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Application of Gegenbauer polynomials for biunivalent functions defined by subordination

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Abstract: We present and investigate a new subclass of biunivalent functions by applying Gegenbouer polynomials in this paper. Also, we find nonsharp estimates on the first two coefficients $|b_0|$ and $|b_1|$ for functions belonging to this subclass. Furthermore, the Fekete–Szegö inequality $\left|b_1 - \eta b_0^2\right|$ for this subclass is obtained. We also point out some consequences of results.

Key words: Gegenbauer polynomials, coefficient estimates, biunivalent functions and subordination

1. Introduction

Let A denote the class of all analytic functions f which are normalized by the conditions $f(0) = 0$ and $f^{'}(0) = 1$ and defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Taylor's series expansion of $f \in A$ is of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we mean the subclass of A containing those functions which are univalent and analytic in U (for detail, see [15]).

Suppose the functions f and F are analytic in U. Then the subordination between f and F is symbolically denoted by $f \prec g$ and is defined as:

The function f is subordinate to F if there exists a Schwarz function w in U with $w(0) = 0$ and $|w(z)| < 1$ satisfying

$$f(z) = F(w(z)).$$

Further, if $F(z)$ is univalent in U, then the following subordination [26] holds:

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(U) \subset F(U) \qquad z \in U.$$

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The Koebe one-quarter theorem (for detail, see [15]) ensures that the image of U under every univalent function $f \in A$ contains a disc of radius $\frac{1}{4}$.

According to this, every univalent function $f \in A$ has an inverse map f^{-1} that satisfies the following conditions:

$$f^{-1}(f(z)) = z \qquad (z \in U),$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f) : r_0(f) \geq \frac{1}{4}).$$

The series representation of an inverse function is

$$g(w) = f^{-1}(w) = w - b_0 w^2 + (2b_0^2 - b_1)w^3 - (5b_0^3 - 5b_0 b_1 + b_4)w^4 + ... \tag{2}$$

A function $f \in A$ is said to be biunivalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of biunivalent functions in U.

It is worth mentioning that the familiar Koebe function $k(z) = \frac{z}{(1-z)}$ is not in Σ, since it maps the unit disc U which is univalent onto the entire complex plane except the part of the negative real axis from $-1/4$ to $-\infty$. For detail and in-depth study about biunivalent functions, we refer to ([6]–[10]) and references cited therein.

The class of analytic biunivalent functions was first introduced and studied by Lewin [25] and it was shown that $|b_0| < 1.51$. Afterward, the result of Lewin is modified to $|b_0| < \sqrt{2}$, see for example, [9]. Many subclasses of biunivalent functions were explored by many authors and found nonsharp estimates on $|b_0|$ and $|b_1|$. Further, for detailed description, we refer to ([16], [19], [21], [23]- [29], [34], [35], [37]).

The most important and fundamental subclass of the class S is the class $S^*(\zeta)$ of starlike functions of order $\zeta, 0 \leq \zeta < 1$ in U and the class $k(\zeta)$ of the convex function of order ζ in U. We have,

$$S^*(\zeta) = \left\{ f : f \in S \text{ and } Re\left(\frac{zf'(z)}{f(z)}\right) > \zeta, \qquad (z \in U; 0 \leq \zeta < 1) \right\},$$

and

$$k(\zeta) = \left\{ f : f \in S \text{ and } Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \zeta, \quad (z \in U; 0 \leq \zeta < 1) \right\}.$$

For $0 \leq \zeta < 1$, a function $f \in \Sigma$ is in the class $S^*_\Sigma(\zeta)$ of the bistarlike function of order ζ or $k_\Sigma(\zeta)$ of the biconvex function of order ζ if both f and f^{-1} are, respectively, starlike or convex functions of order ζ.

The main objective of our present investigation is on Gegenbauer polynomial denoted by $H_\alpha(x, z)$, very recently, was introduced by Amourah [2] and it is given by the following recurrence relation:

For nonzero real constant α, a generating function of Gegenbauer polynomial is defined by

$$H_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^\alpha}, \tag{3}$$

where $x \in [-1,1]$ and $z \in U$. For fixed x, the function H_α is analytic in U, so it can be expanded as

$$H_\alpha(x,z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n, \tag{4}$$

where $C_n^\alpha(x)$ is Gegenbauer polynomial of degree n.

If $\alpha = 0$, then $H_0(x,z)$ no polynomial cannot be generated. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$H_\alpha(x,z) = 1 - \log(1 - 2x + z^2) = \sum_{n=0}^{\infty} C_n^0(x) z^n,$$

for $\alpha = 0$. Moreover, it is worth mentioning that normalization of α to be greater than $-1/2$ is desirable ([14], [31]). Gegenbauer polynomial satisfies the following relation

$$C_n^\alpha(x) = \frac{1}{n}[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-1)C_{n-1}^\alpha(x)],$$

with first three Gegenbauer polynomials are:

$$C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = 2\alpha(1+\alpha)x^2 - \alpha. \tag{5}$$

By taking $\alpha = 1/2$ and $\alpha = 1$ Gegenbauer polynomial reduces to Legendre and Chebyshev polynomials.

Recently, many researchers have been exploring biunivalent functions associated with orthogonal polynomials, few to mention ([1]–[5], [32], [33]). For Gegenbauer polynomial, as far as we know, there is little work associated with biunivalent functions in the literature.

In this section, we are ready to define some basic concepts of q-calculus or quantum calculus, that is, the main objective of the current section.

Quantum calculus, sometimes called calculus without limits, dates back to the eighteenth century. It plays an important role in many areas of mathematical, physical, and engineering sciences. In geometric function theory, subclasses of Σ of biunivalent functions have been studied from different viewpoints. The q-calculus provided important tools that have been used to investigate various subclasses of class Σ. Very a number of incredible mathematicians considered the concepts of q-derivative, for example by Aral et al. [4], Gasper and Rahman [18] and many others ([11]–[13], [30], [36]). We provide some basic definitions and concepts of q-calculus which help us in our subsequent work.

Definition 1.1 *([20], [22]) For a function $f \in \Sigma$ and given by (1) and $(0 < q < 1)$, the q-derivative of function f is defined by*

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)}; & z \neq 0 \\ f'(z) \quad ; & z = 0 \end{cases}. \tag{6}$$

By definition, we write

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{7}$$

where

$$[n]_q = \frac{1-q^n}{1-q}. \tag{8}$$

Noted that:

(i) For $n=2$, in (8) we have, $[2]_q = 1+q$ and when $q \to 1^-$, then $[2]_1 = 2$,

(ii) For $n=3$, in (8) we have, $[3]_q = q^2+q+1$ and when $q \to 1^-$, then $[3]_1 = 3$, and so on.

In terms of q-derivative, the inverse function $F(w)$ defined by (2) can be written as:

$$D_q g(w) = 1 - [2]_q b_0 w + [3]_q (2b_0^2 - b_1) w^2 - [4]_q (5b_0^3 - 5b_0 b_1 + b_4) w^3 + \ldots. \tag{9}$$

We note that:

1. In (6), $\lim\limits_{q\to 1^-} D_q f(z) = f(z)$, for $f \in \Sigma$,

2. In (8), $q \to 1^-$, then $[n]_q \to n$.

Based on q-derivative, we have the following definition:

Definition 1.2 *A function* $f \in \Sigma$ *given by* (1) *is said to be in class* $\beta^{\mu}_{q,\Sigma}(\lambda, x, \alpha)$ *and* $(\lambda \geq 1, \mu \geq 0, 0 < q < 1, 1/2 < x < 1)$, *if the following conditions are satisfied:*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda D_q f(z) \left(\frac{f(z)}{z}\right)^{\mu-1} \prec H_{\alpha}(x,z), \tag{10}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda D_q g(w) \left(\frac{g(w)}{w}\right)^{\mu-1} \prec H_{\alpha}(x,w), \tag{11}$$

where the functions g *is defined by* (2) *and* H_{α} *is the generating function of the Gegenbauer polynomial given by* (4).

Note that:

1. When $q \to 1^-$, the class $\beta^{\mu}_{1,\Sigma}(\lambda, x, \alpha)$ reduced to $\beta^{\mu}_{\Sigma}(\lambda, x, \alpha)$ which was introduced by Almourah [3];

2. When $q \to 1^-$ and $\alpha = 1$, the class $\beta^{\mu}_{1,\Sigma}(\lambda, x, 1)$ reduced to $\beta^{\mu}_{\Sigma}(\lambda, x)$ which was introduced by Bulut [7];

3. When $q \to 1^-$ and $\mu = \alpha = 1$, the class $\beta^{1}_{1,\Sigma}(\lambda, x, 1)$ reduced to $\beta_{\Sigma}(\lambda, x)$ which was introduced by Bulut [8].

In this paper, we introduce a new subclass $\beta^{\mu}_{q,\Sigma}(\lambda, x, \alpha)$ of biunivalent functions by using Gegenbauer polynomials and obtain the estimates on the initial coefficients $|b_0|$ and $|b_1|$ for this subclass.

In the remaining sections, we assume $\lambda \geq 1, \mu \geq 0, 0 < q < 1, 1/2 < x < 1$, and α is a nonzero real constant.

2. Coefficient bounds of the functions class $\beta^{\mu}_{q,\Sigma}(\lambda, x, \alpha)$

Theorem 2.1 *Let the function* $f \in \Sigma$ *given by* (1) *belong to class* $\beta^{\mu}_{q,\Sigma}(\lambda, x, \mu)$. *Then*

$$|b_0| \leq \frac{2\,|\alpha|\,x\sqrt{2\,|\alpha|\,x}}{\sqrt{|\beta_1|}}, \tag{12}$$

and

$$|b_1| \leq \frac{2|\alpha| x}{(\mu+\lambda([3,q]-1)} + \frac{8|\alpha|^3 x^3}{|\beta_1|}, \tag{13}$$

where

$$\beta_1 = \left\{ \begin{array}{c} \alpha(\mu+\lambda([2,q]-1))^2 - 2[\alpha^2((\mu-1)(\mu+2\lambda[2,q]) \\ +2(\mu(1-\lambda)+\lambda[3,q])- \\ (\mu+\lambda([2,q]-1))^2)- \\ \alpha(\mu+\lambda([2,q]-1))^2]x^2 \end{array} \right\}. \tag{14}$$

Proof It follows from (10) and (11) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda D_q f(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = 1 + C_1^{\alpha}(x)u(z) + C_2^{\alpha}(x)u^2(z) + ..., \tag{15}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda D_q g(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = 1 + C_1^{\alpha}(x)v(w) + C_2^{\alpha}(x)v^2(w) + ..., \tag{16}$$

where u(z) and v(w) are analytic functions of the following form

$$u(z) = c_1 z + c_2 z^2 + c_3 z^3 + ... \ (z \in U), \tag{17}$$

and

$$v(w) = d_1 w + d_2 w^2 + d_3 w^3 + ...(w \in U), \tag{18}$$

respectively, and $u(0) = v(0) = 0$ and $|u(z)| < 1$, $|v(w)| < 1$ for all $z, w \in U$.

Thus, in view of (17) and (18), (15) and (16) become

$$\left\{ \begin{array}{c} (1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \\ \lambda D_q f(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \end{array} \right\} = \left\{ \begin{array}{c} 1 + C_1^{\alpha}(x)c_1 z + \\ (C_1^{\alpha}(x)c_2 + C_2^{\alpha}(x)c_1^2)z^2 + ... \end{array} \right\}, \tag{19}$$

and

$$\left\{ \begin{array}{c} (1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \\ \lambda D_q g(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \end{array} \right\} = \left\{ \begin{array}{c} 1 + C_1^{\alpha}(x)c_1 z + \\ (C_1^{\alpha}(x)c_2 + C_2^{\alpha}(x)c_1^2)z^2 + ... \end{array} \right\}. \tag{20}$$

Now, equating the corresponding coefficients in (19) and (20), we get

$$(\mu+\lambda([2,q]-1))b_0 = C_1^{\alpha}(x)c_1. \tag{21}$$

$$\left\{ \begin{array}{c} \left(\frac{\mu-1}{2}\right)[(\mu+2\lambda([2,q]-1)]b_0^2 + \\ [(\mu+\lambda([3,q]-1)]b_1 \end{array} \right\} = C_1^{\alpha}(x)c_2 + C_2^{\alpha}(x)c_1^2, \tag{22}$$

$$-(\mu+\lambda([2,q]-1))b_0 = C_1^{\alpha}(x)d_1, \tag{23}$$

and

$$\left\{\begin{array}{c}\frac{1}{2}[(\mu(1-\lambda)(\mu+3)+\lambda(\mu-1)\\(\mu+2([2,q]+1))\\+4\lambda[3,q]]b_0^2-\\ [(\mu+\lambda([3,q]-1)]b_1\end{array}\right\}=C_1^\alpha(x)d_2+C_2^\alpha(x)d_1^2. \tag{24}$$

From (21) and (23), we have

$$c_1=-d_1, \tag{25}$$

and

$$b_0=\frac{[C_1^\alpha(x)]^2(c_1^2+d_1^2)}{2(\mu+\lambda([2,q]-1))^2}. \tag{26}$$

By summing (22) and (24), we get

$$\left\{\begin{array}{c}[(\mu-1)(\mu+2\lambda[2,q])+\\2(\mu(1-\lambda)+\lambda[3,q])]b_0^2\end{array}\right\}=C_1^\alpha(x)(c_2+d_2)+C_2^\alpha(x)(c_1^2+d_1^2). \tag{27}$$

It follows from (26) that

$$\left\{\begin{array}{c}(\mu-1)(\mu+2\lambda[2,q])+2(\mu(1-\lambda)+\lambda[3,q])-\\\frac{2C_2^\alpha(x)}{[C_1^\alpha(x)]^2}(\mu+\lambda([2,q]-1))^2\end{array}\right\}b_0^2=C_1^\alpha(x)(c_2+d_2). \tag{28}$$

Also, we know that, if $|u(z)|<1,\ (z\in U)$ and $|v(w)|<1,\ (w\in U)$, then

$$|c_i|\le 1 \text{ and } |d_i|\le 1 \text{ for all } i\in N. \tag{29}$$

By using (5) and (29) in (28), we obtained the desired inequality (12) with β_1 defined in (14).

Now, by subtracting (24) from (22), we get

$$2(\mu+\lambda([3,q]-1))b_1-2(\mu+\lambda([3,q]-1))b_0^2)=C_1^\alpha(x)(c_2-d_2)+C_2^\alpha(x)(c_1^2-d_1^2). \tag{30}$$

Moreover, in view of (25), (30) becomes

$$b_1=\frac{C_1^\alpha(x)}{2(\mu+\lambda([3,q]-1))}(c_2-d_2)+b_0^2. \tag{31}$$

Hence, by using (5), (29), and (26), we obtained the desired inequality (13) with β_1 defined in (14), which completes the proof. □

In this part of the paper, we discuss some consequences deduced from Theorem 1. Sitting $q\to 1^-$ in Theorem 1, we obtain the following consequence.

Corollary 2.2 *[3] Let the function $f\ (z)$ given by (1) belong to class $\beta_\Sigma^\mu(\lambda,x,\alpha)$. Then*

$$|b_0|\le\frac{2\,|\alpha|\,x\sqrt{2\,|\alpha|\,x}}{\sqrt{|\alpha(\mu+\lambda)^2-2[\alpha(1+\alpha)(\mu+\lambda)^2-\alpha^2(\mu+2\lambda)(1+\mu)]x^2|}}, \tag{32}$$

and

$$|b_1|\le\frac{2\,|\alpha|\,x}{(\mu+2\lambda)}+\frac{4\,|\alpha|^2\,x^2}{(\mu+\lambda)^2}. \tag{33}$$

Sitting $q \to 1^-$ and $\alpha = 1$ in Theorem 1, we obtain the following consequence.

Corollary 2.3 *[7] Let the function* $f\ (z)$ *given by* (1) *belong to class* $\beta_{\Sigma}^{\mu}(\lambda, x)$. *Then*

$$|b_0| \leq \frac{2x\sqrt{2x}}{\sqrt{|(\mu+\lambda)^2 - 2[2(\mu+\lambda)^2 - (\mu+2\lambda)(1+\mu)]x^2|}}, \tag{34}$$

and

$$|b_1| \leq \frac{2x}{(\mu+2\lambda)} + \frac{4x^2}{(\mu+\lambda)^2}. \tag{35}$$

Sitting $q \to 1^-$ and $\mu = \alpha = 1$ in Theorem 1, we obtain the following consequence.

Corollary 2.4 *[8] Let the function* $f\ (z)$ *given by* (1) *belong to class* $\beta_{\Sigma}(\lambda, x)$. *Then*

$$|b_0| \leq \frac{2x\sqrt{2x}}{\sqrt{|(1+\lambda)^2 - 4\lambda^2 x^2|}}, \tag{36}$$

and

$$|b_1| \leq \frac{2x}{(1+2\lambda)} + \frac{4x^2}{(1+\lambda)^2}. \tag{37}$$

3. Fekete–Szegö inequality for the function class $\beta_{q,\Sigma}^{\mu}(\lambda, x, \alpha)$

In this section of the paper, we are trying to prove the second theorem, which is based on Fekete–Szegö functional $\left|b_1 - \eta a_2^2\right|$. In Mathematics, the main objective of the inequality is to obtain the coefficients of the univalent functions, which is due to Fekete–Szegö [17].

Theorem 3.1 *Let the function* $f\ (z)$ *given by* (1) *belong to class* $\beta_{q,\Sigma}^{\mu}(\lambda, x, \mu)$. *Then*

$$\left|b_1 - \eta b_0^2\right| \leq \left\{ \begin{array}{lll} \frac{2|\alpha| x}{(\mu+\lambda([3,q]-1)}, & |\eta - 1| & \leq \left|\frac{\beta_1}{4\alpha(\mu+\lambda([3,q]-1)x^2}\right| \\ \frac{8|\alpha|^3 x^3 |1-\eta|}{|\beta_1|}, & |\eta - 1| & \geq \left|\frac{\beta_1}{4\alpha(\mu+\lambda([3,q]-1)x^2}\right| \end{array} \right. . \tag{38}$$

where β_1 *is defined by (14).*

Proof From (28) and (31),

$$b_1 - \eta b_0^2 = C_1^{\alpha}(x) \left\{ \begin{array}{c} \left[h(\eta) + \frac{1}{2(\mu+\lambda([3,q]-1)}\right] c_2 + \\ \left[h(\eta) - \frac{1}{2(\mu+\lambda([3,q]-1)}\right] d_2 \end{array} \right\}, \tag{39}$$

where

$$h(\eta) = \frac{[C_1^{\alpha}(x)]^2(1-\eta)}{\xi}, \tag{40}$$

and

$$\xi = \left\{ \begin{array}{c} [C_1^{\alpha}(x)]^2[(\mu-1)(\mu+2\lambda[2,q]) + 2(\mu(1-\lambda) + \lambda[3,q])] - \\ 2C_2^{\alpha}(x)[\mu+\lambda([2,q]-1)]^2 \end{array} \right\}. \tag{41}$$

Then, in view of (5), we obtained the desired inequality (38). □

We end this section by defining some consequences related to Theorem 2. These are in the following corollaries. Sitting $q \to 1^-$ in Theorem 2, we obtain the following consequence.

Corollary 3.2 *[3] Let the function $f(z)$ given by (1) belong to class $\beta_\Sigma^\mu(\lambda, x, \alpha)$. Then*

$$\left|b_1 - \eta b_0^2\right| \leq \begin{cases} \frac{2|\alpha|x}{(\mu+2\lambda)}, & |\eta - 1| \leq \left|\frac{\alpha(\mu+\lambda)^2 - 2[\alpha(1+\alpha)(\mu+\lambda)^2 - \alpha^2(\mu+2\lambda)(1+\mu)]x^2}{4\alpha(\mu+2\lambda)x^2}\right| \\ \frac{8|\alpha|^3 x^3 |1-\eta|}{|\alpha(\mu+\lambda)^2 - 2[\alpha(1+\alpha)(\mu+\lambda)^2 - \alpha^2(\mu+2\lambda)(1+\mu)]x^2|}, & |\eta - 1| \geq \left|\frac{\alpha(\mu+\lambda)^2 - 2[\alpha(1+\alpha)(\mu+\lambda)^2 - \alpha^2(\mu+2\lambda)(1+\mu)]x^2}{4\alpha(\mu+2\lambda)x^2}\right| \end{cases} . \tag{42}$$

Sitting $q \to 1^-$ and $\alpha = 1$ in Theorem 1, we obtain the following consequence.

Corollary 3.3 *[8] Let the function $f\ (z)$ given by (1) belong to class $\beta_\Sigma^\mu(\lambda, x)$. Then*

$$\left|b_1 - \eta b_0^2\right| \leq \begin{cases} \frac{2x}{(\mu+2\lambda)}, & |\eta - 1| \leq \left|\frac{(\mu+\lambda)^2 - 2[2(\mu+\lambda)^2 - (\mu+2\lambda)(1+\mu)]x^2}{4(\mu+2\lambda)x^2}\right| \\ \frac{8x^3|1-\eta|}{|(\mu+\lambda)^2 - 2[2(\mu+\lambda)^2 - (\mu+2\lambda)(1+\mu)]x^2|}, & |\eta - 1| \geq \left|\frac{(\mu+\lambda)^2 - 2[2(\mu+\lambda)^2 - (\mu+2\lambda)(1+\mu)]x^2}{4(\mu+2\lambda)x^2}\right| \end{cases} . \tag{43}$$

Sitting $q \to 1^-$ and $\mu = \alpha = 1$ in Theorem 1, we obtain the following consequence.

Corollary 3.4 *[7] Let the function $f\ (z)$ given by (1) belong to class $\beta_\Sigma(\lambda, x)$. Then*

$$\left|b_1 - \eta b_0^2\right| \leq \begin{cases} \frac{2x}{(1+2\lambda)}, & |\eta - 1| \leq \left|\frac{(1+\lambda)^2 - 4\lambda^2 x^2}{4(\mu+2\lambda)x^2}\right| \\ \frac{8x^3|1-\eta|}{|(1+\lambda)^2 - 4\lambda^2 x^2|}, & |\eta - 1| \geq \left|\frac{(1+\lambda)^2 - 4\lambda^2 x^2}{4(1+2\lambda)x^2}\right| \end{cases} . \tag{44}$$

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