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Existence and transportation inequalities for fractional stochastic differential equations

Abdelghani OUAHAB[1,*], **Mustapha BELABBAS**[2], **Johnny HENDERSON**[3], **Fethi SOUNA**[4]

[1]Laboratory of Mathematics, University of Djillali Liabes, Sidi Bel-Abbès, Algeria
[2]Laboratory of Statistics & Stochastic Processes, University of Djillali Liabs, Sidi Bel-Abbès, Algeria
[3]Department of Mathematics, Baylor University Waco, Texas, USA
[4]Laboratory of Biomathematics, University of Djillali Liabes, Sidi Bel Abès, Algeria

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Abstract: In this work, we establish the existence and uniqueness of solutions for a fractional stochastic differential equation driven by countably many Brownian motions on bounded and unbounded intervals. Also, we study the continuous dependence of solutions on initial data. Finally, we establish the transportation quadratic cost inequality for some classes of fractional stochastic equations and continuous dependence of solutions with respect Wasserstein distance.

Key words: Fractional differential equations, fractional integral, fractional derivative, Mittag–Leffler functions, fixed point, stochastic equation, transportation inequality, Wasserstein distance, entropy

1. Introduction

The theory of stochastic differential equations has become an active area of investigation due to their applications in the fields such as chemistry, mechanics, electrical engineering, medical biology, economical systems, finance and several fields in engineering, etc. One can find detailed information in [19, 22–24, 29, 31, 33, 34] and references therein.

In recent years, the subject of differential and integral equations via different types of fractional derivatives has received much attention because of its applications in various areas of sciences. For more information on applications we refer the reader to [1, 14, 25, 28, 36, 42] and references therein.

The existence and uniqueness of solutions for some classes of stochastic differential equations with fractional order derivative by employing the fixed point theory have been discussed in [2, 13, 15, 17, 18, 35, 40] and the references therein.

Because the modeling of a great many problems in real situations is described by stochastic differential equations, rather than deterministic equations, it is of great importance to study fractional differential equations with stochastic effects. Consider the following stochastic fractional differential equations:

$$\begin{cases} {}^cD^\alpha X_t &= \displaystyle\sum_{l=1}^{\infty} f_l(t, X_t)dW_t^l + g(t, X_t)dt, \quad t \in [0, \infty), \\ X_0 &= x \in \mathbb{R}, \end{cases} \tag{1.1}$$

*Correspondence: agh_ouahab@yahoo.tr

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where $\alpha \in (\frac{1}{2}, 1)$, $(f_l)_{l\in\mathbb{N}}, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are given functions and $(W_t^i)_{i\in\mathbb{N}}$ is an infinite sequence of independent standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). An $\mathbb{R}-$valued random variable is an $\mathcal{F}_t-$measurable function $X_t : \Omega \to \mathbb{R}$ and the collection of random variables,

$$S = \{X(t, \omega) : \Omega \to \mathbb{R} |\ t \in [0, \infty)\},$$

is called a stochastic process. Generally, we write X_t instead of $X(t, \omega)$.

Set

$$\begin{cases} f(\cdot, x) = (f_1(\cdot, x), f_2(\cdot, x), \ldots), \\ \|f(\cdot, x)\| = \Big(\sum_{l=1}^{\infty} f_i^2(\cdot, x)\Big)^{\frac{1}{2}} \end{cases} \tag{1.2}$$

where $f(\cdot, x) \in \ell^2$ for all $x \in \mathbb{R}$ and

$$\ell^2 = \left\{\varphi = (\varphi_l)_{l\geq 1} : \mathbb{R}_+ \to \mathbb{R} \quad : \|\varphi(t)\|^2 = \sum_{l=1}^{\infty} |\varphi_l(t)|^2 < \infty\right\}.$$

Some existence results of solutions for differential equations and inclusions with infinite Brownian or fractional Brownian motion were obtained in [5, 6, 12, 20, 30].

The remainder of this work is organized as follows. Some auxiliary results from stochastic analysis and fractional calculus are gathered together in Section 2. In Section 3, we present results on the existence and continuous dependence of solutions on initial data. We end the paper with a transportation inequality of some classes of fractional stochastic differential equations.

2. Introduction

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Because the modeling of a great many problems in real situations is described by stochastic differential equations, rather than deterministic equations, it is of great importance to study fractional differential equations with stochastic effects. Consider the following stochastic fractional differential equations:

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where $\alpha \in (\frac{1}{2}, 1)$, $(f_l)_{l\in\mathbb{N}}, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are given functions and $(W_t^i)_{i\in\mathbb{N}}$ is an infinite sequence of independent standard Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). An $\mathbb{R}-$valued random variable is an $\mathcal{F}_t-$measurable function $X_t : \Omega \to \mathbb{R}$ and the collection of random variables,

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3. Preliminaries

For each $t \in \mathbb{R}_+$, let $\mathbb{L}^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denote the space of all $\mathcal{F}_t-$measurable, mean square integrable functions $X : \Omega \to \mathbb{R}$, i.e.

$$\mathbb{E}\|X\|^2 < \infty, \quad \text{for all} \quad X \in \mathbb{L}^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}).$$

We shall write $\mathbb{L}^2(\mathcal{F}_t)$ instead of $\mathbb{L}^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Definition 3.1 *A $(\mathcal{F}_t)_{t\geq 0}-$adapted process $X : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ is called a solution of equation (2.1) with initial condition $X_0 = x \in \mathbb{R}$ if, for all $t \geq 0$, the following integral stochastic equation holds,*

$$X_t = x + \frac{1}{\Gamma(\alpha)} \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1} f_l(s, X_s) dW_s^l + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, X_s) ds,$$

where $\Gamma(\alpha) := \int_0^\infty s^{\alpha-1} e^{-s} ds$ is the Gamma function.

Let $T > 0$. $\mathbb{H}_2$ stands for the Banach space of adapted processes X, equipped with the norm $\|\cdot\|_{\mathbb{H}_2}$ such that

$$\|X\|_{\mathbb{H}_2} = \sup_{t\in[0,T]} (E\|X_t\|^2)^{1/2} < \infty.$$

We define for all $\gamma > 0$ the weighted norm $\|\cdot\|_\gamma$ by

$$\|X\|_\gamma := \sup_{t\in[0,T]} \sqrt{\frac{\mathbb{E}\|X_t\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}} \quad for \quad all \quad X \in \mathbb{H}_2,$$

where $E_{2\alpha-1}(.)$ is the Mittag–Leffler function such that

$$E_{2\alpha-1}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((2\alpha-1)k+1)} \quad for \quad all \quad t \in \mathbb{R}.$$

For more details about the Mittag–Leffler functions, see [14]. Obviously, $(\mathbb{H}_2, \|\cdot\|_\gamma)$ is a Banach space, since the norms $\|\cdot\|_{\mathbb{H}_2}$ and $\|\cdot\|_\gamma$ are equivalent.

Lemma 3.2 *For all $\alpha \in (0,1]$ and $\gamma > 0$, the following inequality holds:*

$$\frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(\gamma s^\alpha) ds \le E_\alpha(\gamma t^\alpha).$$

Proof Let $0 < \alpha \le 1$. We consider first the linear problem

$$^cD^\alpha y(t) = \gamma y(t), \quad t \in \mathbb{R}_+. \tag{3.1}$$

From [14, Theorem 7.2 and Remark 7.1], the function $y(t) = E(\gamma t^\alpha)$ is a solution of (3.1), and for any $t \in \mathbb{R}_+$, we have

$$E(\gamma t^\alpha) = 1 + \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(\gamma s^\alpha) ds.$$

This concludes the proof of the lemma. □

We recall Gronwall's lemma for singular kernels, whose proof can be found in [41, Lemma 7.1.1].

Lemma 3.3 *Let $v : [0,b) \to [0,\infty)$ be a real function and $w(\cdot)$ be a nonnegative, locally integrable function on $[0,b)$,(some $b \le +\infty$)) and $a(t)$ be a nonnegative, nondecreasing continuous function defined on $0 \le t < b$, with $a(t) \le M$ (constant), and suppose $v(t)$ is nonnegative and locally integrable on $0 \le t < b$. Assume $\gamma > 0$ such that*

$$v(t) \le w(t) + a(t) \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds.$$

Then

$$v(t) \le w(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(a(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma-1} w(s) ds,$$

for every $t \in [0,b)$.

In the following, we state standing hypotheses holding for the coefficients f and g in our model of this paper.

(H_1) There exists $K > 0$ such that for all $x, y \in \mathbb{R}$ and $t \in [0,\infty)$

$$\|f(t,x) - f(t,y)\| + |g(t,x) - g(t,y)| \le K|x-y|, \quad x, y \in \mathbb{R}.$$

(H_2) $\|f(\cdot,0)\|_\infty := ess\sup_{s\in[0,\infty)} \|f(s,0)\| < \infty$ *and* $\int_0^\infty |g(s,0)|^2 ds < \infty$.

4. Main results

The following result is one of the elementary properties of square-integrable stochastic processes [29].

Lemma 4.1 *(Itô Isometry for Elementary Processes) Let $(\mathcal{F}_t)_{t\geq 0}$ satisfy the usual conditions and be generated by $(W_t^i)_{i\in\mathbb{N}^\star}$. Given two sequences of measurable $(\mathcal{F}_t)_{t\geq 0}-$ adapted processes X_i and Y_i, set*

$$\begin{cases} M_t = \displaystyle\sum_{i=1}^{\infty}\int_0^t (t-s)^{\alpha-1}X_i(s)dW_s^i, \\ N_t = \displaystyle\sum_{i=1}^{\infty}\int_0^t (t-s)^{\alpha-1}Y_i(s)dW_s^i. \end{cases}$$

If $\sum_{i=1}^{\infty}\|X_i\|_{\mathbb{H}_2}^2 < \infty$, then M is a continuous $\mathbb{L}^2(\mathcal{F}_t)-$ martingale. The quadratic variation of M denoted by $[M]_t$ is

$$[M]_t = \int_0^t (t-s)^{2\alpha-2}|X(s)|^2 ds, \quad for \quad all \quad t\geq 0,$$

where $|X(s)|^2 = \sum_{i=1}^{\infty} X_i^2(s)$. And the cross variation of M and N, denoted by $[M,N]_t$, is

$$[M,N]_t = \sum_{i=1}^{\infty}\int_0^t (t-s)^{2\alpha-2}X_i(s)Y_i(s)ds, \quad for \quad all \quad t\geq 0.$$

Proof Let $n\geq 1$, we put

$$M_t^n = \sum_{i=1}^{n}\int_0^t (t-s)^{\alpha-1}X_i(s)dW_s^i.$$

For all $t\geq 0$, we have

$$\begin{aligned} \mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds &= \int_0^t (t-s)^{2\alpha-2}\sum_{i=1}^{n}\mathbb{E}\|X_i(s)\|^2 ds \\ &= \int_0^t J(s)\sum_{i=1}^{n}\left(\sqrt{\frac{\mathbb{E}\|X_i(s)\|^2}{E_{2\alpha-1}(\gamma s^{2\alpha-1})}}\right)^2 ds, \end{aligned}$$

where

$$J(s) = (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1}).$$

Then, by the definition of $\|\cdot\|_\gamma$, we have

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds \leq \sum_{i=1}^{n}\|X_i\|_\gamma^2\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})ds.$$

Then Lemma 2.1 implies that

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds \leq \frac{\Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1})}{\gamma}\sum_{i=1}^{n}\|X_i\|_\gamma^2.$$

Choose and fix a positive constant γ such that

$$\gamma = \Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1}).$$

Then

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds \leq \sum_{i=1}^{n}\|X_i\|_\gamma^2.$$

Since $\sum_{i=1}^{n}\|X_i\|_\gamma^2 < \infty$ for all $n \geq 1$, then

$$\mathbb{E}\sum_{i=1}^{n}\int_0^t (t-s)^{2\alpha-2}\|X_i(s)\|^2 ds < \infty.$$

Consequently, M^n is a continuous $\mathbb{L}^2(\mathcal{F}_t)-$martingale for all $n \geq 1$, and its quadratic variation is $[M^n]_t$ such that

$$[M^n]_t = \int_0^t \sum_{i=1}^{n}(t-s)^{2\alpha-2}X_i^2(s)ds.$$

By the Burkholder–Davis–Gundy inequality [10, 11], we have, for some positive real constant C,

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \leq C\int_0^t (t-s)^{2\alpha-2}\sum_{i=n\wedge m+1}^{n\vee m}\mathbb{E}\|X_i(s)\|^2 ds.$$

By the definition of $\|\cdot\|_\gamma$ and Lemma 4.1, we obtain that

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \leq \frac{C\Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1})}{\gamma}\sum_{i=n\wedge m+1}^{n\vee m}\|X_i\|_\gamma^2.$$

Choose and fix a positive constant γ such that

$$\gamma = C\Gamma(2\alpha-1)E_{2\alpha-1}(\gamma T^{2\alpha-1}).$$

Then

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \leq \sum_{i=n\wedge m+1}^{n\vee m}\|X_i\|_\gamma^2.$$

Since $\sum_{i=1}^{\infty}\|X_i\|_\gamma^2 < \infty$, we have

$$\mathbb{E}\sup_{s\in[0,t]}(M_s^n - M_s^m)^2 \leq \sum_{i=n\wedge m+1}^{n\vee m}\|X_i\|_\gamma^2 \longrightarrow 0 \quad \text{as} \quad n,m \longrightarrow \infty,$$

where

$$n \wedge m = \min(n,m), \quad n \vee m = \max(n,m).$$

So M^n is a Cauchy sequence with respect to the norm $(\mathbb{E}\sup_{t\in[0,T]}(\cdot)^2)^{\frac{1}{2}}$ for any bounded time interval $[0,T]$. Denote its limit by M. Consequently, by the continuity of M^n, we obtain

$$\lim_{n\longrightarrow\infty} \mathbb{E}(\mathbb{E}(M_t^n/\mathcal{F}_s) - \mathbb{E}(M_t/\mathcal{F}_s))^2 = 0, \quad for \quad all \quad s < t,$$

and

$$\lim_{n\longrightarrow\infty} \mathbb{E}(M_s^n - M_s)^2 = 0.$$

Since $\mathbb{E}(M_t^n/\mathcal{F}_s) = M_s^n$ for all $s < t$ and $n \geq 1$, by the two previous limits, we have

$$\mathbb{E}(M_t/\mathcal{F}_s) = M_s, \quad for \quad all \quad s < t.$$

Hence, M is a continuous $\mathbb{L}^2(\mathcal{F}_t)-$martingale. Moreover, $[M^n]_t$ converges to $[M]_t$ as $n \longrightarrow \infty$ in probability, for all $t \geq 0$, i.e.

$$[M]_t = \int_0^t (t-s)^{2\alpha-2}|X(s)|^2 ds, \quad where \quad |X(s)|^2 = \sum_{i=1}^{\infty} X_i^2(s).$$

Similarly, the cross variation of M and N for all $t \geq 0$ is

$$[M,N]_t = \sum_{i=1}^{\infty} \int_0^t (t-s)^{2\alpha-2} X_i(s)Y_i(s)ds.$$

□

Now, we define the operator L on $\mathbb{H}_2$ by

$$(LX)(t) = x + \frac{1}{\Gamma(\alpha)} \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1} f_l(s,X_s)dW_s^l + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,X_s)ds.$$

Lemma 4.2 *The operator L is well-defined on $\mathbb{H}_2([0,T])$.*

Proof Let $X \in \mathbb{H}_2$, then for all $t \in [0,T]$, we get

$$\begin{aligned} \mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\textstyle\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1} f_l(s,X_s)dW_s^l\right\|^2 \\ &\quad + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\textstyle\int_0^t (t-s)^{\alpha-1} g(s,X_s)ds\right\|^2. \end{aligned}$$

By Lemma 4.1 and Hölder's inequality, we obtain

$$\begin{aligned} \mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t (t-s)^{2\alpha-2}|f(s,X_s)|^2 ds \\ &\quad + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\|\textstyle\int_0^t (t-s)^{\alpha-1} g(s,X_s)ds\right\|^2 \\ &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t (t-s)^{2\alpha-2}|f(s,X_s)|^2 ds \\ &\quad + \tfrac{3t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\textstyle\int_0^t \|g(s,X_s)\|^2 ds. \end{aligned}$$

From (H_1) and (H_2), we derive

$$\begin{aligned}\mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t (t-s)^{2\alpha-2}\times \\ &\quad 2(|f(s,X_s)-f(s,0)|^2+|f(s,0)|^2)ds \\ &\quad + \tfrac{3t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\int_0^t 2(\|g(s,X_s)-g(s,0)\|^2+\|g(s,0)\|^2)ds \\ &\leq 3\mathbb{E}\|x\|^2 + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\int_0^t (t-s)^{2\alpha-2}2(K^2\|X_s\|^2+|f(.,0)|_\infty^2)ds \\ &\quad + \tfrac{3t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}\int_0^t 2(K^2\|X_s\|^2+\|g(s,0)\|^2)ds.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\|(LX)(t)\|^2 &\leq 3\mathbb{E}\|x\|^2 + \tfrac{6t^{2\alpha-1}|f(.,0)|_\infty^2}{(2\alpha-1)\Gamma^2(\alpha)} + \tfrac{6t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \|g(s,0)\|^2ds \\ &\quad + \tfrac{6K^2t^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)}\|X\|_{\mathbb{H}_2}^2 + \tfrac{6K^2t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\|X\|_{\mathbb{H}_2}^2.\end{aligned}$$

Moreover,

$$\begin{aligned}\sup_{t\in[0,T]}\left(\mathbb{E}\|(LX)(t)\|^2\right)^{\frac{1}{2}} &\leq 3\mathbb{E}\|x\|^2 + \frac{6T^{2\alpha-1}|f(.,0)|_\infty^2}{(2\alpha-1)\Gamma^2(\alpha)} \\ &\quad + \frac{6T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^T \|g(s,0)\|^2ds \\ &\quad + \left(\frac{6K^2T^{2\alpha}}{(2\alpha-1)\Gamma^2(\alpha)} + \frac{6K^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\right)\|X\|_{\mathbb{H}_2}^2.\end{aligned}$$

Therefore, $\|LX\|_{\mathbb{H}_2} < \infty$. Hence, the map L is well-defined. □

Theorem 4.3 *Assume that* (H_1) *and* (H_2) *hold. Then problem (2.1) has a unique global solution on* $[0,\infty)$.

Proof We show that, for every $T>0$, the operator L is a contractive map with respect to some Bielecki-type norm on $\mathbb{H}_2$ which will be defined later. Let $X, Y \in \mathbb{H}_2$ and $t\in[0,T]$. Then

$$\begin{aligned}\mathbb{E}\|(LX)(t)-(LY)(t)\|^2 &= \mathbb{E}\left\|\frac{1}{\Gamma(\alpha)}\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}(f_l(s,X_s)-f_l(s,Y_s))dW_s^l\right. \\ &\quad \left.+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right\|^2 \\ &\leq \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}(f_l(s,X_s)-f_l(s,Y_s))dW_s^l\right\|^2 \\ &\quad +\frac{2}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right\|^2.\end{aligned}$$

By Lemma 4.1 and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|(LX)(t)-(LY)(t)\|^2 &\leq \tfrac{2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\sum_{l=1}^{\infty}(f_l(s,X_s)-f_l(s,Y_s))^2ds \\ &\quad +\tfrac{2t}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,X_s)-g(s,Y_s)\|^2ds \\ &\leq \tfrac{2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}|f(s,X_s)-f(s,Y_s)|^2ds \\ &\quad +\tfrac{2t}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|g(s,X_s)-g(s,Y_s)\|^2ds.\end{aligned}$$

From (H_1), we derive

$$\begin{aligned}\mathbb{E}\|(LX)(t)-(LY)(t)\|^2 &\leq \tfrac{2K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s-Y_s\|^2 ds\\ &\quad +\tfrac{2tK^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s-Y_s\|^2 ds\\ &= \tfrac{2K^2}{\Gamma^2(\alpha)}(t+1)\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s-Y_s\|^2 ds.\end{aligned}$$

Moreover,

$$\begin{aligned}\tfrac{\mathbb{E}\|(LX)(t)-(LY)(t)\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} &\leq \tfrac{2(t+1)K^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})\times\\ &\quad \left(\sup_{s\in[0,T]}\sqrt{\frac{\mathbb{E}\|X_s-Y_s\|^2}{E_{2\alpha-1}(\gamma s^{2\alpha-1})}}\right)^2 ds.\end{aligned}$$

If we choose $\|\cdot\| = \|\cdot\|_\gamma$ for the Bielecki-type norm on $\mathbb{H}_2$, then the definition of $\|\cdot\|_\gamma$ and the Lemma 4.1 imply that

$$\begin{aligned}\tfrac{\mathbb{E}\|(LX)(t)-(LY)(t)\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} &\leq \tfrac{2(t+1)K^2}{\Gamma^2(\alpha)}\|X-Y\|_\gamma^2\left(\tfrac{\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})ds}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\right)\\ &\leq \tfrac{2(t+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}\|X-Y\|_\gamma^2.\end{aligned}$$

In particular,

$$\|L(X)-L(Y)\|_\gamma \leq \lambda\|X-Y\|_\gamma, \quad where \quad \lambda = \sqrt{\frac{2(T+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}}.$$

Choose and fix a positive constant γ such that

$$\gamma > \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}.$$

Then $\lambda < 1$, and therefore, L is a contraction mapping. According to the Banach fixed point theorem, the unique fixed point of this map is the unique solution on $\mathbb{H}_2$ of problem (2.1). □

We are now in the position to prove the continuous dependence of solutions on the initial data on bounded intervals for the problem (2.1).

Theorem 4.4 *Assume that (H_1) holds. Then for any bounded time interval $[0,T]$ the solution of problem (2.1) depends continuously on x, i.e.*

$$\lim_{x\longrightarrow\eta}\|X^x - X^\eta\|_{\mathbb{H}_2} = 0.$$

Proof Fix $T>0$ and $x,\eta\in\mathbb{R}$. Let X_t^x and X_t^η be two solutions of problem (2.1), i.e.

$$X_t^x = x + \frac{1}{\Gamma(\alpha)}\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}f_l(s,X_s^x)dW_s^l + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,X_s^x)ds,$$

$$X_t^\eta = \eta + \frac{1}{\Gamma(\alpha)}\sum_{l=1}^{\infty}\int_0^t (t-s)^{\alpha-1}f_l(s,X_s^\eta)dW_s^l + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,X_s^\eta)ds.$$

It follows that

$$\begin{aligned}\mathbb{E}\|X_t^x - X_t^\eta\|^2 &= \mathbb{E}\left\| x - \eta + \tfrac{1}{\Gamma(\alpha)} \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1}(f_l(s,X_s^x) - f_l(s,X_s^\eta))dW_s^l \right. \\ &\quad \left. + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(g(s,X_s^x) - g(s,X_s^\eta))ds \right\|^2.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\|X_t^x - X_t^\eta\|^2 &\leq 3\mathbb{E}\|x-\eta\|^2 \\ &\quad + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\| \sum_{l=1}^{\infty} \int_0^t (t-s)^{\alpha-1}(f_l(s,X_s^x) - f_l(s,X_s^\eta))dW_s^l \right\|^2 \\ &\quad + \tfrac{3}{\Gamma^2(\alpha)}\mathbb{E}\left\| \int_0^t (t-s)^{\alpha-1}(g(s,X_s^x) - g(s,X_s^\eta))ds \right\|^2.\end{aligned}$$

By Lemma 4.1 and Hölder's inequality, from (H_1), we get

$$\mathbb{E}\|X_t^x - X_t^\eta\|^2 \leq 3\mathbb{E}\|x-\eta\|^2 + \frac{3K^2}{\Gamma^2(\alpha)}(t+1)\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|X_s^x - X_s^\eta\|^2 ds.$$

By the definition of $\|\cdot\|_\gamma$, we have

$$\frac{\mathbb{E}\|X_t^x - X_t^\eta\|^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \leq 3\mathbb{E}\|x-\eta\|^2 + \frac{3(t+1)K^2}{\Gamma^2(\alpha)}\|X^x - X^\eta\|_\gamma^2 \left(\frac{\int_0^t (t-s)^{2\alpha-2}E_{2\alpha-1}(\gamma s^{2\alpha-1})ds}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\right).$$

Therefore, using Lemma 4.1, we obtain

$$\|X^x - X^\eta\|_\gamma^2 \leq 3\mathbb{E}\|x-\eta\|^2 + \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}\|X^x - X^\eta\|_\gamma^2.$$

Since $\gamma > \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}$, we have

$$\left(1 - \frac{3(T+1)K^2\Gamma(2\alpha-1)}{\gamma\Gamma^2(\alpha)}\right)\|X^x - X^\eta\|_\gamma^2 \leq 3\mathbb{E}\|x-\eta\|^2.$$

We conclude

$$\lim_{x\longrightarrow\eta}\|X^x - X^\eta\|_{\mathbb{H}_2} = 0.$$

The proof is complete. □

5. Talagrand transportation inequalities

Let (X,d) be a metric space and $\mathcal{A}$ be the Borel sets on X. We denote by $\mathcal{P}_p(X)$ the space of probability measure. For every $p \in [1,\infty)$ we define the Wasserstein distance $W_p : \mathcal{P}_p(X)\times\mathcal{P}_p(X) \to \mathbb{R}_+$ by

$$W_p(\mu,\nu) = \inf_{\pi\in\prod(\mu,\nu)}\left(\int_{X\times X} d(x,y)^p d\pi(x,y)\right)^{\frac{1}{p}}$$

, where $\prod(\mu,\nu)$ is the set of all probability measures on the product space $X\times X$ with marginals μ and ν.

Definition 5.1 *Let μ and ν two measures on $(X, \mathcal{A})$. A measure ν is called absolutely continuous with respect μ, if we have*

$$\forall A \in \mathcal{A}, \mu(A) = 0 \Longrightarrow \nu(A) = 0.$$

We denote this by $\nu \ll \mu$.

For investigation of the Monge–Kantorovich optimal transportation problem, this distance plays an important role in the minimal cost to transport distribution μ into ν at the cost rate (cost function) d. Proprieties and some applications of the Wasserstein distance can be found in the important contribution by Ambrosio et al. [3] and Villani [39].

The relative entropy of the probability measure ν with respect to μ is defined by

$$H(\mu|\nu) = \begin{cases} \int_X \ln \frac{d\nu}{d\mu} d\nu, & \nu \ll \mu, \\ \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 5.2 *Given probability measure μ, if there exists $C > 0$ such that for every probability measure ν,*

$$W_2(\mu, \nu) \leq \sqrt{CH(\mu|\nu)},$$

then we say μ satisfies the transportation and entropy inequality.

In this section we study the transportation inequality of the the following problem:

$$\begin{cases} {}^cD^\alpha X_t &= f(t, X_t)dW_t + g(t, X_t)dt, \quad t \in [0, T], \\ \\ X_0 &= x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where $\alpha \in \left(\frac{1}{2}, 1\right)$, $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In 1996, Talagrand [38] estimated the transportation distance (or Wasserstein distance) with a quadratic cost of the standard Guassian measure by the entropy functional. Transportation-cost inequalities have been recently deeply studied, because of their connection between the concentrations of measure phenomenon, or for deviation inequalities for Markov processes [21, 26]. The Talagrand inequality was generalized by Otto and Villani [32].

By means of Girsanov's formula, Djellout et al. [16] obtained a direct proof of Talagrand's transportation inequality for the law of a diffusion process. This idea was used for stochastic differential equations [4, 7–9, 27].

5.1. Transportation inequality

Now, we will establish the transport inequality for the solution of (5.1).

Theorem 5.3 *Assume that the conditions (H_1) and that there exists $M > 0$ such that*

$$|f(t, x)| \leq M, \quad \textit{for all } (t, x) \in [0, T] \times \mathbb{R},$$

hold, and let $\mathbb{P}_x$ be a law of the proesses $X_t(x, \cdot)$ solution of the problem (5.1). Then

$$W_2(\mathbb{P}_x, \mathbb{Q}) \leq \sqrt{2CH(\mathbb{P}|\mathbb{Q})}.$$

Proof Let $\mathbb{Q} \in \mathcal{P}(C([0,T],\mathbb{R}))$ such that $\mathbb{Q} \ll \mathbb{P}_x$. Consider

$$\widehat{\mathbb{Q}} = \frac{d\mathbb{Q}}{d\mathbb{P}_x}(X.(x,\cdot))\mathbb{P}.$$

Then

$$\begin{aligned} H(\widehat{\mathbb{Q}}|\mathbb{P}) &= \int_\Omega \ln\left(\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right) d\widehat{\mathbb{Q}} \\ &= \int_\Omega \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X.(x,\cdot))\right)\frac{d\mathbb{Q}}{d\mathbb{P}_x}(X.(x,\cdot))d\mathbb{P} \\ &= \int_{C([0,T],\mathbb{R})} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}\right)\frac{d\mathbb{Q}}{d\mathbb{P}_x}d\mathbb{P}_x \\ &= \int_{C([0,T],\mathbb{R})} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_x}\right)d\mathbb{Q} \\ &= H(\mathbb{Q}|\mathbb{P}_x). \end{aligned}$$

As in [16], there exists $\widehat{f} \in L^2([0,T],\mathbb{R})$ with $\int_0^T |\widehat{f}(s)|^2 ds < \infty$, $\mathbb{P}$-almost surely, such that

$$H(\widehat{\mathbb{Q}}|\mathbb{P}) = H(\mathbb{Q}|\mathbb{P}_x) = \frac{1}{2}\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\int_0^T |\widehat{f}(s)|^2 ds\right).$$

By the Girsanov theorem, the following process $\widehat{W_t}$ defined by

$$\widehat{W_t} = W_t - \int_0^t \widehat{f}(s)ds$$

is a Brownian motion with respect the filtration $(\mathcal{F}_t)_{t\geq 0}$ on the probability space $(\Omega,\mathcal{F},\widehat{\mathbb{Q}})$. We consider the following problem for the fractional stochastic differential equation

$$\begin{cases} {}^cD^\alpha Y_t &= f(t,Y_t)d\widehat{W_t} + g(t,Y_t)dt, \quad t\in[0,T], \\ Y_0 &= x \in \mathbb{R}. \end{cases} \tag{5.2}$$

From Theorem 4.3, there exists a unique solution $Y \in \mathbb{H}_2([0,T])$ such that

$$Y_t = x + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,Y_s)d\widehat{W}_s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,Y_s)ds.$$

Under $\widehat{\mathbb{Q}}$, the law of $(Y_t)_{t\in[0,T]}$ is exactly $\mathbb{P}_x$. Hence (X,Y), under $\widehat{\mathbb{Q}}$, is a coupling of $(\mathbb{Q},\mathbb{P}_x)$. This implies that

$$W_2(\mathbb{Q},\mathbb{P}_x)^2 \leq \mathbb{E}_{\widehat{\mathbb{Q}}}\|X-Y\|_\infty^2.$$

Let $t \in [0,T]$. Then

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 &= \mathbb{E}_{\widehat{\mathbb{Q}}}\bigg|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s,X_s)-f(s,Y_s))dW_s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\bigg|^2 \\
&\leq \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1}(f(s,X_s)-f(s,Y_s))dW_s\right|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds\right|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 &\leq \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2 ds \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds\right|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1}(g(s,X_s)-g(s,Y_s))ds\right|^2.
\end{aligned}$$

It follows from Hölder's inequality,

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 &\leq \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2 ds \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\left|\int_0^t (t-s)^{\alpha-1} f(s,Y_s)\widehat{f}(s)ds\right|^2 \\
&\quad + \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}ds\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2 ds \\
&\leq \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2 ds \\
&\quad + \frac{3K^2M^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}ds\,\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2} \\
&\quad + \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}ds\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s-Y_s|^2 ds.
\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \quad &\leq \quad \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds \\ &+\frac{3K^2M^2}{(2\alpha-1)\Gamma^2(\alpha)}t^{2\alpha-1}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2} \\ &+\frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}ds\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds.\end{aligned}$$

Further, we have

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \quad &\leq \quad \frac{3K^2M^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2} + \frac{3K^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds \\ &+\frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds.\end{aligned}$$

Let

$$\mathcal{V}(t) = C_1 + C_2\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds, \quad t \in [0,b].$$

Then

$$\mathcal{V}'(t) = C_2\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2, \quad \mathcal{V}(0) = C_1$$

and

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \leq \mathcal{V}(t) + C_3\int_0^t (t-s)^{\beta-1}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds,$$

where

$$C_1 = \frac{3K^2M^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2},\ C_2 = \frac{3K^2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)},\ C_3 = \frac{3K^2}{\Gamma^2(\alpha)},\ \beta = 2\alpha - 1.$$

Furthermore, by Lemma 3.3

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \quad &\leq \quad \mathcal{V}(t) + \int_0^t \sum_{n=1}^{\infty}\frac{(C_3\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}\mathcal{V}(s)ds \\ &\leq \quad \mathcal{V}(t) + \int_0^t \sum_{n=1}^{\infty}\frac{(C_3\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}\mathcal{V}(t)ds.\end{aligned}$$

Therefore

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \leq \left[1 + \sum_{n=1}^{\infty}\frac{(C_3\Gamma(\alpha)T^{\beta})^n}{\Gamma(n\beta+1)}\right]\mathcal{V}(t).$$

Then

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \leq C_1E(C_3\Gamma(\beta)T^{\alpha}) + C_2E(C_3\Gamma(\beta)T^{\alpha})\int_0^t \mathbb{E}_{\widehat{\mathbb{Q}}}|X_s - Y_s|^2 ds.$$

By Gronwall's Lemma

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \leq C_1 E(C_3\Gamma(\beta)T^\alpha)e^{tC_2E(C_3\Gamma(\beta)T^\alpha)} \quad t \in [0,T].$$

This means that

$$\mathbb{E}_{\widehat{\mathbb{Q}}}|X_t - Y_t|^2 \leq \frac{C}{2}\mathbb{E}_{\widehat{\mathbb{Q}}}\|\widehat{f}\|^2_{L^2} \quad t \in [0,T],$$

where

$$C = \frac{6K^2M^2T^{2\alpha-1}e^{TC_2E(C_3\Gamma(\beta)T^\alpha)}}{(2\alpha-1)\Gamma^2(\alpha)}E(C_3\Gamma(\beta)T^\alpha).$$

Thus it follows,

$$W_2^2(\mathbb{P}_x, \mathbb{Q}) \leq CH(\mathbb{P}_x|\mathbb{Q}).$$

The proof of this lemma is complete. □

Now, we give the continuity dependance result via the Wasserstein distance.

Theorem 5.4 *Assume that the condition (H_1) holds. Then, for every pair of solutions X_t, Y_t to (5.1), with respective laws $\mu_t, \nu_t \in P_2(\mathbb{H}_2([0,T]))$, such that the initial data $X_0, Y_0 \in L^2(\Omega, \mathbb{P})$, we have*

$$W_2(\mu_t, \nu_t) \leq C(t)W_2(\mu_0, \nu_0),$$

where μ_0, ν_0 are laws of X_0, Y_0, respectively, and $C \in C([0,T], \mathbb{R})$.

Proof From [39], it is clear that we can rewrite W_2 in the following form,

$$W_2(\mu_t, \nu_t) = \inf\{\left[E\|X_\cdot - Y_\cdot\|^2_\infty\right]^2 : \quad \text{law}(X_t) = \mu_t, \ \text{law}(Y_t) = \nu_t\}.$$

Since X_t, Y_t are solutions of (5.1), then

$$X_t = X_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, X_s)dW_s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, X_s)ds$$

and

$$Y_t = Y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, Y_s)dW_s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, Y_s)ds.$$

Thus

$$\begin{aligned}
\mathbb{E}|X_t - Y_t|^2 &= \mathbb{E}\left|X_0 - Y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s,X_s) - f(s,Y_s))dW_s \right.\\
&\quad \left. + \int_0^t (t-s)^{\alpha-1}(g(s,X_s) - g(s,Y_s))ds\right|^2 \\
&\leq 3\mathbb{E}|X_0 - Y_0|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\left|\int_0^t (t-s)^{\alpha-1}(f(s,X_s) - f(s,Y_s))dW_s\right|^2 \\
&\quad + \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\left|\int_0^t (t-s)^{\alpha-1}(g(s,X_s) - g(s,Y_s))ds\right|^2 \\
&\leq 3\mathbb{E}|X_0 - Y_0|^2 + \frac{3K^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}\mathbb{E}|X_s - Y_s|^2 ds \\
&\quad + \frac{3K^2 t^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\int_0^t \mathbb{E}|X_s - Y_s|^2 ds.
\end{aligned}$$

By the same argument of Theorem 4.4, we can prove that there exist $M_1, M_2 \geq 0$ such that

$$\mathbb{E}|X_t - Y_t|^2 \leq M_1 e^{M_2 t}\mathbb{E}|X_0 - Y_0|^2, \quad t \in [0,T].$$

Since $\text{law}(X_t) = \mu_t$ and $\text{law}(X_t) = \mu_t$, then

$$W_2^2(\mu_t, \nu_t) \leq \mathbb{E}|X_t - Y_t|^2.$$

By taking the infimum over X_0 and Y_0, we obtain

$$W_2^2(\mu_t, \nu_t)^2 \leq M_1 e^{M_2 t} W_2^2(\mu_0, \nu_0), \quad t \in [0,T].$$

□

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