Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 746 – 752
© TÜBİTAK
doi:10.3906/mat-2105-60

Non-solvable groups all of whose indices are odd-square-free

Sajjad MAHMOOD ROBATI[1,*], **Roghayeh HAFEZIEH BALAMAN**[2]
[1]Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
[2]Department of Mathematics, Gebze Technical University, P.O.Box 41400, Gebze, Turkey

Received: 18.05.2021 • **Accepted/Published Online:** 06.01.2022 • **Final Version:** 11.03.2022

Abstract: Given a finite group G and $x \in G$, the class size of x in G is called odd-square-free if it is not divisible by the square of any odd prime number. In this paper, we show that if G is a nonsolvable finite group, all of whose class sizes are odd-square-free, then we have some control on the structure of G, which is an answer to the dual of the question mentioned by Huppert in [5].

Key words: Finite groups, nonsolvable groups, conjugacy class, index

1. Introduction

Given a finite group G, a deep-rooted area of research in finite group theory relates to the connection of the structure of the group G and the set of positive integers, which can inherently be associated to G. The set of the conjugacy class sizes, sometimes called indices denoted by $cs(G)$, is one of these sets of positive integers. The main question, which arises in this area of research is which information one can obtain from the size of conjugacy classes. In view of this question, Sylow in 1872 discussed the case where there was information about the size of all conjugacy classes. In contrast, Burnside in 1904 commented on the strong influence of the size of just one conjugacy class on the structure of the group. On the other hand, a group of authors exchanged views about the set of irreducible character degrees of a finite group G, which is denoted by $cd(G)$, and tried to explain how much this set may control the group theoretical structure of G.

In order to progress of the above cases, the authors mostly considered two different cases, regarding solvability or nonsolvability of G. As an example, it is straightforward to observe that if S is a finite solvable group, all of whose irreducible character degrees are prime-square-free numbers, which are not divisible by any prime in $\{2,3,5,7\}$, then $\mathrm{Alt}(7) \times S$ is a nonsolvable group, all of whose irreducible character degrees are prime-square-free. With regard to this example in [7], considering the case where all irreducible character degrees of a given nonsolvable group G are odd-square-free, Huppert and Manz proved that $G \cong \mathrm{Alt}(7) \times S$, where S is a solvable group with the above mentioned properties. Instead, in [8], Lewis considered those finite groups whose character degrees are 4-free. Moreover, in the conclusion section of the paper [5], Huppert posed the following question:

Question 1 *What are those simple groups whose all character degrees are of the form $2^k p_1 p_2 ... p_n$, where p_i s are distinct odd primes and k and n are positive integers?*

*Correspondence: sajjad.robati@gmail.com, mahmoodrobati@sci.ikiu.ac.ir
2010 *AMS Mathematics Subject Classification:* 20E45

This work is licensed under a Creative Commons Attribution 4.0 International License.

As an answer to the above mentioned question, Lewis and White in [9], classified those nonsolvable groups all of whose character degrees are odd-square-free.

It should be mentioned that the dual question has been taken into consideration for the set of class sizes of a finite group. In particular, Chillag and Herzog in [2], clarified that if the class sizes of a finite group G are all prime-square-free, then G is supersolvable.

Motivated by the results in [9], in this paper, we aim to characterize those nonsolvable groups, all of whose class sizes are odd-square-free, which is an answer to the dual of Question 1.

Notation. Suppose n is a positive integer. n is called prime-square-free if it is not divisible by a square of a prime number. In particular, n is called odd-square-free if it is not divisible by a square of an odd prime number. Given a finite group G, we denote the conjugacy class of $x \in G$ in G by x^G, and the set of conjugacy class sizes by $cs(G)$.

2. Simple groups

Let N be a normal subgroup of G. The notable facts that for $x \in N$, $|x^N|$ divides $|x^G|$ and if $x \in G$, then $|xN^{G/N}|$ divides $|x^G|$, showing that, in particular, if G is a finite group all of whose class sizes are odd-square-free, then so are N and G/N. We proceed by expressing some results on nonabelian finite simple groups.

According to the classification of finite simple groups, it is a well-known result that if S is a nonabelian finite simple group, then S is an alternating group $\mathrm{Alt}(n)$ where $n \geq 5$, or a classical group of Lie type, or an exceptional group of Lie type, or a sporadic simple group. In this section, considering these four cases, we classify those finite nonabelian simple groups, all of whose class sizes are odd-square-free.

Lemma 2.1 *Suppose $S \cong \mathrm{Alt}(n)$, where $n \geq 5$. All class sizes of S are odd-square-free if and only if $S \cong \mathrm{Alt}(5)$.*

Proof First, suppose $n \geq 10$. Let $\alpha := (1\,2\,3\,4\,5)(6\,7)(8\,9)$. As a consequence of a famous result, we obtain that

$$|\alpha^{\mathrm{Alt}(n)}| = \frac{n!}{1^{n-9}(n-9)!2^2 2!5^1 1!} = \frac{n(n-1)...(n-8)}{2^3 5}$$

Therefore, we have the following three cases where for an integer k, $(k)_3$ denotes the largest 3-power, which divides k:

- If $n \equiv 0, \quad \mod(3)$, then $(n(n-3)(n-6))_3 \geq 3^3$;
- If $n \equiv 1, \quad \mod(3)$, then $((n-1)(n-4)(n-7))_3 \geq 3^3$;
- If $n \equiv 2, \quad \mod(3)$, then $((n-2)(n-5)(n-8))_3 \geq 3^3$;

This confirms that, for $n \geq 10$, the alternating group $\mathrm{Alt}(n)$ includes a permutation whose index is not odd-square-free. Furthermore, the use of GAP with $n < 10$ proves the claim. □

Lemma 2.2 *Let S be a sporadic simple group all of whose class sizes are odd-square-free. Then $S \cong J_1$.*

Proof Let r be the largest prime divisor of $|S|$. By the application of the results in ATLAS of finite groups [3], we conclude that S has a cyclic Sylow r-subgroup, say R, which is self-centralizer. Considering a as a generator of the cyclic group R, it can be noticed that the index of a in S is odd-square-free if and only if $|S|$ is odd-square-free. Therefore, $S \cong J_1$. □

Lemma 2.3 *Let S be a classical group of Lie type. The class sizes of S are odd-square-free if and only if S is isomorphic to one of the following groups:*

(i) $\mathrm{PSL}(2,q)$*, where $q \geq 5$ is an odd prime and q^2-1 is odd-square-free.*

(ii) $\mathrm{PSL}(2,2^k)$*, where $n \geq 2$ and $2^{2n}-1$ is prime-square-free.*

Proof It is a long-familiar fact that while q is odd and $q \equiv \varepsilon \bmod (4)$, we have

$$cs(\mathrm{PSL}(2,q)) = \{1, (q^2-1)/2, q(q+\varepsilon)/2, q(q-\varepsilon)/2, q(q+\varepsilon)\}$$

Suppose the class sizes of S are all odd-square-free. The matter that q is a divisor of some class sizes of S determines that q is a prime. Besides, as $gcd(q^2-1, q) = 1$ and $\varepsilon = \pm 1$, we come to the conclusion that q^2-1 is odd-square-free.

Considering the case where q is even, we have

$$cs(\mathrm{PSL}(2,q)) = \{1, q^2-1, q(q+1), q(q-1)\}$$

Seeing that q^2-1 is an odd number, it can be verified that the class sizes of S are odd-square-free if and only if q^2-1 is prime-square-free.

If S is not isomorphic to $\mathrm{PSL}(2,q)$, the application of Theorem 4.1 and Lemma 4.6 [1] shows the existence of a cyclic subgroup T of S with the property that for some $g \in T \setminus \{1\}$, we have $T = \mathrm{C}_S(g)$. Now, we should deal with the following eight cases:

- If $S \cong \mathrm{PSL}(3,q)$, where $q \geq 3$, then $|T| = \frac{q^3-1}{(q-1)gcd(3,q-1)}$, which implies that $q^3(q-1)^2$ divides some conjugacy class size of S.

- If $S \cong \mathrm{PSL}(n,q)$, where $n \geq 4$, then $|T| = \frac{q^n-1}{(q-1)gcd(n,q-1)}$, which implies that $q^3(q-1)^2$ divides some conjugacy class size of S.

- If $S \cong \mathrm{PSU}(3,q)$, where $q > 2$, then $|T| = \frac{q^3+1}{(q+1)gcd(3,q+1)}$, which implies that $q^3(q+1)^2$ divides some conjugacy class size of S.

- If $S \cong \mathrm{PSU}(n,q)$, where $n \geq 4$, then $|T| = \frac{q^n+(-1)^{n-1}}{(q+1)gcd(n,q+1)}$, $q^3(q+1)^2$ divides some conjugacy class size of S.

- If $S \cong \mathrm{PSP}(2m,q)$, where $m \geq 2$ and $(m,q) \neq (2,2)$, then $|T| = \frac{q^m-1}{gcd(2,q-1)}$, which implies that $q^4(q^2-1)^2$ divides some conjugacy class size of S.

- If $S \cong \mathrm{P\Omega}(2m+1,q)$, where $m \geq 3$ and q is odd, then $|T| = \frac{q^m+1}{2}$, which implies that $q^4(q^2-1)^2$ divides some conjugacy class size of S.

- If $S \cong P\Omega^+(2m,q)$, where $m \geq 4$, $m \equiv 0(2)$, then $|T| = \frac{(q^{m-1}+1)(q+1)}{(2,q-1)^2}$, which implies that $q^4(q^2-1)^2$ divides some conjugacy class size of S.
- If $S \cong P\Omega^+(2m,q)$, where $m \geq 5$, $m \equiv 1(2)$, then $|T| = \frac{(q^{m-1}+1)(q+1)}{(4,q^m-1)}$, which implies that $q^4(q^2-1)^2$ divides some conjugacy class size of S.
- If $S \cong P\Omega^-(2m,q)$, where $m \geq 4$, then $|T| = \frac{q^m+1}{(4,q^m+1)}$, which implies that $q^4(q^2-1)^2$ divides some conjugacy class size of S.

Considering these cases, we notice that none of the above groups satisfy the odd-square-free condition on its class sizes, which proves that the case that S is not isomorphic to $\mathrm{PSL}(2,q)$ is impossible, and the proof is now complete. □

Lemma 2.4 *Suppose S is an exceptional group of Lie type. The class sizes of S are all odd-square-free if and only if $S \cong^2 B_2(q)$, where $q = 2^{2m+1} \geq 8$ and $(q-1)(q^2-1)$ is prime-square-free.*

Proof The results in [10] confirms the truth that while $S \cong^2 B_2(q)$ we have:

$$cs(S) = \{q^2(q^2+1), q^2(q-1)(q-\sqrt{2q}+1), q^2(q-1)(q+\sqrt{2q}+1)\}$$

Since $gcd(q-1, q^2+1) = 1$ and $q^2+1 = (q-\sqrt{2q}+1)(q+\sqrt{2q}+1)$, it can be deduced that the condition that all class sizes of S are odd-square-free occurs if and only if the odd number $(q-1)(q^2+1)$ is prime-square-free.

Otherwise, if S is not isomorphic with ${}^2B_2(q)$, Theorem 3.1 of [1] shows the existence of a cyclic subgroup of S, say T, such that, for some $g \in T \setminus \{1\}$ we have $T = \mathrm{C}_S(g)$. Now considering the orders of such cyclic subgroups, we have the following cases, which demonstrates that this event will not occur:

- If $S \cong^2 D_4(q)$, then $|T| = q^4 - q^2 + 1$, which implies that $q^{12}(q-1)^2$ divides some class sizes of S.
- If $S \cong G_2(q)$, where $q \geq 3$, then $|T| = q^2 + q + 1$, which implies that $q^6(q-1)^2$ divides some class sizes of S.
- If $S \cong^2 G_2(q)$, where $q = 3^{2k+1} \geq 27$, then $|T| = q + \sqrt{3q} + 1$, which implies that q^3 divides some class sizes of S.
- If $S \cong F_4(q)$, then $|T| = q^4 - q^2 + 1$, which implies that $q^{24}(q-1)^4$ divides some class sizes of S.
- If $S \cong^2 F_4(q)$, where $q = 2^{2k+1} \geq 8$, then $|S| = q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1$, which implies that $q^{12}(q-1)^2$ divides some class sizes of S.
- If $S \cong E_6(q)$, then $|T| = \frac{q^6+q^3+1}{gcd(3,q-1)}$, which implies that $q^{36}(q-1)^4$ divides some class sizes of S.
- If $S \cong^2 E_6(q)$, then $|T| = \frac{q^6-q^3+1}{gcd(3,q-1)}$, which implies that $q^{36}(q^2-1)^2$ divides some class sizes of S.
- If $S \cong E_7(q)$, then $|T| = \frac{(q^7-1)}{(2,q-1)}$, which implies that $q^{63}(q-1)^4$ divides some class sizes of S.
- If $S \cong E_8(q)$, then $|T| = q^8 - q^4 + 1$, which implies that $q^{120}(q-1)^8$.

- If $S \cong {}^2F_4(2)'$, then $|T| = 13$, which implies that $2^3 3^2$ divides some class sizes of S.

The proof is complete now. □

Theorem 2.5 *Let S be a finite nonabelian simple group. Then the class sizes of S are all odd-square-free if and only if one of the following statements holds:*

(1) $S \simeq \mathrm{Alt}(5)$;

(2) $S \simeq J_1$;

(3) $S \simeq \mathrm{PSL}(2, q)$, where $q = p \geq 5$ is an odd prime and $q^2 - 1$ is odd-square-free.

(4) $S \simeq \mathrm{PSL}(2, 2^n)$, where $n \geq 2$ and $2^{2n} - 1$ is prime-square-free.

(5) $S \simeq {}^2B_2(q)$, where $q = 2^{2m+1} \geq 8$ and $(q-1)(q^2-1)$ is prime-square-free.

Proof Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 2.4 finalizes the proof. □

Considering the orders of finite nonabelian simple groups and the results in Theorem 2.5, it is straightforward to deduce the following corollary:

Corollary 2.6 *Suppose S is a finite nonabelian simple group. The class sizes of S are all odd-square-free if and only if $|S|$ is odd-square-free.*

Corollary 2.7 *Suppose S is a finite nonabelian simple group, which is not isomorphic to* $\mathrm{Alt}(7)$. *All class sizes of S are odd-square-free if and only if all irreducible character degrees of S are odd-square-free.*

Proof Theorem 2.5 and Theorem A of [9] complete the proof. □

3. Main theorems

Lemma 3.1 *Let G be a finite group all of whose class sizes are odd-square-free. If M and N are normal subgroups of G such that $N > M$ and N/M is a non-abelian chief factor, then N/M is a simple group all of whose class sizes are odd-square-free.*

Proof Since the order of each conjugacy class of N/M divides the order of a conjugacy class of G, then all class sizes of N/M are odd-square-free. There is a non-abelian simple group S such that $N/M \cong S_1 \times ... \times S_n$ and $S_i \cong S$ for each i.

Obviously there is a class size of S that is divisible by an odd prime q. If $n \geq 2$, then q^2 divides some class size in N/M. Thus, we must have $n = 1$, and the lemma is proved. □

Theorem 3.2 *Suppose that G is a finite group all of whose class sizes are odd-square-free and such that all minimal normal subgroups of G are non-abelian. Then we have one of the following cases:*

(i) G has a unique minimal normal subgroup S and S is a simple group all of whose class sizes are odd-square-free. In particular, $S \leq G \leq \mathrm{Aut}(S)$;

(ii) *G has exactly two minimal normal subgroups S and T, these are simple groups all of whose class sizes are odd-square-free and $\pi(S) \cap \pi(T) = \{2\}$. In particular, $S \times T \leq G \leq \mathrm{Aut}(S) \times \mathrm{Aut}(T)$.*

Proof Let N be a minimal normal subgroup of G. Then N is a simple group, all of whose class sizes are odd-square-free by Lemma 3.1. Let $C := \mathrm{C}_G(N)$. Then $C \cap N = 1$ and C is normal in G. If M is another minimal normal subgroup of G, then $M \cap N = 1$ and $M \leq C$. If $C = 1$, then G is an almost simple group with socle N and (i) holds with $N = S$. Suppose now that $C \neq 1$. Then there exists a minimal normal subgroup of G, say M, contained in C. Similar to the case of N, we can observe that M is one of the non-abelian simple groups mentioned in Theorem 2.5. Let $D := \mathrm{C}_G(M)$. Likewise, one can see that $D \cap M = 1$, $D \lhd G$, G/D can be embedded in $\mathrm{Aut}(M)$, and $N \subseteq D$. The fact that $N \times M \lhd G$, leads to the conclusion that all nontrivial class sizes of $N \times M$ must be odd-square-free. Thus, $\pi(N) \cap \pi(M) = \{2\}$. It can be viewed that $G/C \cap D$ is isomorphic with a subgroup of $\mathrm{Aut}(N) \times \mathrm{Aut}(M)$. For this reason, if $C \cap D = 1$,

$$N \times M \leq G \leq \mathrm{Aut}(N) \times \mathrm{Aut}(M)$$

, which implies part (ii) for $S := N$ and $T := M$.

We state that the case $C \cap D \neq 1$ is impossible. Otherwise, $C \cap D$ contains a minimal normal subgroup of G, say T. In the manner of the previous cases, we can observe that $N \times M \times T \lhd G$, where T is one of the simple groups mentioned in Theorem 2.5. Using the classification of finite simple groups, we know that every nonabelian simple group has a class, which is divisible by 3 or 5. This assures the existence of a class in G whose size is divisible by 3^2 or 5^2, which is a contradiction. Hence, $C \cap D = 1$ which completes the proof. □

Theorem 3.3 *Let G be a finite nonsolvable group, all of whose class sizes are odd-square-free. Then, G has normal subgroups N and R such that R is a solvable group whose class sizes are all odd-square-free, and one of the following holds for $\overline{G} := G/R$:*

(i) *$\overline{G}$ is an almost simple group with socle $\overline{N}$, which is a simple group all of whose class sizes are odd-square-free and is the only minimal normal subgroup of $\overline{G}$ and*

$$\overline{N} \leq \overline{G} \leq \mathrm{Aut}(\overline{N}),$$

(ii) *$\overline{N} = S \times T$, where S, T are simple groups all of whose class sizes are odd- square-free, are the only non-abelian chief factors of G, and $\pi(S) \cap \pi(T) = \{2\}$. Moreover*

$$S \times T \leq \overline{G} \leq \mathrm{Aut}(S) \times \mathrm{Aut}(T).$$

Proof Let R be the solvable radical of G which is maximal among solvable normal subgroups of G. The condition on class sizes of G clarifies that both R and $\overline{G} := G/R$ are groups whose all class sizes are odd-square-free. Moreover, it can be seen that every minimal normal subgroup of $\overline{G}$ is non-abelian. Now Theorem 3.2 implies that either $\overline{G}$ is an almost simple group whose socle which is its unique minimal normal subgroup is isomorphic with one of the groups mentioned in Theorem 2.5; or, $\overline{G}$ has exactly two minimal normal subgroups, say $\overline{S}$ and $\overline{T}$ with the following properties:

(i) $\overline{S}$ and $\overline{T}$ are one of the groups mentioned in Theorem 2.5;

(ii) $\pi(\overline{S}) \cap \pi(\overline{T}) = \{2\}$;

(ii) $\overline{S} \times \overline{T} \leq \overline{G} \leq \mathrm{Aut}(\overline{S}) \times \mathrm{Aut}(\overline{T})$.

□

Acknowledgment

The authors are grateful to the referee for the valuable comments.

References

[1] Babai L, Palfy PP, Saxl J. On the number of p-regular elements in finite simple groups. LMS journal of Computation and Mathematics 2009; 12: 82-119. doi: 10.1112/S1461157000000036

[2] Chillag D, Herzog M. On the length of the conjugacy classes of finite groups. Journal of Algebra 1990; 131 (1): 110-125. doi: 10.1016/0021-8693(90)90168-N

[3] Conway JH, Curtis RT, Norton SP, Parker RA, Wilson RA. ATLAS of finite groups, Clarendon Press, Oxford, 1985.

[4] The GAP group, GAP-Groups, Algorithms and Programming, version 4.7.4, 2014.

[5] Huppert B. Some simple groups which are determined by the set of their character degrees II. Rendiconti del Seminario Matematico della Università di Padova 2006; 115: 1-113.

[6] Huppert B. Endliche Gruppen, Springer-Verlag, Berlin, 1967.

[7] Huppert B, Manz O. Degree-problems I: Square-free character degrees. Archive der Mathematics 1985; 45: 125-132. doi: 10.1007/BF01270483

[8] Lewis ML. Generalizing a theorem of Huppert and Manz. Journal of Algebra and its Applications 2007; 6: 687-695. doi: 10.1142/S0219498807002442

[9] Lewis ML, White DL. Non-solvable groups all of whose character degrees are odd-square-free. Communication in Algebra 2011; 39: 1273-1292. doi: 10.1080/00927871003652652

[10] Suzuki M. On a class of double transitive groups. Annals of Mathematics 1962; 75: 105-145. doi: 10.2307/1970423