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# Oscillation criteria for third-order neutral differential equations with unbounded neutral coefficients and distributed deviating arguments

**Yibing SUN**[1,*], **Yige ZHAO**[1], **Qiangqiang XIE**[2]
[1]School of Mathematical Sciences, University of Jinan, Jinan, China
[2]Department of Natural Resources, Inspur Software Technology Co., Ltd., Jinan, China

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**Abstract:** This paper focuses on the oscillation criteria for the third-order neutral differential equations with unbounded neutral coefficients and distributed deviating arguments. Using comparison principles, new sufficient conditions improve some known existing results substantially due to less constraints on the considered equation. At last, two examples are established to illustrate the given theorems.

**Key words:** Oscillation, third-order, neutral, unbounded neutral coefficients, distributed deviating arguments

## 1. Introduction

Differential equations have played important roles in several practical applications, such as population growth, economics, engineering and neural networks. The study of oscillation criteria for first-, second-, third-, and high-order differential equations are significant for practical reasons and the development of qualitative theory; see [3, 9, 25, 37]. The problem of establishing sufficient conditions for the oscillatory and asymptotic behavior of third-order neutral functional differential equations and dynamic equations on time scales has attracted much research attention during the last few decades, and we refer the reader to the papers [1, 2, 4–8, 12, 14–16, 20, 22, 24, 26, 27, 32, 35, 36]. Chatzarakis et al. [14] considered the following differential equation

$$z'''(t) + q(t)x^{\lambda}(g(t)) = 0, \tag{1.1}$$

and explored some new oscillation criteria for solutions of (1.1), where $z(t) = x(t)+p(t)x(\eta(t))$, and $p(t) \geq 1$ is the unbounded neutral coefficient. The obtained results essentially improved the related results in the literature.

Recently, much research activity has been focused on the oscillation of various of classes of third-order neutral differential and dynamic equations with distributed deviating arguments; see for example [10, 11, 13, 17, 23]. In what follows, the following differential equation

$$\big(b(t)(y''(t))^{\alpha}\big)' + \int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi = 0, \tag{1.2}$$

and its special cases have been well studied, where $y(t) = x(t)+p(t)x(\tau(t))$ or $y(t) = x(t)+\int_a^b p(t,\xi)x(\tau(t,\xi))d\xi$. Sun and Zhao [31], Tian et al. [33], Xiang [38] and Zhang et al. [39] studied the oscillation of (1.2) under the

*Correspondence: sun_yibing@126.com
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assumption $0 \le p(t) \le P < 1$ or $0 \le \int_a^b p(t,\xi)d\xi \le P < 1$, while Jiang and Li [18] investigated the oscillatory and asymptotic behavior of (1.2) by assuming that $0 \le p(t) \le p_0 < \infty$. We can see that the neutral coefficient $p(t)$ in such papers are bounded. To the best of our knowledge, there are few results dealing with the oscillation of third-order differential equations with distributed deviating arguments and unbounded neutral coefficients. In this case, Sun and Zhao [28–30] and Tunç [34] gave some oscillation criteria for third-order differential equations with distributed deviating arguments and assumed $p(t) \ge 1$. On the other hand, some additional conditions are also needed in certain papers, such as Jiang et al. [17], Sun and Zhao [28] and Tian et al. [33] needed $\alpha \ge 1$, which is a ratio of odd positive integers.

In this paper, we will consider (1.2), where $y(t) = x(t) + m(t)x(\tau(t))$, $t \ge t_0 > 0$, $c < d$, and $\alpha$ is the ratio of odd positive integers. In this paper, the following conditions are assumed to hold:

(D1) $b(t)$, $m(t) \in C([t_0,\infty),\mathbb{R})$, $b(t) > 0$, $\int_{t_0}^{\infty} b^{-\frac{1}{\alpha}}(t)dt = \infty$, $m(t) \ge 1$ and $m(t) \not\equiv 1$;

(D2) $q(t,\xi) \in C([t_0,\infty) \times [c,d],[0,\infty))$ and $q(t,\xi) \not\equiv 0$ for large $t$;

(D3) $\tau(t) \in C([t_0,\infty),\mathbb{R}) \le t$ is a strictly increasing function, $\lim_{t\to\infty} \tau(t) = \infty$, and $\sigma(t,\xi) \in C([t_0,\infty) \times [c,d],\mathbb{R})$ is nonincreasing for $\xi$ satisfying $\liminf_{t\to\infty} \sigma(t,\xi) = \infty$ for $\xi \in [c,d]$;

(D4) $f(x) \in C(\mathbb{R},\mathbb{R})$ and there exists $K > 0$ such that $f(x)/x^{\alpha} \ge K$ for any variable $x \ne 0$.

A function $x(t)$ is a solution of (1.2), if it satisfies (1.2) on the interval $[T_x,\infty)$ and has the properties $y(t)$, $y'(t)$ and $b(t)(y''(t))^{\alpha} \in C^1([T_x,\infty),\mathbb{R})$, respectively. We consider only those solutions to (1.2) here, which satisfy the condition $\sup\{|x(t)| : t \ge T\} > 0$ for every $T \ge T_x$ and assume that (1.2) possesses such solutions. A solution of (1.2) is said to be oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise it is called nonoscillatory. Equation (1.2) is said to be oscillatory if all its solutions are oscillatory.

The objective of this paper is to obtain some new oscillation criteria for solutions of (1.2) by using comparison principles. Moreover, our proposed results improve a number of related results in the literature due to less restrictive assumptions on the coefficients of (1.2), and can easily be extended to more general third-order differential equations with unbounded neutral coefficients and distributed deviating arguments. In what follows, all functional inequalities are assumed to hold eventually.

## 2. Some lemmas

We begin this section with some lemmas, which are essential in establishing our main results. For simplicity, the following notations are used:

$$\bar{\sigma}(t) = \sigma(t,c),\ \tilde{\sigma}(t) = \sigma(t,d),$$

$$\delta_1(t,t_1) = \int_{t_1}^{t} b^{-\frac{1}{\alpha}}(s)ds \quad \text{for} \quad t \ge t_1 > t_0,$$

$$\delta_2(t,t_2) = \int_{t_2}^{t} \delta_1(s,t_1)ds \quad \text{for} \quad t \ge t_2 > t_1.$$

Furthermore, we assume that

$$m_1(t) = \frac{1}{m(\tau^{-1}(t))}\Big(1 - \frac{1}{m(\tau^{-1}(\tau^{-1}(t)))}\Big) \geq 0, \tag{2.1}$$

$$m_2(t) = \frac{1}{m(\tau^{-1}(t))}\Big(1 - \frac{\delta_2(\tau^{-1}(\tau^{-1}(t)), t_2)}{m(\tau^{-1}(\tau^{-1}(t)))\delta_2(\tau^{-1}(t), t_2)}\Big) \geq 0, \tag{2.2}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Let

$$Q_1(t) = K\int_c^d q(t,\xi)m_1^{\alpha}(\sigma(t,\xi))d\xi,$$

$$Q_2(t) = K\int_c^d q(t,\xi)m_2^{\alpha}(\sigma(t,\xi))d\xi.$$

**Lemma 2.1** *Assume that conditions (D1)-(D4) hold, and $x(t)$ is an eventually positive solution of (1.2). Then for sufficiently large $t$, $y(t)$ satisfies either*

*(I) $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$ and $\big(b(t)(y''(t))^{\alpha}\big)' \leq 0$;*

*or*

*(II) $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$ and $\big(b(t)(y''(t))^{\alpha}\big)' \leq 0$.*

The proof of this lemma is similar to that of some existing papers, such as Elabbasy and Moaaz ([10], Lemma 2.1), Sun and Zhao ([29], Lemma 2.1), and Tian et al. ([33], Lemma 2.1). So it is omitted.

**Lemma 2.2** *Assume that conditions (D1)-(D4) and (2.2) hold, and $x(t)$ is an eventually positive solution of (1.2) with $y(t)$ satisfying case (I) of Lemma 2.1. Then for sufficiently large $t$, we get*

$$\big(b(t)(y''(t))^{\alpha}\big)' + Q_2(t)y^{\alpha}(\tau^{-1}(\tilde{\sigma}(t))) \leq 0. \tag{2.3}$$

**Proof** Since $x(t)$ is an eventually positive solution of (1.2), there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$ for $t \geq t_1$. It is easy to yield $y(t) > x(t) > 0$ eventually, and

$$y(\tau^{-1}(t)) = x(\tau^{-1}(t)) + m(\tau^{-1}(t))x(t), \tag{2.4}$$

based on the definition of $y(t)$. Following from (2.4), we have

$$\begin{aligned} x(t) &= \frac{1}{m(\tau^{-1}(t))}\Big(y(\tau^{-1}(t)) - x(\tau^{-1}(t))\Big) \\ &= \frac{y(\tau^{-1}(t))}{m(\tau^{-1}(t))} - \frac{y(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))m(\tau^{-1}(\tau^{-1}(t)))} \\ &\geq \frac{y(\tau^{-1}(t))}{m(\tau^{-1}(t))} - \frac{y(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))m(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{2.5}$$

From (1.2), (D2) and (D4), we get

$$\big(b(t)(y''(t))^{\alpha}\big)' \leq 0,$$

and $b(t)(y''(t))^\alpha$ is nonincreasing for $t \geq t_1$, from which we have

$$y'(t) = y'(t_1) + \int_{t_1}^{t} \frac{\left(b(s)(y''(s))^\alpha\right)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}(s)} ds \geq b^{\frac{1}{\alpha}}(t) y''(t)\delta_1(t, t_1), \tag{2.6}$$

since $y(t)$ satisfies the case (I) of Lemma 2.1. We can see from (2.6) that

$$\left(\frac{y'(t)}{\delta_1(t, t_1)}\right)' \leq 0 \quad \text{for} \quad t \geq t_2 > t_1,$$

which yields that $y'(t)/\delta_1(t, t_1)$ is nonincreasing for $t \geq t_2$. Furthermore, we obtain

$$\begin{aligned} y(t) &= y(t_2) + \int_{t_2}^{t} \frac{y'(s)}{\delta_1(s, t_1)} \delta_1(s, t_1) ds \\ &\geq \frac{\delta_2(t, t_2)}{\delta_1(t, t_1)} y'(t) \quad \text{for} \quad t \geq t_2, \end{aligned} \tag{2.7}$$

which leads to

$$\left(\frac{y(t)}{\delta_2(t, t_2)}\right)' \leq 0 \quad \text{for} \quad t \geq t_3 > t_2.$$

Then we have

$$y(\tau^{-1}(\tau^{-1}(t))) \leq \frac{\delta_2(\tau^{-1}(\tau^{-1}(t)), t_2) y(\tau^{-1}(t))}{\delta_2(\tau^{-1}(t), t_2)} \tag{2.8}$$

for $t \geq t_3$, in views of the fact that $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$. Substituting (2.8) into (2.5), we get

$$x(t) \geq m_2(t) y(\tau^{-1}(t)),$$

which yields that

$$\begin{aligned} x(\sigma(t, \xi)) &\geq m_2(\sigma(t, \xi)) y(\tau^{-1}(\sigma(t, \xi))) \\ &\geq m_2(\sigma(t, \xi)) y(\tau^{-1}(\tilde{\sigma}(t))). \end{aligned} \tag{2.9}$$

Combining (1.2) and (2.9), we arrive at (2.3), due to the monotonicity of $y(t)$ and $\tau(t)$. □

## 3. Main results

**Theorem 3.1** *Assume that conditions (D1)-(D4) hold. Furthermore, assume that there exists a function $g(t) \in C([t_0, \infty), \mathbb{R})$ such that $\sigma(t, \xi) \leq g(t) < \tau(t)$ for $t \geq t_0$. If the two first-order delay differential equations*

$$z'(t) + Q_2(t)\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_2) z(\tau^{-1}(\tilde{\sigma}(t))) = 0 \tag{3.1}$$

*and*

$$\omega'(t) + Q_1(t) G^\alpha(t) \omega(\tau^{-1}(g(t))) = 0 \tag{3.2}$$

*oscillate for large $t \geq t_2 > t_1 \geq t_0$, where*

$$G(t) = \int_{\tau^{-1}(\tilde{\sigma}(t))}^{\tau^{-1}(g(t))} \delta_1(\tau^{-1}(g(t)), l) dl,$$

*then (1.2) is oscillatory.*

**Proof** Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.2). In general, we assume that $x(t)$ is eventually positive, since $-x(t)$ is also a solution of (1.2), and the proof is similar when $x(t)<0$ eventually. Then there exists $t_1 \geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t,\xi))>0$, $\xi \in [c,d]$ for $t \geq t_1$. $y(t)$ satisfies two possible cases (I) and (II) following from Lemma 2.1.

Firstly, if case (I) holds, then we obtain (2.3), (2.5) and (2.6), proceeding as in the proof of Lemma 2.2. Integrating (2.6) from $t_2 (t_2 > t_1)$ to $t$, we get

$$y(t) \geq b^{\frac{1}{\alpha}}(t) y''(t) \delta_2(t,t_2). \tag{3.3}$$

Substituting (3.3) into (2.3), we have

$$\big(b(t)(y''(t))^\alpha\big)' + Q_2(t) b(\tau^{-1}(\tilde{\sigma}(t)))(y''(\tau^{-1}(\tilde{\sigma}(t))))^\alpha \delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)),t_2) \leq 0$$

with $\tau^{-1}(\tilde{\sigma}(t)) \geq t_2$. Letting $z(t)=b(t)(y''(t))^\alpha$, we can see that $z(t)$ is a positive solution of the following first-order delay differential inequality

$$z'(t) + Q_2(t)\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(t)),t_2) z(\tau^{-1}(\tilde{\sigma}(t))) \leq 0.$$

We conclude from [21] that (3.1) also has a positive solution, which contradicts the fact that (3.1) oscillates.

Secondly, we assume that case (II) holds. Since $y'(t)<0$ and $\tau(t) \leq t$, we have

$$y(\tau^{-1}(t)) \geq y(\tau^{-1}(\tau^{-1}(t))). \tag{3.4}$$

Combining (2.5) and (3.4), we get

$$x(t) \geq m_1(t) y(\tau^{-1}(t)),$$

and

$$x(\sigma(t,\xi)) \geq m_1(\sigma(t,\xi)) y(\tau^{-1}(\sigma(t,\xi))). \tag{3.5}$$

Substitute (3.5) into (1.2) and use (D4) to obtain

$$\big(b(t)(y''(t))^\alpha\big)' + K\int_c^d q(t,\xi) m_1^\alpha(\sigma(t,\xi)) y^\alpha(\tau^{-1}(\sigma(t,\xi)))d\xi \leq 0,$$

which indicates that

$$\big(b(t)(y''(t))^\alpha\big)' + Q_1(t) y^\alpha(\tau^{-1}(\bar{\sigma}(t))) \leq 0. \tag{3.6}$$

Since $y(t)>0$, $y'(t)<0$, $y''(t)>0$ and $\big(b(t)(y''(t))^\alpha\big)' \leq 0$, one can get for $t_2 \leq u \leq v$,

$$y'(v) - y'(u) = \int_u^v \frac{\big(b(s)(y''(s))^\alpha\big)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}(s)} ds,$$

which yields that

$$-y'(u) \geq b^{\frac{1}{\alpha}}(v) y''(v) \delta_1(v,u).$$

Integrating the above inequality from $u$ to $v$, we obtain

$$\begin{aligned} y(u) &\geq y(v) + b^{\frac{1}{\alpha}}(v)y''(v)\int_u^v \delta_1(v,l)dl \\ &\geq b^{\frac{1}{\alpha}}(v)y''(v)\int_u^v \delta_1(v,l)dl. \end{aligned} \tag{3.7}$$

From the fact that $\bar{\sigma}(t) \leq g(t)$ and $\tau(t)$ is strictly increasing, we get $\tau^{-1}(\bar{\sigma}(t)) \leq \tau^{-1}(g(t))$. Substituting $u = \tau^{-1}(\bar{\sigma}(t))$ and $v = \tau^{-1}(g(t))$ into (3.7), we have

$$y(\tau^{-1}(\bar{\sigma}(t))) \geq G(t)b^{\frac{1}{\alpha}}(\tau^{-1}(g(t)))y''(\tau^{-1}(g(t))). \tag{3.8}$$

Using (3.6) and (3.8), one can see that

$$\big(b(t)(y''(t))^{\alpha}\big)' + Q_1(t)G^{\alpha}(t)b(\tau^{-1}(g(t)))(y''(\tau^{-1}(g(t))))^{\alpha} \leq 0.$$

Letting $\omega(t) = b(t)(y''(t))^{\alpha}$, we deduce that $\omega(t)$ is a positive solution of the following differential inequality

$$\omega'(t) + Q_1(t)G^{\alpha}(t)\omega(\tau^{-1}(g(t))) \leq 0.$$

The rest of the proof is similar to that of the former case, and so we omit it here. This completes the proof. □

Next, we introduce a well-known result. Consider the first-order delay differential equation

$$x'(t) + R(t)x(\tau(t)) = 0, \tag{3.9}$$

under the conditions $R(t) \in C([t_0,\infty),[0,\infty))$, $\tau(t) \in C([t_0,\infty),\mathbb{R})$, $\tau(t) < t$ and $\lim_{t\to\infty}\tau(t) = \infty$. From [19], if

$$\liminf_{t\to\infty}\int_{\tau(t)}^t R(s)ds > \frac{1}{e},$$

then (3.9) is oscillatory. Hence, combine Theorem 3.1 and this result to derive the following result.

**Corollary 3.2** *Assume that conditions (D1)-(D4) hold. Furthermore, assume that there exists a function $g(t) \in C([t_0,\infty),\mathbb{R})$ such that $\sigma(t,\xi) \leq g(t) < \tau(t)$ for $t \geq t_0$. If for large $t \geq t_2 > t_1 \geq t_0$,*

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\tilde{\sigma}(t))}^t Q_2(s)\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_2)ds > \frac{1}{e},$$

*and*

$$\liminf_{t\to\infty}\int_{\tau^{-1}(g(t))}^t Q_1(s)G^{\alpha}(s)ds > \frac{1}{e}, \tag{3.10}$$

*where $G(t)$ is defined as in Theorem 3.1, then (1.2) is oscillatory.*

Directly from Theorem 3.1, we get an oscillatory criterion as below.

**Theorem 3.3** *Assume that conditions (D1)-(D4) hold, and $\sigma(t,\xi) < \tau(t)$ for $t \geq t_0$. If the two first-order delay differential equations*

$$z'(t) + Q_2(t)\delta_2^{\alpha}(\tilde{\sigma}(t), t_2)z(\tilde{\sigma}(t)) = 0 \tag{3.11}$$

*and*

$$\omega'(t) + Q_1(t)\bar{G}^{\alpha}(t)\omega(\tau^{-1}(\bar{\sigma}(t))) = 0 \tag{3.12}$$

*oscillate for large $t \geq t_2 > t_1 \geq t_0$, where*

$$\bar{G}(t) = \int_{\tau^{-1}(\bar{\sigma}(t))}^{\infty} \delta_1(\tau^{-1}(\bar{\sigma}(t)), s)ds,$$

*then (1.2) is oscillatory.*

**Proof** Let $x(t)$ be a nonoscillatory solution of (1.2). We may assume that $x(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$ for $t \geq t_1$. Lemma 2.1 yields that $y(t)$ satisfies either case (I) or case (II).

Firstly, we assume that case (I) satisfies. Proceeding as in the proof of Theorem 3.1, we get (3.3). Since $\tau(t) \leq t$, we can see that $t \leq \tau^{-1}(t)$ and $\tilde{\sigma}(t) \leq \tau^{-1}(\tilde{\sigma}(t))$. Then we obtain $y(\tilde{\sigma}(t)) \leq y(\tau^{-1}(\tilde{\sigma}(t)))$, due to $y'(t) > 0$. Based on (2.3) and this inequality, we have

$$\big(b(t)(y''(t))^{\alpha}\big)' + Q_2(t)y^{\alpha}(\tilde{\sigma}(t)) \leq 0. \tag{3.13}$$

Substituting (3.3) into (3.13) yields that

$$\big(b(t)(y''(t))^{\alpha}\big)' + Q_2(t)\delta_2^{\alpha}(\tilde{\sigma}(t), t_2)b(\tilde{\sigma}(t))(y''(\tilde{\sigma}(t)))^{\alpha} \leq 0$$

for $\tilde{\sigma}(t) \geq t_2 > t_1$. Letting $z(t) = b(t)(y''(t))^{\alpha}$, we conclude that $z(t)$ is a positive solution of

$$z'(t) + Q_2(t)\delta_2^{\alpha}(\tilde{\sigma}(t), t_2)z(\tilde{\sigma}(t)) \leq 0,$$

which is a contradiction to that (4.1) oscillates.

Secondly, we assume that case (II) holds. Proceeding as in the proof of Theorem 3.1, we have (3.6). Since $y(t) > 0$, $y'(t) < 0$, $y''(t) > 0$ and $\big(b(t)(y''(t))^{\alpha}\big)' \leq 0$, we get for $t_2 \leq s \leq t$,

$$y'(t) - y'(s) = \int_s^t \frac{\big(b(u)(y''(u))^{\alpha}\big)^{\frac{1}{\alpha}}}{b^{\frac{1}{\alpha}}(u)}du,$$

which means that

$$-y'(s) \geq b^{\frac{1}{\alpha}}(t)y''(t)\delta_1(t,s). \tag{3.14}$$

Integrating (3.14) from $t$ to $v$ $(v > t)$ and letting $v \to \infty$, we get

$$y(t) \geq b^{\frac{1}{\alpha}}(t)y''(t)\int_t^{\infty} \delta_1(t,s)ds.$$

and then

$$y(\tau^{-1}(\bar{\sigma}(t))) \geq \bar{G}(t)b^{\frac{1}{\alpha}}(\tau^{-1}(\bar{\sigma}(t)))y''(\tau^{-1}(\bar{\sigma}(t))). \tag{3.15}$$

Substitute (3.15) into (3.6) to obtain

$$\big(b(t)(y''(t))^{\alpha}\big)' + Q_1(t)\bar{G}^{\alpha}(t)b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^{\alpha} \le 0.$$

Similarly, letting $\omega(t) = b(t)(y''(t))^{\alpha}$, we get a contradiction to the fact that (4.2) oscillates. This completes the proof. □

**Remark 3.4** *The result of case (I) in Theorem 3.3 is directly obtained from Theorem 3.1. Hence, the corresponding results are similar. The main difference between Theorems 3.1 and 3.3 is the result about case (II). Especially, (4.2) does not need the reference function $g(t)$.*

Analogously, based on Theorem 3.3 and the well-known result in [19], we can get the following result.

**Corollary 3.5** *Assume that conditions (D1)-(D4) hold, and $\sigma(t,\xi) < \tau(t)$ for $t \ge t_0$. If for large $t \ge t_2 > t_1 \ge t_0$,*

$$\liminf_{t\to\infty} \int_{\tilde{\sigma}(t)}^{t} Q_2(s)\delta_2^{\alpha}(\tilde{\sigma}(s), t_2)ds > \frac{1}{e},$$

*and*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\bar{\sigma}(t))}^{t} Q_1(s)\bar{G}^{\alpha}(s)ds > \frac{1}{e},$$

*where $\bar{G}(t)$ is defined as in Theorem 3.3, then (1.2) is oscillatory.*

In the following interesting result, we need to assume that the function $\sigma(t,\xi)$ is nondecreasing for $t$.

**Theorem 3.6** *Assume that conditions (D1)-(D4) hold. Furthermore, assume that $\sigma(t,\xi)$ is nondecreasing for $t$ with $\sigma(t,\xi) < \tau(t)$, $t \ge t_0$. If*

$$\limsup_{t\to\infty} \delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)), t_1) \int_t^{\infty} Q_2(s)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)), t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)), t_1)}ds > 1, \tag{3.16}$$

*and*

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\bar{\sigma}(t))}^{t} Q_1(s)H^{\alpha}(t,s)ds > 1 \tag{3.17}$$

*for large $t \ge t_2 > t_1 \ge t_0$, where*

$$H(t,s) = \int_{\tau^{-1}(\bar{\sigma}(s))}^{\tau^{-1}(\bar{\sigma}(t))} \delta_1(\tau^{-1}(\bar{\sigma}(t)), l)dl,$$

*then (1.2) is oscillatory.*

**Proof** Let $x(t)$ be a nonoscillatory solution of (1.2). We may assume that $x(t)$ is eventually positive. Then there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$ for $t \ge t_1$. Lemma 2.1 yields that $y(t)$ satisfies either case (I) or case (II).

Firstly, we assume that case (I) satisfies. Proceeding as in the proof of Lemma 2.2, we have (2.7), which means that

$$y(\tau^{-1}(\tilde{\sigma}(t))) \geq \frac{\delta_2(\tau^{-1}(\tilde{\sigma}(t)),t_2)}{\delta_1(\tau^{-1}(\tilde{\sigma}(t)),t_1)} y'(\tau^{-1}(\tilde{\sigma}(t))) \tag{3.18}$$

for $t \geq t_2 > t_1$. Substitute (3.18) into (2.3) to get

$$\big(b(t)(y''(t))^{\alpha}\big)' + Q_2(t)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_1)}(y'(\tau^{-1}(\tilde{\sigma}(t))))^{\alpha} \leq 0. \tag{3.19}$$

Letting $z(t) = y'(t)$, we have

$$z(t) > 0, \; z'(t) > 0, \; \big(b(t)(z'(t))^{\alpha}\big)' \leq 0,$$

and (3.19) can be written as

$$\big(b(t)(z'(t))^{\alpha}\big)' + Q_2(t)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_1)} z^{\alpha}(\tau^{-1}(\tilde{\sigma}(t))) \leq 0. \tag{3.20}$$

Integrating (3.20) from $t$ to $u$ $(u > t)$ and letting $u \to \infty$, we obtain

$$\begin{aligned} b(t)(z'(t))^{\alpha} &\geq \int_t^{\infty} Q_2(s)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_1)} z^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)))ds \\ &\geq z^{\alpha}(\tau^{-1}(\tilde{\sigma}(t))) \int_t^{\infty} Q_2(s)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_1)}ds. \end{aligned} \tag{3.21}$$

(2.6) can be rewritten as

$$z(t) \geq b^{\frac{1}{\alpha}}(t) z'(t)\delta_1(t,t_1). \tag{3.22}$$

Substitute (3.22) into (3.21) to get

$$\begin{aligned} b(t)(z'(t))^{\alpha} \geq & b(\tau^{-1}(\tilde{\sigma}(t)))(z'(\tau^{-1}(\tilde{\sigma}(t))))^{\alpha}\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_1) \\ & \times \int_t^{\infty} Q_2(s)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_1)}ds. \end{aligned} \tag{3.23}$$

Since $\big(b(t)(z'(t))^{\alpha}\big)' \leq 0$ and $\tilde{\sigma}(t) < \tau(t)$, one can see that

$$b(t)(z'(t))^{\alpha} \leq b(\tau^{-1}(\tilde{\sigma}(t)))(z'(\tau^{-1}(\tilde{\sigma}(t))))^{\alpha}. \tag{3.24}$$

Hence, (3.23) and (3.24) yield that

$$\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_1)\int_t^{\infty} Q_2(s)\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_1)}ds \leq 1,$$

which contradicts (3.16).

Secondly, we assume that case (II) holds. Proceeding as in the proof of Theorem 3.1, we have (3.6) and (3.7). For $t_2 \leq s \leq t$, one can see that $\tau^{-1}(\bar{\sigma}(t)) \geq \tau^{-1}(\bar{\sigma}(s))$ due to the monotonicity of $\bar{\sigma}(t)$ and $\tau^{-1}(t)$. Substitute $u = \tau^{-1}(\bar{\sigma}(s))$ and $v = \tau^{-1}(\bar{\sigma}(t))$ into (3.7) to obtain

$$y(\tau^{-1}(\bar{\sigma}(s))) \geq H(t,s) b^{\frac{1}{\alpha}}(\tau^{-1}(\bar{\sigma}(t)))y''(\tau^{-1}(\bar{\sigma}(t))). \tag{3.25}$$

Integrating (3.6) from $\tau^{-1}(\bar{\sigma}(t))$ to $t$, we get

$$b(t)(y''(t))^{\alpha} - b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^{\alpha}$$
$$+ \int_{\tau^{-1}(\bar{\sigma}(t))}^{t} Q_1(s) y^{\alpha}(\tau^{-1}(\bar{\sigma}(s)))ds \leq 0.$$

Combining (3.25) and the above inequality, we conclude that

$$b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^{\alpha} \geq b(\tau^{-1}(\bar{\sigma}(t)))(y''(\tau^{-1}(\bar{\sigma}(t))))^{\alpha}$$
$$\times \int_{\tau^{-1}(\bar{\sigma}(t))}^{t} Q_1(s) H^{\alpha}(t,s)ds,$$

and then

$$\int_{\tau^{-1}(\bar{\sigma}(t))}^{t} Q_1(s) H^{\alpha}(t,s)ds \leq 1.$$

Take limsup as $t \to \infty$ to yield a contradiction to (3.17). This completes the proof. □

## 4. Examples

**Example 4.1.** Consider the following equation

$$\left((y''(t))^{\frac{1}{3}}\right)' + \int_{\frac{1}{2}}^{1} 8q_0 \xi t^{\frac{1}{3}} x^{\frac{1}{3}}\left(\frac{t}{2} - \frac{\xi}{2}\right)d\xi = 0, \ t > 8, \tag{4.1}$$

where

$$y(t) = x(t) + \frac{9t+8}{t+1}x\left(\frac{t}{2}\right), \ m(t) = \frac{9t+8}{t+1}, \ q(t,\xi) = 8q_0\xi t^{\frac{1}{3}},$$

$b(t) = 1$, $\alpha = 1/3$, $\tau(t) = t/2$, $f(x) = x^{1/3}$, $\sigma(t,\xi) = t/2 - \xi/2$, $c = 1/2$, $d = 1$ and $t_0 = 1$. We note that conditions (D1)-(D4) are satisfied. Choose $t_1 = 2$ and $t_2 = 4$. Then we get $8 \leq m(t) < 9$,

$$\bar{\sigma}(t) = \sigma(t, 1/2) = \frac{t}{2} - \frac{1}{4}, \ \tilde{\sigma}(t) = \sigma(t,1) = \frac{t}{2} - \frac{1}{2},$$

$$\delta_1(t,t_1) = \int_2^t ds = t - 2,$$

$$\delta_2(t,t_2) = \int_4^t (s-2)ds = \frac{1}{2}t^2 - 2t.$$

Furthermore, we deduce that

$$m_1(t) > \frac{1}{9}\left(1 - \frac{1}{8}\right) = \frac{7}{72} > 0,$$

$$m_2(t) > \frac{1}{9}\left(1 - \frac{1}{8} \cdot \frac{\frac{1}{2}(4t)^2 - 2\cdot 4t}{\frac{1}{2}(2t)^2 - 2 \cdot 2t}\right) > \frac{1}{24} > 0,$$

$$Q_1(t) > \int_{\frac{1}{2}}^{1} \left(\frac{7}{72}\right)^{\frac{1}{3}} 8q_0\xi t^{\frac{1}{3}} d\xi = 3q_0\left(\frac{7t}{72}\right)^{\frac{1}{3}},$$

$$Q_2(t) > \int_{\frac{1}{2}}^{1} \left(\frac{1}{24}\right)^{\frac{1}{3}} 8q_0\xi t^{\frac{1}{3}} d\xi = 3q_0\left(\frac{t}{24}\right)^{\frac{1}{3}}.$$

Letting $g(t)=t/2-1/6$, we get

$$\begin{aligned}&\int_{\tau^{-1}(\tilde{\sigma}(t))}^{t} Q_2(s)\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(s)),t_2)ds\\ &> \int_{t-1}^{t} 3q_0\Big(\frac{s}{24}\Big)^{\frac{1}{3}}\Big(\frac{1}{2}(s-1)^2-2(s-1)\Big)^{\frac{1}{3}}ds > 3q_0\Big(\frac{3}{2}\Big)^{\frac{1}{3}} > \frac{1}{e},\end{aligned}$$

if $q_0 > 2^{1/3}/(3^{4/3}e)$, and

$$\begin{aligned}&\int_{\tau^{-1}(g(t))}^{t} Q_1(s)G^{\alpha}(s)ds\\ &> \int_{t-\frac{1}{3}}^{t} 3q_0\Big(\frac{7s}{72}\Big)^{\frac{1}{3}}\Big(\int_{s-\frac{1}{2}}^{s-\frac{1}{3}}(s-\frac{1}{3}-l)dl\Big)^{\frac{1}{3}}ds > q_0\Big(\frac{1}{12}\Big)^{\frac{2}{3}} > \frac{1}{e},\end{aligned}$$

if $q_0 > 12^{2/3}/e$. Hence, by Corollary 3.2, (4.1) is oscillatory with $q_0 > 12^{\frac{2}{3}}/e$.

**Example 4.2.** Consider the equation

$$\big((2t)^{-\frac{5}{3}}(y''(t))^{\frac{5}{3}}\big)' + \int_1^3 \frac{q_0\xi}{4t^{\frac{4}{3}}}x^{\frac{5}{3}}\Big(\frac{t}{3}-\frac{4}{3}+\frac{1}{\xi}\Big)d\xi = 0,\ t>9, \tag{4.2}$$

where $y(t)=x(t)+30x(t/3)$, $m(t)=30$, $b(t)=(2t)^{-5/3}$, $\alpha=5/3$, $\tau(t)=t/3$, $q(t,\xi)=q_0\xi/(4t^{4/3})$, $f(x)=x^{5/3}$, $\sigma(t,\xi)=t/3-4/3+1/\xi$, $c=1$, $d=3$ and $t_0=1$. Then conditions (D1)-(D4) also hold. Choosing $t_1=2$ and $t_2=3$, we obtain

$$\begin{aligned}\bar{\sigma}(t) &= \frac{t}{3}-\frac{1}{3},\ \tilde{\sigma}(t)=\frac{t}{3}-1,\\ \delta_1(t,t_1) &= t^2-4,\ \delta_2(t,t_2)=\frac{1}{3}t^3-4t+3,\\ m_1(t) &= \frac{1}{30}\Big(1-\frac{1}{30}\Big)=\frac{29}{900}>0,\\ m_2(t) &= \frac{1}{30}\Big(1-\frac{1}{30}\cdot\frac{\frac{1}{3}(9t)^3-4\cdot 9t+3}{\frac{1}{3}(3t)^3-4\cdot 3t+3}\Big) > \frac{1}{450} > 0,\\ Q_1(t) &> \int_1^3\Big(\frac{29}{900}\Big)^{\frac{5}{3}}\frac{q_0\xi}{4t^{\frac{4}{3}}}d\xi = \Big(\frac{29}{900}\Big)^{\frac{5}{3}}\frac{q_0}{t^{\frac{4}{3}}},\\ Q_2(t) &> \int_1^3\Big(\frac{1}{450}\Big)^{\frac{5}{3}}\frac{q_0\xi}{4t^{\frac{4}{3}}}d\xi = \Big(\frac{1}{450}\Big)^{\frac{5}{3}}\frac{q_0}{t^{\frac{4}{3}}}.\end{aligned}$$

Letting

$$\rho(t)=\frac{\delta_2^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_2)}{\delta_1^{\alpha}(\tau^{-1}(\tilde{\sigma}(t)),t_1)} = \Big(\frac{\frac{1}{3}(t-3)^3-4(t-3)+3}{(t-3)^2-4}\Big)^{\frac{5}{3}},$$

it is easy to verify that $\rho(t)$ is increasing. Then $\rho(t) > (51/32)^{5/3}$ for $t > 9$. Furthermore, we see that

$$\begin{aligned}
&\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(t)), t_1) \int_t^\infty Q_2(s) \frac{\delta_2^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_2)}{\delta_1^\alpha(\tau^{-1}(\tilde{\sigma}(s)), t_1)} ds \\
>&((t-3)^2-4)^{\frac{5}{3}} \int_t^\infty \Big(\frac{1}{450}\Big)^{\frac{5}{3}} \frac{q_0}{s^{\frac{4}{3}}} \cdot \Big(\frac{51}{32}\Big)^{\frac{5}{3}} ds \\
>&3q_0 \Big(\frac{1}{9 \cdot 32}\Big)^{\frac{5}{3}} \frac{((t-3)^2-4)^{\frac{5}{3}}}{t^{\frac{1}{3}}} > 1,
\end{aligned}$$

if $q_0 > 3^{7/3} \cdot 32^{5/3}$, and

$$\begin{aligned}
&\int_{\tau^{-1}(\tilde{\sigma}(t))}^t Q_1(s) H^\alpha(t,s) ds \\
>&\int_{t-1}^t \Big(\frac{29}{900}\Big)^{\frac{5}{3}} \frac{q_0}{s^{\frac{4}{3}}} \Big(\int_{s-1}^{t-1} (t^2 - l^2) dl\Big)^{\frac{5}{3}} ds \\
>&\Big(\frac{29}{900}\Big)^{\frac{5}{3}} q_0 \int_{t-1}^t s^{\frac{1}{3}} ds > \Big(\frac{3}{100}\Big)^{\frac{5}{3}} \frac{3q_0}{4} > 1,
\end{aligned}$$

if $q_0 > 4 \cdot 100^{\frac{5}{3}} / 3^{\frac{8}{3}}$. Hence, by Theorem 3.6, (4.2) is oscillatory with $q_0 > 3^{7/3} \cdot 32^{5/3}$.


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### Conflict of interest

The authors declare that they have no competing interests.

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