - g(P,Q), \tag{2.5}$$

$$\mathcal{S}(P,\zeta) = -2B(P). \tag{2.6}$$

**Lemma 2.1** *In $N^3$ with a special type of vector field, we have*

$$\mathcal{Q}P = (\frac{r}{2}+1)P - (\frac{r}{2}+3)B(P)\zeta, \tag{2.7}$$

*which implies*

$$\mathcal{S}(P,Q) = (\frac{r}{2}+1)g(P,Q) - (\frac{r}{2}+3)B(P)B(Q). \tag{2.8}$$

**Lemma 2.2** *In $N^3$, we have*

$$\zeta r = 2(r+6). \tag{2.9}$$

*Proof.* *Equation (2.7) implies*

$$\begin{aligned}(\nabla_Q \mathcal{Q})P \quad = \quad & \frac{1}{2}(Qr)P - \frac{1}{2}(Qr)B(P)\zeta \\ & -(\frac{r}{2}+3)[2B(P)B(Q)\zeta - g(P,Q)\zeta - \eta(P)Q].\end{aligned} \tag{2.10}$$

*Contracting $Q$ from the above equation, we get*

$$(\zeta r)B(P) = 2(r+6)B(P), \tag{2.11}$$

*which implies equation (2.9) and hence completes the proof.*

## 3. Ricci-Yamabe solitons

Let $N^3$ obey Ricci-Yamabe solitons. Then equation (1.2) yields

$$(\pounds_X g)(P,Q) = -2\alpha\mathcal{S}(P,Q) + (\beta r - 2\lambda)g(P,Q). \tag{3.1}$$

Taking covariant derivative of (3.1), we obtain

$$(\nabla_G \pounds_X g)(P,Q) = -2\alpha(\nabla_G \mathcal{S})(P,Q) + \beta(Gr)g(P,Q). \tag{3.2}$$

From equation (2.8), we get

$$(\nabla_G \mathcal{S})(P,Q) = \frac{Gr}{2}[g(P,Q) - B(P)B(Q)] + (\frac{r}{2}+3)[g(P,G)B(Q) + g(Q,G)B(P) - 2B(P)B(Q)B(G)]. \tag{3.3}$$

Using (3.3) in (3.2) gives

$$(\nabla_G \pounds_X g)(P,Q) = (\beta-\alpha)(Gr)g(P,Q) + \alpha(Gr)B(P)B(Q) - \alpha(r+6)[g(P,G)B(Q) + g(Q,G)B(P) - 2B(P)B(Q)B(G)]. \tag{3.4}$$

Following Yano [21], the following relation holds:

$$(\pounds_X \nabla_P g - \pounds_P \nabla_X g - \nabla_{[X,P]} g)(Q,G) = -g((\pounds_X \nabla)(P,Q),G) - g((\pounds_X \nabla)(P,G),Q), \tag{3.5}$$

which implies

$$(\nabla_P \pounds_X g)(Q,G) = g((\pounds_X \nabla)(P,Q),G) + g((\pounds_X \nabla)(P,G),Q). \tag{3.6}$$

Since $\pounds_X \nabla$ is a (1,2) type symmetric tensor, that is, $(\pounds_X \nabla)(P,Q) = (\pounds_X \nabla)(Q,P)$, then (3.6) infers

$$g((\pounds_X \nabla)(P,Q),G) = \frac{1}{2}(\nabla_P \pounds_X g)(Q,G) + \frac{1}{2}(\nabla_Q \pounds_X g)(P,G) - \frac{1}{2}(\nabla_G \pounds_X g)(P,Q). \tag{3.7}$$

Using (3.4) in (3.7) entails that

$$2g((\pounds_X \nabla)(P,Q),G) = (\beta-\alpha)[(Pr)g(Q,G) + (Qr)g(P,G) - (Gr)g(P,Q)] + \alpha[(Pr)\eta(Q)\eta(G) + (Qr)\eta(P)\eta(G) - (Gr)\eta(P)\eta(Q)] - 2\alpha(r+6)[g(P,Q)\eta(G) - \eta(P)\eta(Q)\eta(G)]. \tag{3.8}$$

Removing $G$ from the above equation, we acquire

$$2(\pounds_X \nabla)(P,Q) = (\beta-\alpha)[(Pr)Q + (Qr)P - g(P,Q)Dr] + \alpha[(Pr)\eta(Q)\zeta + (Qr)\eta(P)\zeta - \eta(P)\eta(Q)Dr] - 2\alpha(r+6)[g(P,Q)\zeta - \eta(P)\eta(Q)\zeta]. \tag{3.9}$$

Putting $Q = \zeta$ in the foregoing equation gives

$$(\pounds_X \nabla)(P,\zeta) = \frac{\beta}{2}(Pr)\zeta - \frac{\beta}{2}B(P)Dr + (r+6)[(\beta-\alpha)P + \alpha B(P)\zeta]. \tag{3.10}$$

If we take $r$ = constant, then (2.9) implies $r = -6$.
Hence equation (3.10) implies

$$(\pounds_X \nabla)(P, \zeta) = 0, \tag{3.11}$$

which implies

$$(\nabla_Q \pounds_X \nabla)(P, \zeta) = 0. \tag{3.12}$$

Again, we know that

$$(\pounds_X \mathcal{R})(P, Q)G = (\nabla_P \pounds_X \nabla)(Q, G) - (\nabla_Q \pounds_X \nabla)(P, G). \tag{3.13}$$

Substituting $G$ by $\zeta$ in (3.13) entails that

$$(\pounds_X \mathcal{R})(P, Q)\zeta = (\nabla_P \pounds_X \nabla)(Q, \zeta) - (\nabla_Q \pounds_X \nabla)(P, \zeta). \tag{3.14}$$

Using (3.12) in (3.14), we infer

$$(\pounds_X \mathcal{R})(P, Q)\zeta = 0. \tag{3.15}$$

Putting $Q = \zeta$ in (3.15), we get

$$(\pounds_X \mathcal{R})(P, \zeta)\zeta = 0. \tag{3.16}$$

From (2.3) and (2.4), we have

$$\mathcal{R}(P, \zeta)Q = g(P, Q)\zeta - B(Q)P \tag{3.17}$$

and

$$\mathcal{R}(P, \zeta)\zeta = B(P)\zeta - P. \tag{3.18}$$

In view of (2.3), (3.17), and (3.18), we get

$$(\pounds_X \mathcal{R})(P, \zeta)\zeta = ((\pounds_X B)P)\zeta - g(P, \pounds_X \zeta)\zeta. \tag{3.19}$$

Putting $Q = \zeta$ in (3.1), we acquire

$$(\pounds_X g)(P, \zeta) = (4\alpha - 6\beta - 2\lambda)B(P), \tag{3.20}$$

which implies

$$(\pounds_X B)P - g(P, \pounds_X \zeta) = (4\alpha - 6\beta - 2\lambda)B(P). \tag{3.21}$$

Using (3.21) in (3.19), we infer

$$(\pounds_X \mathcal{R})(P, \zeta)\zeta = (4\alpha - 6\beta - 2\lambda)B(P)\zeta. \tag{3.22}$$

In view of (3.16) and (3.22), we obtain

$$(4\alpha - 6\beta - 2\lambda)B(P)\zeta = 0, \tag{3.23}$$

which implies

$$\lambda = 2\alpha - 3\beta. \tag{3.24}$$

Hence we have:

**Theorem 3.1** *If the metric of $N^3$ endowed with a special type of vector field is Ricci-Yamabe solitons, then the constants $\alpha, \beta, \lambda$ are related by the equation*

$$\lambda = 2\alpha - 3\beta,$$

*provided the scalar curvature is constant.*

If we take $\alpha = 1$ and $\beta = 0$, then from (3.24), we get

$$\lambda = 2.$$

Hence we have:

**Corollary 3.2** *If the metric of $N^3$ of constant scalar curvature equipped with a special type of vector field admits a Ricci soliton, then the soliton is expanding.*

In particular, for $\alpha = 0$ and $\beta = 1$ equations (3.24) implies

$$\lambda = -3.$$

Hence we have:

**Corollary 3.3** *If the metric of $N^3$ of constant scalar curvature endowed with a special type of vector field admits a Yamabe soliton, then the soliton is shrinking.*

Again, if we take $\alpha = 1$ and $\beta = -1$, then from (3.24), we acquire

$$\lambda = 5.$$

Hence we have:

**Corollary 3.4** *If the metric of $N^3$ of constant scalar curvature equipped with a special type of vector field admits an Einstein soliton, then the soliton is expanding.*

Let $X$ be pointwise collinear with $\zeta$, that is, $X = b\zeta$, where $b$ is a smooth function. Then (1.2) implies

$$(\pounds_{b\zeta} g)(P, Q) = -2\alpha \mathcal{S}(P, Q) - (2\lambda - \beta r) g(P, Q). \tag{3.25}$$

Now,

$$\begin{aligned} (\pounds_{b\zeta} g)(P, Q) &= (Pb)B(Q) + (Qb)B(P) \\ &\quad + b[g(\nabla_P \zeta, Q) + g(P, \nabla_Q \zeta)], \end{aligned} \tag{3.26}$$

which implies

$$\begin{aligned} (\pounds_{b\zeta} g)(P, Q) &= (Pb)B(Q) + (Qb)B(P) \\ &\quad - 2b[g(P, Q) - B(P)B(Q)]. \end{aligned} \tag{3.27}$$

Using (3.27) in (3.25) entails that

$$(Pb)B(Q) + (Qb)B(P) - 2b[g(P,Q) - B(P)B(Q)] \qquad (3.28)$$
$$= -2\alpha\mathcal{S}(P,Q) - (2\lambda - \beta r)g(P,Q).$$

Setting $Q = \zeta$ in (3.28), we infer

$$(Pb) + (\zeta b)B(P) = 2(2\alpha - \lambda + \frac{\beta}{2}r)B(P). \qquad (3.29)$$

Putting $P = \zeta$ in the foregoing equation gives

$$\zeta b = (2\alpha - \lambda + \frac{\beta}{2}r). \qquad (3.30)$$

In view of (3.29) and (3.30), we get

$$Pb = (2\alpha - \lambda + \frac{\beta}{2}r)B(P). \qquad (3.31)$$

If we take $2\alpha - \lambda + \frac{\beta}{2}r = 0$, then (3.31) implies

$$Pb = 0, \qquad (3.32)$$

which implies $b$ is constant. Hence we have:

**Theorem 3.5** *If the metric of $N^3$ equipped with a special type of vector field is Ricci-Yamabe solitons and the potential vector field $X$ is pointwise collinear with the vector field $\zeta$, then $X$ is a constant multiple of $\zeta$, provided $2\alpha - \lambda + \frac{\beta}{2}r = 0$.*

## 4. Gradient Ricci-Yamabe solitons

Let $N^3$ admit a gradient Ricci-Yamabe soliton. Then (1.3) implies

$$\nabla_P Df = -\alpha\mathcal{Q}P - (\lambda - \frac{\beta}{2}r)P. \qquad (4.1)$$

Taking covariant derivative of (4.1), we obtain

$$\nabla_Q\nabla_P Df = -\alpha\nabla_Q\mathcal{Q}P - (\lambda - \frac{\beta}{2}r)\nabla_Q P + \frac{\beta}{2}(Qr)P. \qquad (4.2)$$

Equation (4.2) gives

$$\nabla_P\nabla_Q Df = -\alpha\nabla_P\mathcal{Q}Q - (\lambda - \frac{\beta}{2}r)\nabla_P Q + \frac{\beta}{2}(Pr)Q. \qquad (4.3)$$

From equation (4.1), we get

$$\nabla_{[P,Q]}Df = -\alpha\mathcal{Q}(\nabla_P Q - \nabla_Q P) - (\lambda - \frac{\beta}{2}r)(\nabla_P Q - \nabla_Q P). \qquad (4.4)$$

In view of (4.2)–(4.4), we infer

$$\mathcal{R}(P,Q)Df = -\alpha[(\nabla_P \mathcal{Q})Q - (\nabla_Q \mathcal{Q})P] + \frac{\beta}{2}[(Pr)Q - (Qr)P]. \tag{4.5}$$

Using (2.10) in the above equation gives

$$\begin{aligned} \mathcal{R}(P,Q)Df &= \frac{(\beta-\alpha)}{2}[(Pr)Q - (Qr)P] \\ &+ \frac{\alpha}{2}[(Pr)B(Q)\zeta - (Qr)B(P)\zeta] \\ &+ \alpha(\frac{r}{2}+3)[B(P)Q - B(Q)P]. \end{aligned} \tag{4.6}$$

Contracting (4.6) yields

$$\mathcal{S}(Q,Df) = (\frac{\alpha}{2} - \beta)Qr. \tag{4.7}$$

Replacing $P$ by $Df$ in (2.8), we obtain

$$\mathcal{S}(Q,Df) = (\frac{r}{2}+1)Qf - (\frac{r}{2}+3)(\zeta f)B(Q). \tag{4.8}$$

Above two equations implies

$$\begin{aligned} (\frac{\alpha}{2} - \beta)Qr &= (\frac{r}{2}+1)Qf \\ &- (\frac{r}{2}+3)(\zeta f)B(Q). \end{aligned} \tag{4.9}$$

Putting $Q = \zeta$ in (4.9) and using (2.9) gives

$$\zeta f = \frac{(2\beta - \alpha)(r+6)}{2}. \tag{4.10}$$

Taking inner product of (4.6) with $\zeta$, we get

$$B(Q)Pf - B(P)Qf = \frac{\beta}{2}[(Pr)B(Q) - (Qr)B(P)]. \tag{4.11}$$

Putting $P = \zeta$ in (4.11) and using (2.9) and (4.10) entails that

$$Qf = \frac{\beta}{2}(Qr) - \frac{\alpha}{2}(r+6)B(Q). \tag{4.12}$$

If we take $r = constant$, then from (2.9), we get $r = -6$. Hence (4.12) implies

$$Qf = 0,$$

which implies $f$ is constant. Therefore, the soliton is trivial. Hence, the manifold is an Einstein manifold. Being 3-dimensional, the manifold becomes a space of constant curvature.
Hence, we have:

**Theorem 4.1** *If the metric of $N^3$ of constant scalar curvature endowed with a special type of vector field is gradient Ricci-Yamabe soliton, then the soliton is trivial and $N^3$ is a space of constant curvature.*

## 5. Example

We consider $N^3 = \{(x,y,z) \in \mathbb{R}\}$, the 3-dimensional manifold, where $(x,y,z) \in \mathbb{R}^3$. Let $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ be three vector fields in $\mathbb{R}^3$ which satisfies

$$\mathrm{v}_1 = e^z \frac{\partial}{\partial x} + e^z \frac{\partial}{\partial y}, \quad \mathrm{v}_2 = e^z \frac{\partial}{\partial x} - e^z \frac{\partial}{\partial y}, \quad \mathrm{v}_3 = \frac{\partial}{\partial z},$$

which implies

$$[\mathrm{v}_1, \mathrm{v}_2] = 0, \quad [\mathrm{v}_1, \mathrm{v}_3] = -\mathrm{v}_1, \quad [\mathrm{v}_2, \mathrm{v}_3] = -\mathrm{v}_2.$$

Let $g$ indicate the Riemannian metric defined by

$$g(\mathrm{v}_1, \mathrm{v}_1) = g(\mathrm{v}_2, \mathrm{v}_2) = g(\mathrm{v}_3, \mathrm{v}_3) = 1 \quad \textit{and} \quad g(\mathrm{v}_1, \mathrm{v}_2) = g(\mathrm{v}_1, \mathrm{v}_3) = g(\mathrm{v}_2, \mathrm{v}_3) = 0.$$

Let $B$ be the 1-form defined by $B(Z) = g(Z, \mathrm{v}_3)$, for any $Z \in \chi(N)$.

Now from Koszul's formula, Levi-Civita connection $\nabla$ is given by

$$\nabla_{\mathrm{v}_1}\mathrm{v}_1 = \mathrm{v}_3, \quad \nabla_{\mathrm{v}_1}\mathrm{v}_2 = 0, \quad \nabla_{\mathrm{v}_1}\mathrm{v}_3 = -\mathrm{v}_1,$$

$$\nabla_{\mathrm{v}_2}\mathrm{v}_1 = 0, \quad \nabla_{\mathrm{v}_2}\mathrm{v}_2 = \mathrm{v}_3, \quad \nabla_{\mathrm{v}_2}\mathrm{v}_3 = -\mathrm{v}_2,$$

$$\nabla_{\mathrm{v}_3}\mathrm{v}_1 = 0, \quad \nabla_{\mathrm{v}_3}\mathrm{v}_2 = 0, \quad \nabla_{\mathrm{v}_3}\mathrm{v}_3 = 0.$$

Hence, the manifold is a Riemannian manifold with a special type of vector field $\mathrm{v}_3$. The components of the curvature tensor and Ricci tensor are

$$\mathcal{R}(\mathrm{v}_1, \mathrm{v}_2)\mathrm{v}_1 = \mathrm{v}_2, \quad \mathcal{R}(\mathrm{v}_1, \mathrm{v}_2)\mathrm{v}_2 = -\mathrm{v}_1, \quad \mathcal{R}(\mathrm{v}_1, \mathrm{v}_2)\mathrm{v}_3 = 0,$$

$$\mathcal{R}(\mathrm{v}_1, \mathrm{v}_3)\mathrm{v}_1 = \mathrm{v}_3, \quad \mathcal{R}(\mathrm{v}_1, \mathrm{v}_3)\mathrm{v}_2 = 0, \quad \mathcal{R}(\mathrm{v}_1, \mathrm{v}_3)\mathrm{v}_3 = -\mathrm{v}_1,$$

$$\mathcal{R}(\mathrm{v}_2, \mathrm{v}_3)\mathrm{v}_1 = 0, \quad \mathcal{R}(\mathrm{v}_2, \mathrm{v}_3)\mathrm{v}_2 = \mathrm{v}_3, \quad \mathcal{R}(\mathrm{v}_2, \mathrm{v}_3)\mathrm{v}_3 = -\mathrm{v}_2$$

and

$$\mathcal{S}(\mathrm{v}_1, \mathrm{v}_1) = -2, \quad \mathcal{S}(\mathrm{v}_2, \mathrm{v}_2) = -2, \quad \mathcal{S}(\mathrm{v}_3, \mathrm{v}_3) = -2.$$

From the above equations, we obtain $r = -6$. Suppose that there exists a function $f$ on $N$ such that

$$\nabla_Z Df = -\alpha \mathcal{Q} Z - (\lambda - \frac{\beta}{2} r)Z. \tag{5.1}$$

Since $Df = (\mathrm{v}_1 f)\mathrm{v}_1 + (\mathrm{v}_2 f)\mathrm{v}_2 + (\mathrm{v}_3 f)\mathrm{v}_3$, we get

$$\nabla_{\mathrm{v}_1} Df = (\mathrm{v}_1(\mathrm{v}_1 f) - \mathrm{v}_3 f)\mathrm{v}_1 + (\mathrm{v}_1(\mathrm{v}_2 f))\mathrm{v}_2 + (\mathrm{v}_1(\mathrm{v}_3 f) + \mathrm{v}_1 f)\mathrm{v}_3,$$

$$\nabla_{\mathrm{v}_2} Df = (\mathrm{v}_2(\mathrm{v}_1 f))\mathrm{v}_1 + (\mathrm{v}_2(\mathrm{v}_2 f) - \mathrm{v}_3 f)\mathrm{v}_2 + (\mathrm{v}_2(\mathrm{v}_3 f) + \mathrm{v}_2 f)\mathrm{v}_3,$$

$$\nabla_{\mathrm{v}_3} Df = (\mathrm{v}_3(\mathrm{v}_1 f))\mathrm{v}_1 + (\mathrm{v}_3(\mathrm{v}_2 f))\mathrm{v}_2 + (\mathrm{v}_3(\mathrm{v}_3 f))\mathrm{v}_3.$$

Thus, $f$ satisfies the following equations

$$\mathrm{v}_1(\mathrm{v}_1 f) - \mathrm{v}_3 f = 2\alpha - 3\beta - \lambda,$$

$$\mathrm{v}_2(\mathrm{v}_2 f) - \mathrm{v}_3 f = 2\alpha - 3\beta - \lambda,$$

$$\mathrm{v}_3(\mathrm{v}_3 f) = 2\alpha - 3\beta - \lambda,$$

$$\mathrm{v}_1(\mathrm{v}_2 f) = 0,$$

$$\mathrm{v}_2(\mathrm{v}_1 f) = 0,$$

$$\mathrm{v}_2(\mathrm{v}_3 f) + \mathrm{v}_2 f = 0,$$

which are equivalent to

$$e^{2z}\frac{\partial^2 f}{\partial x^2} + 2e^{2z}\frac{\partial^2 f}{\partial x \partial y} + e^{2z}\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial z} = 2\alpha - 3\beta - \lambda,$$

$$e^{2z}\frac{\partial^2 f}{\partial x^2} - 2e^{2z}\frac{\partial^2 f}{\partial x \partial y} + e^{2z}\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial z} = 2\alpha - 3\beta - \lambda,$$

$$\frac{\partial^2 f}{\partial z^2} = 2\alpha - 3\beta - \lambda,$$

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0.$$

The above equations imply $f$ is constant for $\lambda = 2\alpha - 3\beta$. Hence, equation (4.1) is satisfied. Thus $g$ is a gradient Ricci-Yamabe soliton with the soliton vector field $X = Df$, where $f =$ constant and $\lambda = 2\alpha - 3\beta$. Thus, Theorem 4.1 is verified.

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