Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1119 – 1131
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doi:10.3906/mat-2112-6

On the inclusion properties for ϑ-spirallike functions involving both Mittag-Leffler and Wright function

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Received: 02.12.2021 • **Accepted/Published Online:** 24.02.2022 • **Final Version:** 11.03.2022

Abstract: By making use of the both Mittag-Leffler and Wright function, we establish a new subfamily of the class S_ϑ of ϑ-spirallike functions. The main object of the paper is to provide sufficient conditions for a function to be in this newly established class and to discuss subordination outcomes.

Key words: Wright function, Mittag-Leffler function, Spirallike function, Subordination, Convolution.

1. Introduction and Preliminaries

The well-known Wright function $W_{\alpha,\beta}(z)\ (\alpha > -1,\ \beta, z \in \mathbb{C})$ is expressed by

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta) n!}. \tag{1.1}$$

It is well known that for $\alpha > -1$ and every $z \in \mathbb{C}$ the series (1.1) is absolutely convergent. In the case $\alpha = -1$, this series is absolutely convergent in $U = \{z \in \mathbb{C} : |z| < 1\}$. For $\alpha > -1$, the function $W_{\alpha,\beta}$ is an entire function [37]. The various properties of the Wright function and its applications were investigated in a series of papers [1], [20–26], [33].

The Mittag-Leffler function $E_\alpha(z)\ (\alpha \in \mathbb{C},\ \Re(\alpha) > 0)$ is expressed by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

and its generalization with two parameters $E_{\alpha,\beta}(z)\ (\alpha, z \in \mathbb{C},\ \Re(\alpha) > 0, \Re(\beta) > 0)$ is expressed by [36]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

where Γ stands for the Gamma function. It is clear that $E_{\alpha,\beta}(z)$ is related $E_\alpha(z)$ with the below relation

$$E_{\alpha,1}(z) = E_\alpha(z).$$

2010 *AMS Mathematics Subject Classification:* 30C45, 33E12

Afterwards, it has been generalized in different ways over the years [7], [10–12], [14, 15], [17], [30]. While the Wright function has a major role due to its applications in partial differential equations of fractional order, the Mittag–Leffler function has gained a major importance in engineering, physics, chemistry, biology as well as mathematics and mathematical physics.

Next, assume that $\mathcal{A}$ is the family of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1.2}$$

which are analytic in U with the normalization $f(0) = f'(0) - 1 = 0$. Furthermore, assume that $\mathcal{S}$ is the subfamily of $\mathcal{A}$ comprising of univalent functions.

If f_1 and f_2 are analytic in U, then we call that f_1 is subordinated to f_2, showed by $f_1 \prec f_2$, for the Schwarz function

$$\varpi(z) = \sum_{n=1}^{\infty} \mathbf{c}_n z^n \quad (\varpi(0) = 0, |\varpi(z)| < 1),$$

analytic in U such that

$$f_1(z) = f_2(\varpi(z)) \qquad (z \in U).$$

On the other hand, the convolution of functions $f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is expressed by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in U).$$

A logarithmic ϑ-spiral is a curve defined by

$$\omega = \omega_0 \exp(-e^{-i\vartheta} t) \qquad \left(-\infty < t < \infty, \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right),$$

where ω_0 is a nonzero complex number. Since $\omega = \omega(t)$ is a logarithmic ϑ-spiral, $\Im\left[e^{i\vartheta} \log \omega(t)\right] = constant$. Observe that 0-spirals are radial half-lines. For each ϑ $\left(|\vartheta| < \frac{\pi}{2}\right)$, there is a unique ϑ-spiral which joins a given point $\omega \neq 0$ to the origin (see [13]). A domain D containing the origin is named ϑ-spirallike if for each point $\omega \neq 0$ in D the arc of the ϑ-spiral from ω to the origin lies entirely in D. That is, D is simply connected. An analytic univalent function f is named ϑ-spirallike if its range is ϑ-spirallike. The class of ϑ-spirallike functions is indicated by $\mathcal{S}_\vartheta$. Analytically, this means that a function $f \in \mathcal{A}$ is in $\mathcal{S}_\vartheta$ iff $\Re\left(e^{i\vartheta} \frac{zf'(z)}{f(z)}\right) > 0$. The class $\mathcal{S}_\vartheta$ of ϑ-spirallike functions was expressed by Špaček [31] and he proved that $\mathcal{S}_\vartheta \subset \mathcal{S}$. Explicitly, let ϑ and σ be real numbers, where $|\vartheta| < \frac{\pi}{2}$ and $0 \leq \sigma < 1$. Libera [19] adopted this concept to functions ϑ-spirallike of order σ indicated by $\mathcal{S}_\vartheta(\sigma)$ as follows (see also [18]):

$$\Re\left(e^{i\vartheta} \frac{zf'(z)}{f(z)}\right) > \sigma \cos \vartheta.$$

Clearly, $\mathcal{S}_\vartheta(\sigma) \subset \mathcal{S}_\vartheta$.

The q-calculus presents different tools which have been studied in special functions. While Jackson was one of the leading authors among mathematicians working on the q-calculus theory [16], the use of q-calculus within the context of Geometric Function Theory was partially provided by Srivastava [32]. In course of this paper, assume $q \in (0,1)$ and the definitions of q-derivative operators are dealt with the complex-valued function f.

Now, we consider the following q-derivative of f introduced by [16]:

$$D_q f(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0 \end{cases}. \tag{1.3}$$

We would like to point out that if f is differentiable at z, then

$$\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} \frac{f(z)-f(qz)}{(1-q)z} = f'(z).$$

Recently, Bansal and Mehrez [4] defined the function

$$T_{\alpha,\beta}(\gamma;z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)\left[\gamma n! + (1-\gamma)\right]},\ (0 \leq \gamma \leq 1)$$

which enables us to discuss both Mittag-Leffler and Wright function, that is,

$$T_{\alpha,\beta}(0;z) = E_{\alpha,\beta}(z),$$

$$T_{\alpha,\beta}(1;z) = W_{\alpha,\beta}(z).$$

Observe that the function $T_{\alpha,\beta}(\gamma;z)$ is not an element of $\mathcal{A}$. Hence, the function $T_{\alpha,\beta}(\gamma;z)$ must be normalized as follows [2]

$$\mathbb{T}_{\alpha,\beta,\gamma}(z) = \Gamma(\beta) z T_{\alpha,\beta}(\gamma;z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)\left[\gamma(n-1)! + (1-\gamma)\right]} z^n.$$

This function holds for complex-valued α, β and $z \in \mathbb{C}$. However, throughout this paper, we restrict our attention to the case real-valued α, β with $\alpha > 0, \beta > 0$ and $z \in U$.

On the other hand, the function $\mathbb{T}_{\alpha,\beta,\gamma}$ is a natural extension of the exponential, hyperbolic and trigonometric functions, and some special cases can be given as follows; for example [2]

$$\mathbb{T}_{0,1,0}(z) = \tfrac{z}{1-z}, \quad \mathbb{T}_{1,2,0}(z) = e^z - 1, \qquad \mathbb{T}_{0,1,1}(z) = ze^z,$$

$$\mathbb{T}_{1,1,0}(z) = ze^z, \quad \mathbb{T}_{2,2,0}(z) = \sqrt{z}\sinh(\sqrt{z}), \quad \mathbb{T}_{2,1,0}(z) = z\cosh(\sqrt{z}).$$

Furthermore, for a function $f \in \mathcal{A}$, we get

$$\mathcal{T}_{\alpha,\beta,\gamma} f(z) = \mathbb{T}_{\alpha,\beta,\gamma}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta) a_n}{\Gamma(\alpha(n-1)+\beta)\left[\gamma(n-1)! + (1-\gamma)\right]} z^n. \tag{1.4}$$

Thus, with the help of (1.3) and (1.4), we get

$$D_q\left(\mathcal{T}_{\alpha,\beta,\gamma} f(z)\right) = 1 + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)\,[n]_q\, a_n}{\Gamma(\alpha(n-1)+\beta)\left[\gamma(n-1)! + (1-\gamma)\right]} z^{n-1},$$

where $[n]_q = \dfrac{1-q^n}{1-q}$.

Recently, a remarkably large number of researchers discussed families of analytic functions involving special functions to present geometric properties such as univalency and starlikeness in U. For convenience, many outcomes are available in the literature ([3], [5, 6], [8], [29], [34, 35]). In this organized research, by inserting the normalized function $\mathcal{T}_{\alpha,\beta,\gamma}f(z)$, we shall introduce a new subfamily $\mathcal{S}_\vartheta(\sigma;\alpha,\beta,\gamma,\mu)$ of $\vartheta-$spirallike functions. Afterwards, we discuss membership conditions, coefficient estimates, the relation of subordination and the Fekete-Szegö functional.

Definition 1.1 *A function $f \in \mathcal{A}$ is in $\mathcal{S}_\vartheta(\sigma;\alpha,\beta,\gamma,\mu)$ provided that*

$$\Re\left(e^{i\vartheta}\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)}\right) > \sigma\cos\vartheta,$$

where $|\vartheta| < \frac{\pi}{2},\ 0 \le \sigma < 1, 0 \le \mu < 1,\ 0 \le \gamma \le 1$.

2. Membership conditions and coefficients estimates

Theorem 2.1 *Assume $f \in \mathcal{A}$ and p is a real number with $0 \le p < 1$. If*

$$\left|\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)}-1\right| \le 1-p, \tag{2.1}$$

then $f \in \mathcal{S}_\vartheta(\sigma;\alpha,\beta,\gamma,\mu)$ provided that $|\vartheta| \le \cos^{-1}\left(\frac{1-p}{1-\sigma}\right)$.

Proof It follows from (2.1) that

$$\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)} = (1-p)\,\varpi(z)+1.$$

Further computations give

$$\begin{aligned}\Re\left\{e^{i\vartheta}\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)}\right\} &= \Re\left\{e^{i\vartheta}(1-p)\,\varpi(z)+e^{i\vartheta}\right\}\\ &= (1-p)\,\Re\left\{e^{i\vartheta}\varpi(z)\right\}+\cos\vartheta\\ &\ge -(1-p)\left|e^{i\vartheta}\varpi(z)\right|+\cos\vartheta\\ &= -(1-p)+\cos\vartheta.\end{aligned}$$

Indeed, $-(1-p)+\cos\vartheta = \sigma\cos\vartheta$ provided that $|\vartheta| \le \cos^{-1}\left(\frac{1-p}{1-\sigma}\right)$. □

Corollary 2.2 *Assume $p = 1-(1-\sigma)\cos\vartheta$. If*

$$\left|\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)}-1\right| \le (1-\sigma)\cos\vartheta,$$

then $f \in \mathcal{S}_\vartheta(\sigma;\alpha,\beta,\gamma,\mu)$.

Next, we discuss another sufficient condition.

Theorem 2.3 *A function* $f \in \mathcal{A}$ *belongs to* $\mathcal{S}_{\vartheta}(\sigma;\alpha,\beta,\gamma,\mu)$ *when*

$$\begin{aligned}&\textstyle\sum_{n=2}^{\infty}\left\{(1-\mu)\left([n]_q-1\right)\sec\vartheta+(1-\sigma)\left(1-\mu+\mu\,[n]_q\right)\right\}\\&\times\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}\,|a_n|\\&\leq 1-\sigma.\end{aligned}\tag{2.2}$$

Proof In view of above Corollary 2.2, since

$$\left|\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)}-1\right|$$

$$\begin{aligned}&= (1-\mu)\left|\frac{\sum_{n=2}^{\infty}\left([n]_q-1\right)\frac{\Gamma(\beta)a_nz^{n-1}}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}}{1+\sum_{n=2}^{\infty}\left(1-\mu+\mu\,[n]_q\right)\frac{\Gamma(\beta)a_nz^{n-1}}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}}\right|\\&< (1-\mu)\frac{\sum_{n=2}^{\infty}\left([n]_q-1\right)\frac{\Gamma(\beta)|a_n|}{\Gamma(\alpha(n-1)+\beta)[\gamma(n-1)!+(1-\gamma)]}}{1-\sum_{n=2}^{\infty}\left(1-\mu+\mu\,[n]_q\right)\frac{\Gamma(\beta)|a_n|}{\Gamma(\alpha(n-1)+\beta)[\gamma(n-1)!+(1-\gamma)]}},\end{aligned}$$

we get

$$\begin{aligned}&\sum_{n=2}^{\infty}(1-\mu)\left([n]_q-1\right)\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}\,|a_n|\\&\leq (1-\sigma)\cos\vartheta\left\{1-\sum_{n=2}^{\infty}\left(1-\mu+\mu\,[n]_q\right)\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}\,|a_n|\right\}\end{aligned}$$

which means

$$\begin{aligned}&\textstyle\sum_{n=2}^{\infty}\left\{(1-\mu)\left([n]_q-1\right)\sec\vartheta+(1-\sigma)\left(1-\mu+\mu\,[n]_q\right)\right\}\\&\times\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}\,|a_n|\\&\leq 1-\sigma.\end{aligned}$$

□

Our next outcome includes the general coefficient estimate.

Corollary 2.4 *If* $f \in \mathcal{S}_{\vartheta}(\sigma;\alpha,\beta,\gamma,\mu)$, *then*

$$|a_n| \leq \frac{1-\sigma}{C_n} \qquad (n \geq 2),$$

where

$$C_n = \left\{(1-\mu)\left([n]_q - 1\right)\sec\vartheta + (1-\sigma)\left(1-\mu+\mu\,[n]_q\right)\right\} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)\,[\gamma(n-1)!+(1-\gamma)]}.$$

The outcome is sharp for $f_n(z) = z + \frac{1-\sigma}{C_n} z^n$.

3. Subordination results

In order to discuss subordination outcomes for the class $\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$, we shall look at some definitions and lemmas due to Wilf [35].

Definition 3.1 *Assume* K *is the subfamily of* $\mathcal{S}$ *including convex functions, defined analytically with the equivalences*

$$h \in K \Leftrightarrow \Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > 0.$$

Definition 3.2 *The complex numbers sequence* $\{t_n\}_{n=1}^{\infty}$ *is named a subordinating factor sequence if, whenever*

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is regular, univalent and convex in U, *we get*

$$\sum_{n=1}^{\infty} b_n t_n z^n \prec h(z) \qquad (z \in U).$$

Lemma 3.3 *The sequence* $\{t_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence iff*

$$\Re\left\{1 + 2\sum_{n=1}^{\infty} t_n z^n\right\} > 0.$$

Theorem 3.4 *Assume* $f \in \mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$ *and* $h \in K$. *Then*

$$\frac{C_2}{2\left(1-\sigma+C_2\right)}\left(f * h\right)(z) \prec h(z), \tag{3.1}$$

where

$$C_2 = \left\{(1-\mu)\,q\sec\vartheta + (1-\sigma)\left(1+\mu q\right)\right\}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and

$$\Re\left\{f(z)\right\} > -\frac{1-\sigma+C_2}{C_2}. \tag{3.2}$$

The constant factor $\frac{C_2}{2(1-\sigma+C_2)}$ *cannot be changed with a larger number.*

Proof From Definition 3.2, the subordination (3.1) will be fulfilled if the sequence

$$\left\{\frac{C_2}{2\left(1-\sigma+C_2\right)}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence with $a_1 = 1$. Then by Lemma 3.3, it is evident to prove

$$\Re\left\{1+2\sum_{n=1}^{\infty}\frac{C_2}{2\left(1-\sigma+C_2\right)}a_n z^n\right\} > 0.$$

In view of (2.2), for $|z| = r < 1$, we get

$$\begin{aligned}\Re\left\{1+\frac{C_2}{\left(1-\sigma+C_2\right)}\sum_{n=1}^{\infty}a_n z^n\right\} &\geq 1-\frac{C_2}{1-\sigma+C_2}\left|\sum_{n=1}^{\infty}a_n z^n\right| \\ &= 1-\frac{C_2}{1-\sigma+C_2}r-\frac{1}{1-\sigma+C_2}\sum_{n=2}^{\infty}\left|a_n\right|C_n r^n \\ &= 1-\frac{1-\sigma+C_2}{1-\sigma+C_2}r > 0.\end{aligned}$$

Hence, equation (3.1) holds. Now, by taking $h(z) = \frac{z}{1-z} = z+\sum_{n=2}^{\infty}z^n$ in (3.1), we obtain the inequality (3.2). That is,

$$\begin{aligned}\frac{C_2}{2\left(1-\sigma+C_2\right)}f(z) &\prec h(z) \\ \frac{C_2}{2\left(1-\sigma+C_2\right)}\Re\left\{f(z)\right\} &> \Re\left\{h(z)\right\} \\ \frac{C_2}{2\left(1-\sigma+C_2\right)}\Re\left\{f(z)\right\} &> -\frac{1}{2}.\end{aligned}$$

Next, for the sharpness, let us assume a function

$$F(z) = z-\frac{1-\sigma}{1-\sigma+C_2}z^2.$$

Clearly $F \in \mathcal{S}_{\vartheta}\left(\sigma;\alpha,\beta,\gamma,\mu\right)$. By (3.1), since

$$\frac{C_2}{2\left(1-\sigma+C_2\right)}\left(F*h\right)(z) \prec \frac{z}{1-z},$$

we infer that

$$\min\left[\Re\left\{\frac{C_2}{2\left(1-\sigma+C_2\right)}F(z)\right\}\right] = -\frac{1}{2}.$$

This shows that the constant $\frac{C_2}{2(1-\sigma+C_2)}$ cannot be changed with any larger one. □

4. The Fekete-Szegö functional

The Fekete-Szegö problem, stated by Fekete and Szegö [9], aims to find the maximum value of the functional $\Theta_\Psi(f) = \left|a_3 - \Psi a_2^2\right|$, where $f \in \mathcal{S}$ and $\Psi \in [-1,1]$.

Recall the following pivotal observation to find sharp upper bounds for functions in $\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$.

Lemma 4.1 *[27] Assume ϖ is a Schwarz function. Then*

$$|\mathbf{c}_1| \leq 1, \qquad |\mathbf{c}_2| \leq 1 - |\mathbf{c}_1|^2$$

and for any complex number Ψ

$$\left|\mathbf{c}_2 - \Psi \mathbf{c}_1^2\right| \leq \max\left\{1, |\Psi|\right\}.$$

This outcome is sharp for $\varpi(z) = z$ and $\varpi(z) = z^2$.

For the constants ϑ $\left(|\vartheta| < \frac{\pi}{2}\right)$ and σ $(0 \leq \sigma < 1)$, assume

$$P(z) = \frac{1 + e^{-i\vartheta}\left(e^{-i\vartheta} - 2\sigma\cos\vartheta\right)z}{1-z} \quad (z \in U).$$

This function maps U onto the half-plane [28]

$$H = \left\{z \in \mathbb{C} : \Re\left(e^{i\vartheta}z\right) > \sigma\cos\vartheta\right\}.$$

If $P(z) = 1 + \sum\limits_{n=1}^{\infty} P_n z^n$, then $P_n = 2e^{-i\vartheta}\left(1-\sigma\right)\cos\vartheta$.

Now, we will solve the Fekete-Szegö problem for the class $\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$ with Ψ real parameter.

Theorem 4.2 *Let $f \in \mathcal{A}$ belong to $\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$ and Ψ be a real number. Then*

$$\left|a_3 - \Psi a_2^2\right| \leq$$

$$\begin{cases} \dfrac{2\Gamma(\alpha+\beta)\left(1+\gamma\right)\left(1-\sigma\right)\cos\vartheta}{\Gamma(\beta)q\left(1+q\right)\left(1-\mu\right)}\left[1 + \dfrac{2(1-\sigma)(1+\mu q)}{q\left(1-\mu\right)} - \Psi\dfrac{2\left(\Gamma(\alpha+\beta)\right)^2\left(1-\sigma\right)\left(1+q\right)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\left(1+\gamma\right)q\left(1-\mu\right)}\right] & \Psi \leq \xi_1 \\ \dfrac{2\Gamma(\alpha+\beta)\left(1+\gamma\right)\left(1-\sigma\right)\cos\vartheta}{\Gamma(\beta)q\left(1+q\right)\left(1-\mu\right)} & \xi_1 \leq \Psi \leq \xi_2 \\ \dfrac{2\Gamma(\alpha+\beta)\left(1+\gamma\right)\left(1-\sigma\right)\cos\vartheta}{\Gamma(\beta)q\left(1+q\right)\left(1-\mu\right)}\left[\Psi\dfrac{2\left(\Gamma(\alpha+\beta)\right)^2\left(1-\sigma\right)\left(1+q\right)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\left(1+\gamma\right)q\left(1-\mu\right)} - \dfrac{2(1-\sigma)(1+\mu q)}{q\left(1-\mu\right)} - 1\right] & \Psi \geq \xi_2 \end{cases},$$

where

$$\xi_1 = \frac{\Gamma(\beta)\Gamma(2\alpha+\beta)\left(1+\gamma\right)\left(1+\mu q\right)}{\left(\Gamma(\alpha+\beta)\right)^2\left(1+q\right)} \tag{4.1}$$

and

$$\xi_2 = \frac{\Gamma(\beta)\Gamma(2\alpha+\beta)\left(1+\gamma\right)\left(1+q-\sigma(1+\mu q)\right)}{\left(\Gamma(\alpha+\beta)\right)^2\left(1+q\right)\left(1-\sigma\right)}. \tag{4.2}$$

All estimates are sharp.

Proof Suppose $f \in \mathcal{S}_{\vartheta}(\sigma;\alpha,\beta,\gamma,\mu)$. From Definition 3.2, we can write

$$\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mu zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)} = P\left(\varpi(z)\right),$$

which gives

$$\begin{aligned}
&1+\frac{\Gamma(\beta)q\,(1-\mu)}{\Gamma(\alpha+\beta)}a_2 z\\
&+\left\{\frac{\Gamma(\beta)q\,(1+q)\,(1-\mu)}{\Gamma(2\alpha+\beta)\,(1+\gamma)}a_3-\left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)^2 q\,(1+\mu q)\,(1-\mu)\,a_2^2\right\}z^2+\cdots\\
&=1+P_1\mathbf{c}_1 z+\left(P_1\mathbf{c}_2+P_2\mathbf{c}_1^2\right)z^2+\cdots.
\end{aligned}$$

Hence, we get

$$a_2=\frac{\Gamma(\alpha+\beta)P_1}{\Gamma(\beta)q\,(1-\mu)}\mathbf{c}_1 \tag{4.3}$$

and

$$a_3=\frac{\Gamma(2\alpha+\beta)\,(1+\gamma)\,P_1}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\left[\mathbf{c}_2+\left(1+\frac{(1+\mu q)\,P_1}{q\,(1-\mu)}\right)\mathbf{c}_1^2\right]. \tag{4.4}$$

Further computations give

$$\begin{aligned}
&\left|a_3-\Psi a_2^2\right|\le\frac{\Gamma(2\alpha+\beta)\,(1+\gamma)\,|P_1|}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\\
&\times\left\{|\mathbf{c}_2|+\left|1+\frac{P_1}{q\,(1-\mu)}\left\{1+\mu q-\Psi\frac{\left(\Gamma(\alpha+\beta)\right)^2(1+q)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\,(1+\gamma)}\right\}\right||\mathbf{c}_1|^2\right\}\\
&\le\frac{2\Gamma(2\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\left[\left(1-|\mathbf{c}_1|^2\right)+\left|1+Be^{-i\vartheta}\cos\vartheta\right||\mathbf{c}_1|^2\right],
\end{aligned}$$

where

$$B=\frac{2(1-\sigma)}{q\,(1-\mu)}\left\{1+\mu q-\Psi\frac{\left(\Gamma(\alpha+\beta)\right)^2(1+q)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\,(1+\gamma)}\right\}.$$

Or, equivalently

$$\left|a_3-\Psi a_2^2\right|\le\frac{2\Gamma(2\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\left[1+\left(\sqrt{1+B(2+B)\cos^2\vartheta}-1\right)|\mathbf{c}_1|^2\right].$$

Let us put

$$F(x,y)=1+\left(\sqrt{1+B(2+B)x^2}-1\right)y^2,$$

where $x=\cos\vartheta,\ y=|\mathbf{c}_1|^2$ and $(x,y)\in[0,1]\times[0,1]$.

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)\times(0,1)$. Therefore, the maximum must be attained at the boundary point. Since $F(x,0)=1$, $F(0,y)=1$ and $F(1,1)=|1+B|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0)=1$ or $F(1,1)=|1+B|$. So,

$$\left|a_3-\Psi a_2^2\right| \leq \frac{2\Gamma(\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\max\left\{1,|1+B|\right\}. \tag{4.5}$$

Case 1 If $\Psi \leq \xi_1$, where ξ_1 is given by (4.1), then $B\geq 0$ implies $|1+B|\geq 1$. Now from (4.5) we find

$$\left|a_3-\Psi a_2^2\right| \leq \frac{2\Gamma(\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\left[1+\frac{2(1-\sigma)(1+\mu q)}{q\,(1-\mu)}-\Psi\frac{2\left(\Gamma(\alpha+\beta)\right)^2(1-\sigma)\,(1+q)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\,(1+\gamma)\,q\,(1-\mu)}\right].$$

Case 2 If $\Psi \geq \xi_2$, where ξ_2 is given by (4.2), then $B\leq -2$. Now from (4.5) we find

$$\left|a_3-\Psi a_2^2\right| \leq \frac{2\Gamma(\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\left[\Psi\frac{2\left(\Gamma(\alpha+\beta)\right)^2(1-\sigma)\,(1+q)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\,(1+\gamma)\,q\,(1-\mu)}-\frac{2(1-\sigma)(1+\mu q)}{q\,(1-\mu)}-1\right].$$

Case 3 If $\xi_1\leq\Psi\leq\xi_2$, then $|1+B|\leq 1$. Now from (4.5) we find

$$\left|a_3-\Psi a_2^2\right| \leq \frac{2\Gamma(\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}.$$

□

In view of Lemma 4.1, the outcomes are sharp for $\varpi(z)=z$ and $\varpi(z)=z^2$ or one of their rotations.

Next, we consider the Fekete-Szegö problem for the class $\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$ with Ψ complex parameter.

Theorem 4.3 *Let $f\in\mathcal{A}$ belong to $\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$ and Ψ be a complex number. Then*

$$\left|a_3-\Psi a_2^2\right| \leq \frac{2\Gamma(\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\max\left\{1,|\Omega|\right\},$$

where

$$\Omega=\frac{2e^{-i\vartheta}\,(1-\sigma)\cos\vartheta}{q\,(1-\mu)}\left\{\Psi\frac{\left(\Gamma(\alpha+\beta)\right)^2(1+q)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\,(1+\gamma)}-\mu q-1\right\}-1. \tag{4.6}$$

Proof Suppose $f\in\mathcal{S}_\vartheta\left(\sigma;\alpha,\beta,\gamma,\mu\right)$. Making use of (4.3) and (4.4), we obtain

$$\begin{aligned}\left|a_3-\Psi a_2^2\right| &\leq \frac{\Gamma(2\alpha+\beta)\,(1+\gamma)\,|P_1|}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\\ &\times\left|\mathbf{c}_2-\left[\frac{P_1}{q\,(1-\mu)}\left\{\Psi\frac{\left(\Gamma(\alpha+\beta)\right)^2(1+q)}{\Gamma(\beta)\Gamma(2\alpha+\beta)\,(1+\gamma)}-\mu q-1\right\}-1\right]\mathbf{c}_1^2\right|\\ &\leq \frac{2\Gamma(2\alpha+\beta)\,(1+\gamma)\,(1-\sigma)\cos\vartheta}{\Gamma(\beta)q\,(1+q)\,(1-\mu)}\left|\mathbf{c}_2-\Omega\mathbf{c}_1^2\right|,\end{aligned}$$

where Ω is given by (4.6). The desired inequality follows by appying Lemma 4.1. □

5. Concluding remarks and future perspectives

In the current paper, we have introduced the family $\mathcal{S}_{\vartheta}(\sigma;\alpha,\beta,\gamma,\mu)$ of $\vartheta-$spirallike functions. We have discussed membership conditions, coefficient estimates, the relation of subordination and the Fekete-Szegö functional. Some corollaries are also obtained. Furthermore, the paper includes several various outcomes for special values of the parameters:

Example 5.1 *If $\mu=0$, the class $\mathcal{S}_{\vartheta}(\sigma;\alpha,\beta,\gamma)$ can be expressed by*

$$\Re\left(e^{i\vartheta}\frac{zD_q\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)}{\mathcal{T}_{\alpha,\beta,\gamma}f(z)}\right)>\sigma\cos\vartheta,$$

where $|\vartheta|<\frac{\pi}{2},\ 0\leq\sigma<1,\ 0\leq\gamma\leq1$.

Example 5.2 *If $q\to1^{-}$, we get the class $\mathcal{SL}_{\vartheta}^{\sigma}(\alpha,\beta,\gamma,\mu)$ can be expressed by*

$$\Re\left(e^{i\vartheta}\frac{z\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)'}{\mu z\left(\mathcal{T}_{\alpha,\beta,\gamma}f(z)\right)'+(1-\mu)\,\mathcal{T}_{\alpha,\beta,\gamma}f(z)}\right)>\sigma\cos\vartheta,$$

where $|\vartheta|<\frac{\pi}{2},\ 0\leq\sigma<1, 0\leq\mu<1,\ 0\leq\gamma\leq1$.

On the other hand, by choosing special values of α,β and γ, we obtain the similar outcomes only for Mittag-Leffler function or only for Wright function. Hence, the other interesting remarks of our main result can be obtained similarly. The related results may be left as a practice for the researchers.

Finally, many other problems like Hankel determinant inequalities can be determined for the class $\mathcal{S}_{\vartheta}(\sigma;\alpha,\beta,\gamma,\mu)$ as a future work. Also, the approach presented here has been extended to establish new subfamilies of univalent functions with the other special functions.

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