Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2022) 46: 768 – 784
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doi:10.3906/mat-2110-44

Some notes on crossed semimodules

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Received: 11.10.2021 • **Accepted/Published Online:** 10.01.2022 • **Final Version:** 11.03.2022

Abstract: In this paper, we introduce the notion of lifting via a homomorphism of monoids for a crossed semimodule and give some properties. Further, we characterize actions and coverings of Schreier internal categories in the category Mon of monoids and prove the natural equivalence between their categories. Then, we prove that liftings of a certain crossed semimodule are naturally equivalent to the actions of Schreier internal category in Mon, where the Schreier internal category corresponds to the crossed semimodule. Finally, we give a relation between crossed semimodules and simplicial monoids.

Key words: Crossed semimodule, Schreier internal category, covering, action, lifting, simplicial monoid

1. Introduction

Crossed modules were defined by Whitehead in [30, 31] as an algebraic model for homotopy 2-types and have been used with several branches of mathematics such as homotopy theory [12], homological algebra [18], and algebraic K-theory [20]. The categorical equivalence between crossed modules and internal categories in the category of groups (which are known under the names 2-groups [7], group-groupoids or $\mathcal{G}-$groupoids [11]) is well-known. This equivalence was generalized in different ways: for instance, by taking other type of internal categories like Schreier internal categories in the category Mon of monoids by Patchkoria [25] or even by taking 2-groupoids with one object [6] (see also [28]). In [22], the Patchkoria's result was extended to a more general case of an internal category in an algebraic category of monoid with operations, which includes monoids, commutative monoids, semirings, join-semilattices with a bottom element and distributive lattices with a bottom element. This study generalizes, at the same time, Porter's result [26] concerning groups with operations. Patchkoria also defined Schreier internal groupoids in Mon and prove that they are naturally equivalent to the category of crossed semimodules $\mathcal{C} = (A, B, \partial)$ where A is a group.

A simplicial object in an arbitrary category $\mathcal{C}$ is a functor from the opposite of the simplicial indexing category to $\mathcal{C}$. A simplicial group is a simplicial object in the category Gp of groups. Crossed modules over groups which model connected homotopy 2-types were known to be equivalent to that of simplicial groups whose Moore complex has length 1 [21]. In [14], Conduché introduced the notion of 2-crossed modules as a model of connected homotopy 3-types and he proved in [15] the relations between crossed squares (i.e., crossed 2-cubes), 2-crossed modules and simplicial groups having a Moore complex of length 2. Arvasi et al. extended these studies to the notions of 3-crossed modules [2] (see also [3–5]). For further properties of simplicial structures, see [16] and [23].

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2010 *AMS Mathematics Subject Classification:* 20L05, 18G30, 18B40, 20L99, 18B99

The notions of covering of a groupoid and of action of a groupoid over a set have been studied by Brown et al. because of their connections with algebraic topology (see, for example, [9, 17]). For another applications in category theory on coverings and actions, see Borceux and Janelidze's book on Galois theories [8]. It is well-known that for a groupoid $\mathcal{G}$, the category $Act(\text{GPD})/\mathcal{G}$ of groupoid actions of $\mathcal{G}$ on sets (which are also called $\mathcal{G}$-sets or operations) is equivalent to the category $Cov(\text{GPD})/\mathcal{G}$ of covering groupoids of $\mathcal{G}$ (see [13] for topological aspect of this equivalence). The same is true where $\mathcal{G}$ is a category. Also Brown & Mucuk [10] proved that if $\mathcal{G}$ is a group-groupoid, then the category $Cov(\text{GPGD})/\mathcal{G}$ of group-groupoid coverings of $\mathcal{G}$ is equivalent to the category $Act(\text{GPGD})(\mathcal{G})$ of group-groupoid actions of $\mathcal{G}$ on groups. Unfortunately, action group-groupoids and covering group-groupoids have very complicated and difficult structures. But fortunately, recently, Mucuk & Şahan [24] have described the notion of action group-groupoid in the category of crossed modules using the well known categorical equivalence between group-groupoids and crossed modules. They called this new structure "lifting crossed module". Lifting crossed modules have much more easier structures than action group-groupoids and are much more understandable. See also [27] for further reading on lifting of crossed modules over groups.

This manuscript is devoted to introduce the notion of coverings and actions of Schreier internal categories and Schreier internal groupoids in MON and prove a categorical equivalence using underlying categories. Then, we define liftings of crossed semimodules and prove a categorical equivalence between liftings of crossed semimodules and actions of Schreier internal categories in MON using the equivalence between categories of crossed semimodules and of Schreier internal categories. In Section 5, we give an equivalence between crossed semimodules and simplicial monoids under some conditions. Hence, we generalize the equivalence between crossed modules over groups and simplicial groups whose Moore complex has length 1.

2. Preliminaries

A category $\mathcal{C} = (C_0, C)$ consists of a class C_0 of objects, a class $C = \cup_{x,y\in C_0} C(x,y)$ of morphisms where $C(x,y)$ is the class of morphisms from x to y as

$$x \xrightarrow{c} y$$

with the source and the target maps $d_0, d_1 \colon C \to C_0$, respectively, such that $d_0(c) = x, d_1(c) = y$, the associative composition map $C(y,z) \times C(x,y) \to C(x,z),\ (d,c) \mapsto d \circ c$ and the unit map $\varepsilon \colon C_0 \to C,\ x \mapsto 1_x \in C(x,x)$ such that $c \circ 1_x = c$ and $1_x \circ c' = c'$, where $d_1(c') = x$. We write $St_{\mathcal{C}}x$ for $d_0^{-1}(x)$ and call it the star of $\mathcal{C}$ at $x \in C_0$.

A groupoid $\mathcal{G} = (G_0, G)$ is a category in which all morphisms are invertible. For further details, see [9, 19].

An internal category is a category object in a category, which has all finite products, pullbacks, and a terminal object. For further details, see [19].

Let $\mathcal{M} = (M_0, M)$ be an internal category in the category MON of monoids. If for any $c \in M$ there exists a unique $\widehat{c} \in \ker d_0$ such that

$$c = \widehat{c} \cdot 1_{d_0(c)},$$

then $\mathcal{M}$ is called a *Schreier internal category* in MON and this condition is called *the Schreier condition* [25].

In a Schreier internal category, the monoid product is an internal functor giving the following interchange rule

$$(d \circ c) \cdot (d' \circ c') = (d \cdot d') \circ (c \cdot c') \tag{2.1}$$

whenever compositions are defined. A *Schreier internal groupoid* in Mon is a Schreier internal category whose all morphisms are invertible.

Recall the definition of a crossed semimodule from [25]. A crossed semimodule $\mathcal{C} = (A, B, \partial)$ consists of a pair of monoids A, B and a homomorphism $\partial\colon A \to B$ of monoids with an action $\bullet\colon B \times A \to A$ of B on A satisfying

[CSM 1] $\partial(b \bullet a)b = b\partial(a)$,

[CSM 2] $(\partial(a) \bullet a_1)a = aa_1$, for $a, a_1 \in A$ and $b \in B$.

Let $\mathcal{C} = (A, B, \partial)$ and $\mathcal{C}' = (A', B', \partial')$ be crossed semimodules. A crossed semimodule morphism is a mapping $\lambda = \langle \lambda_1, \lambda_2 \rangle\colon \mathcal{C} \to \mathcal{C}'$ where $\lambda_1\colon A \to A'$ and $\lambda_2\colon B \to B'$ are monoid homomorphisms such that $\lambda_1(b \bullet a) = \lambda_2(b) \bullet' \lambda_1(a)$ and $\lambda_2\partial = \partial'\lambda_1$. Hence, crossed semimodules and their morphisms form a category denoted by Csm.

The following theorem was proved by Patchkoria in [25]. Since we need some details of the proof, we give a sketch proof in terms of our notation.

Theorem 2.1 *The category Sic of Schreier internal categories in Mon is equivalent to the category Csm of crossed semimodules.*

Proof A functor $\delta\colon \text{Sic} \to \text{Csm}$ is defined as an equivalence of categories. Given a Schreier internal category $\mathcal{M} = (M_0, M)$ in Mon, then $\delta(\mathcal{M}) = (A, B, \partial)$ is a crossed semimodule where $A = \ker d_0, B = M_0, \partial = d_1|_{\ker d_0}$ and action of M_0 on $\ker d_0$ is defined by $(x \bullet c)1_x = 1_x c$, for all $x \in M_0$ and $c \in \ker d_0$.

A functor $\gamma\colon \text{Csm} \to \text{Sic}$ is defined as a weak inverse of δ. Given a crossed semimodule (A, B, ∂), then $\gamma(A, B, \partial) = (B, B \ltimes A)$ is a Schreier internal category in Mon where $B \ltimes A$ is the semi-direct product of monoids with the multiplication $(b, a)(b', a') = (bb', a(b \bullet a'))$. The structure maps are given by $d_0(b, a) = b$, $d_1(b, a) = \partial(a)b$, $\varepsilon(b) = (b, e_A)$ where e_A is the identity element of A, and the composition of morphisms is defined by $(\partial(a)b, a_1) \circ (b, a) = (b, a_1 a)$.

In order to prove $1_{\text{Sic}} \cong \gamma\delta$, we must define a natural transformation $\eta\colon \mathcal{M} \to \gamma\delta(M)$ via a mapping $\eta_{\mathcal{M}}$ such that $\eta_{\mathcal{M}}(c) = (d_0(c), \widehat{c})$. In order to define a natural transformation $\mu\colon 1_{\text{Csm}} \to \delta\gamma$, a map $\mu_{(A,B,\partial)}$ is defined to be identity on B where e_B is the identity element of B and is defined by $a \mapsto (e_B, a)$ on A. □

The following results can be obtained as restrictions of this equivalence. Let Csm* be the category of crossed semimodules (A, B, ∂) where A is a group.

Corollary 2.2 *[25] The category Sig of Schreier internal groupoids in Mon is equivalent to Csm*.*

Theorem 2.3 *[11] The category of group-groupoids (2-groups) is equivalent to the category of crossed modules.*

2.1. Coverings and actions of categories

Some properties of covering categories are examined in [17]. In this section, we shall define actions of categories in terms of our notations.

Definition 2.4 *Let* $\mathcal{C}, \widetilde{\mathcal{C}}$ *be two categories and* $p\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ *be a functor. If for each* $\widetilde{x} \in \widetilde{C}_0$ *the restriction*

$$St_{\widetilde{\mathcal{C}}}(\widetilde{x}) \to St_{\mathcal{C}}p(\widetilde{x})$$

is bijective, then p *is called covering functor and* $\widetilde{\mathcal{C}}$ *is called a covering category of* $\mathcal{C}$*.*

Example 2.5 *[17] Let* T *be a trivial category and* A *be an arbitrary category. Then, the projection* $A \times T \to A$ *is a covering functor.*

Definition 2.6 *Let* $p\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ *and* $q\colon \widetilde{\mathcal{C}}' \to \mathcal{C}$ *be two covering morphisms. A morphism* $\widetilde{p}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}'$ *such that* $q\widetilde{p} = p$ *is called a morphism of coverings of* $\mathcal{C}$*. Hence, coverings of* $\mathcal{C}$ *and their morphisms form a category denoted by Cov(*$\textsc{Cat}$*)/*$\mathcal{C}$*.*

Definition 2.7 *Let* $\mathcal{C}$ *be a category,* S *be a set and* $\omega\colon S \to C_0$ *be a map. An action of* $\mathcal{C}$ *on* S *via* ω *is a mapping*

$$C_{d_0} \times_\omega S \to S, \quad (c,s) \mapsto c \centerdot s$$

where $C_{d_0} \times_\omega S = \{(c,s) | d_0(c) = \omega(s)\}$ *satisfying following conditions*

[AC 1] $\omega(c \centerdot s) = d_1(c)$,

[AC 2] $1_{\omega(s)} \centerdot s = s$,

[AC 3] $(d \circ c) \centerdot s = d \centerdot (c \centerdot s)$

whenever $d \circ c$ *and* $c \centerdot s$ *are defined. This action of* $\mathcal{C}$ *on* S *via* ω *is denoted by* (S, ω)*. We also say* $\mathcal{C}$ *acts on* S *via* ω *or* S *is a* $\mathcal{C}$*-set.*

Given such an action, the semidirect product category $(S, S \ltimes C)$ is defined to be a category whose morphisms are the pairs (s,c) as follows.

$$s \xrightarrow{(s,c)} c \centerdot s$$

The composition of morphisms is defined by $(c \centerdot s, d) \circ (s,c) = (s, d \circ c)$. Given such a category, the projection $p\colon S \ltimes C \to C$, $(s,c) \mapsto c$ is a covering functor where p is given on objects by $\omega\colon S \to C_0$. For further details, see [9].

Definition 2.8 *Let* $\mathcal{C}$ *acts on* S *and* S' *via* ω *and* ω'*, respectively. A morphism* $f\colon (S,\omega) \to (S', \omega')$ *of such actions is a map* $f\colon S \to S'$ *such that* $\omega' f = \omega$ *and* $f(c \centerdot s) = c \centerdot f(s)$ *whenever* $c \centerdot s$ *is defined.*

Hence, actions of $\mathcal{C}$ on sets and their morphisms form a category denoted by $Act(\textsc{Cat})/\mathcal{C}$.

Theorem 2.9 *Let* $\mathcal{C}$ *be a category. Then, the categories Cov(*$\textsc{Cat}$*)/*$\mathcal{C}$ *and Act(*$\textsc{Cat}$*)/*$\mathcal{C}$ *are equivalent.*

Proof A functor $\theta\colon Act(\textsc{Cat})/\mathcal{C} \to Cov(\textsc{Cat})/\mathcal{C}$ is an equivalence of categories. Given an object (S,ω) of $Act(\textsc{Cat})/\mathcal{C}$, then $\theta(S,\omega) = (S, C \ltimes S, p)$. Here, $(S, C \ltimes S)$ is a category with the set S of objects and the set $C \ltimes S = \{(c,s)|d_0(c) = \omega(s)\}$ of morphisms where the source and the target maps are defined by $d_0(c,s) = s,\ d_1(c,s) = c \centerdot s$, respectively, and the composition map is given by

$$(c_1, s_1) \circ (c, s) = (c_1 \circ c, s)$$

whenever $s_1 = c \centerdot s$. The identity map is defined by $\varepsilon(s) = (1_{\omega(s)}, s)$. On the other hand, $(S, C \ltimes S, p)$ is covering category of $\mathcal{C}$ with the functor $p\colon (S, C \ltimes S) \to (C_0, C)$ which is defined on objects by $p_0(s) = \omega(s)$ and the projection $p(c,s) = \pi_1(c,s) = c$ on morphisms.

Given a mophism $f\colon (S,\omega) \to (S',\omega')$ of $Act(\textsc{Cat})/\mathcal{C}$, then $\theta(f) = (f, 1_C \times f)$ is a morphism of $Cov(\textsc{Cat})/\mathcal{C}$.

Let $p = (p_0, p)\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ be a covering functor. We define a functor

$$\psi\colon Cov(\textsc{Cat})/\mathcal{C} \to Act(\textsc{Cat})/\mathcal{C}$$

as a weak inverse of θ such that $\psi(\widetilde{\mathcal{C}}, p) = (\widetilde{C_0}, p_0)$ and $\psi(\widetilde{p}) = \widetilde{p}_0$ where $\widetilde{p} = (\widetilde{p}_0, \widetilde{p})$ is a morphism of $Cov(\textsc{Cat})/\mathcal{C}$. We will verify that $\psi(\widetilde{\mathcal{C}}, p)$ is an object of $Act(\textsc{Cat})/\mathcal{C}$ where action of $\mathcal{C}$ on $\psi(\widetilde{\mathcal{C}}, p)$ is defined by $c \centerdot \widetilde{x} = d_1(\widetilde{c})$, for $x \xrightarrow{c} y$, $\widetilde{x} \xrightarrow{\widetilde{c}} \widetilde{y}$ and $p(\widetilde{c}) = c$.

[AC 1] $p_0(c \centerdot \widetilde{x}) = p_0 d_1(\widetilde{c}) = p_0(\widetilde{y}) = y = d_1(c)$

[AC 2] $1_{p_0(\widetilde{x})} \centerdot \widetilde{x} = 1_x \centerdot \widetilde{x} = d_1(\widetilde{1_x}) = \widetilde{x}$,

[AC 3] $(d \circ c) \centerdot \widetilde{x} = d_1(\widetilde{d} \circ \widetilde{c}) = d_1(\widetilde{d}) = d \centerdot (c \centerdot \widetilde{x})$.

It is easy to check that $\psi\theta \cong 1$. To show $1 \cong \theta\psi$, a natural equivalence $\xi\colon 1_{Cov(\textsc{Cat})/\mathcal{C}} \to \theta\psi$ is defined via a mapping $\xi_{\widetilde{C}}$ to be identity on objects, and $\xi_{\widetilde{C}}(\widetilde{c}) = (p(\widetilde{c}), d_0(\widetilde{c}))$ on morphisms. Since p is bijective, $\xi_{\widetilde{C}}^{-1}$ is defined. We also verify that $\xi_{\widetilde{C}}$ preserves the composition.

$$\xi_{\widetilde{C}}(\widetilde{d} \circ \widetilde{c}) = (p(\widetilde{d} \circ \widetilde{c}), d_0(\widetilde{d} \circ \widetilde{c})) = (p(\widetilde{d}) \circ p(\widetilde{c}), d_0(\widetilde{c})) = (p(\widetilde{d}), d_0(\widetilde{d})) \circ (p(\widetilde{c}), d_0(\widetilde{c})) = \xi_{\widetilde{C}}(\widetilde{d}) \circ \xi_{\widetilde{C}}(\widetilde{c})$$

This completes the proof. □

3. Actions and coverings of Schreier internal categories

It was proved in [10] that, for a group-groupoid $\mathcal{G}$, the category $Act(\textsc{GpGpd})/\mathcal{G}$ of group-groupoid actions of $\mathcal{G}$ on sets is equivalent to the category $Cov(\textsc{GpGpd})/\mathcal{G}$ of covering group-groupoids of $\mathcal{G}$. In this section, we generalize this result to the notions of Schreier internal categories in MON.

Definition 3.1 *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be two Schreier internal categories in* MON *and $p\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ be a morphism of Schreier internal categories. If p is a covering functor of underlying categories, then $\widetilde{\mathcal{M}}$ is called covering Schreier internal category of $\mathcal{M}$.*

Definition 3.2 *Given a Schreier internal category $\mathcal{M}$ in* Mon*, an action of $\mathcal{M}$ on a monoid $(S,+)$ via a homomorphism $\omega\colon S \to M_0$ of monoids is defined by an action of the underlying category $\mathcal{M}$ on the underlying set S satisfying the following interchange rule*

$$(c \centerdot s) + (c' \centerdot s') = (c \cdot c') \centerdot (s + s') \tag{3.1}$$

whenever both sides are defined.

Proposition 3.3 *Let $\mathcal{M}$ be a Schreier internal category in* Mon *with an action on a monoid $(S,+)$ via ω satisfying the following interchange rule*

$$(c \centerdot s) + (c' \centerdot s') = (c \cdot c') \centerdot (s + s')$$

whenever both sides are defined. Then,

[1] the semidirect product category $(S, S \ltimes M)$ is a Schreier internal category in Mon *with the product*

$$(s,c) \odot (s',c') = (s+s', c \cdot c'),$$

[2] the projection $p\colon S \ltimes M \to M$ is a covering Schreier internal category of $\mathcal{M}$.

Proof

[1] The pairs (s,c) are morphisms of the semidirect product category $(S, S \ltimes C)$ as follows.

$$s \xrightarrow{(s,c)} c \centerdot s$$

whenever the composition of morphisms is defined by $(c \centerdot s, d) \circ (s,c) = (s, d \circ c)$. The structure maps are defined as $d_0(s,c) = s, d_1(s,c) = c \centerdot s$ and $1_s = (s, 1_{\omega(s)})$. Using 2.1, we obtain the interchange rule for the semidirect product category $(S, S \ltimes M)$ as follows. Let $r = c \centerdot s, r' = c' \centerdot s'$. Then,

$$\begin{aligned}
\Big[(r,d) \odot (r',d')\Big] \circ \Big[(s,c) \odot (s',c')\Big] &= (r+r', d \cdot d') \circ (s+s', c \cdot c') \\
&= \Big(s+s', (d \cdot d') \circ (c \cdot c')\Big) \\
&= \Big(s+s', (d \circ c) \cdot (d' \circ c')\Big) \\
&= (s, d \circ c) \odot (s', d' \circ c') \\
&= \Big[(r,d) \circ (s,c)\Big] \odot \Big[(r',d') \circ (s',c')\Big]
\end{aligned}$$

whenever all compositions are defined. Since $d_0(c) = \omega(s)$,

$$(s,c) = (0+s, \widehat{c} \cdot 1_{d_0(c)}) = (0, \widehat{c}) \odot (s, 1_{d_0(c)}) = (0, \widehat{c}) \odot (s, 1_{\omega(s)})$$

and so all pairs satisfy the Schreier condition. Hence, the semidirect product category $(S, S \ltimes M)$ is a Schreier internal category in Mon.

[2] Since the projection $p\colon S \ltimes M \to M$ is a covering functor of underlying categories, then $(S, S \ltimes M)$ is a covering Schreier internal category of $\mathcal{M}$.

□

Similar results can be obtained for Schreier internal groupoids in Mon. Note that the inverse morphism of (s,c) is defined by $(c \centerdot s, c^{-1})$ where c^{-1} is the inverse morphism of c in the Schreier internal groupoid $\mathcal{M}$ in Mon.

Definition 3.4 *Let $\mathcal{M}$ be a Schreier internal category in Mon and acts on monoids S, S' via ω, ω', respectively. A morphism $f\colon (S,\omega) \to (S',\omega')$ of such actions is a homomorphism $f\colon S \to S'$ of monoids such that $\omega' f = \omega$ and $f(c \centerdot s) = c \centerdot f(s)$ whenever $c \centerdot s$ is defined. Hence actions of $\mathcal{M}$ on monoids and their morphisms form a category which will be denoted by Act(Sic)$/\mathcal{M}$.*

Let $Cov(\text{Sic})/\mathcal{M}$ be the category of covering categories of $\mathcal{M}$. Then we give the following theorem.

Theorem 3.5 *Let $\mathcal{M}$ be a Schreier internal category in Mon. Then, the categories Cov(Sic)$/\mathcal{M}$ and Act(Sic)$/\mathcal{M}$ are naturally equivalent.*

Proof We can use the same functor $\psi\colon Cov(\text{Sic})/\mathcal{M} \to Act(\text{Sic})/\mathcal{M}$ given in the proof of the Theorem 2.9. So, we only need to prove the interchange rule (eq.(3.1)) between the action and the monoid multiplications. Let $x \xrightarrow{c} y$, $\widetilde{x} \xrightarrow{\widetilde{c}} \widetilde{y}$, $x' \xrightarrow{c'} y'$, $\widetilde{x'} \xrightarrow{\widetilde{c'}} \widetilde{y'}$ and $p(\widetilde{c}) = c$, $p(\widetilde{c'}) = c'$. Since

$$d_0(\widetilde{c} \cdot \widetilde{c'}) = d_0(\widetilde{c}) + d_0(\widetilde{c'}) = \widetilde{x} + \widetilde{x'},$$

then $(c \cdot c') \centerdot (\widetilde{x} + \widetilde{x'})$ is defined. Hence, we write

$$(c \cdot c') \centerdot (\widetilde{x} + \widetilde{x'}) = d_1(\widetilde{c} \cdot \widetilde{c'}) = d_1(\widetilde{c}) + d_1(\widetilde{c'}) = (c \centerdot \widetilde{x}) + (c' \centerdot \widetilde{x'}).$$

For the functor $\theta\colon Act(\text{Sic})/\mathcal{M} \to Cov(\text{Sic})/\mathcal{M}$, $\theta(S,\omega) = (S, M \ltimes S)$ according to the proof of the Theorem 2.9, the monoid operation of $M \ltimes S$ is defined by

$$(c,s) \odot (c',s') = (c \cdot c', s + s')$$

where $d_0(c) = \omega(s), d_0(s') = \omega(s')$. Since $(c \centerdot s) + (c' \centerdot s') = (c \cdot c') \centerdot (s + s')$, the target map d_1 of $\theta(S,\omega)$ is a homomorphism of monoids. Now, we will verify that the composition and the monoid product satisfy the usual interchange rule.

$$\begin{aligned}
\Big[(c_1,s_1) \circ (c,s)\Big] \odot \Big[(c_1',s_1') \circ (c',s')\Big] &= (c_1 \circ c, s) \odot (c_1' \circ c', s') \\
&= \Big((c_1 \circ c) \cdot (c_1' \circ c'), s + s'\Big) \\
&= \Big((c_1 \cdot c_1') \circ (c \cdot c'), s + s'\Big) \\
&= (c_1 \cdot c_1', s_1 + s_1') \circ (c \cdot c', s + s') \\
&= \Big[(c_1,s_1) \odot (c_1',s_1')\Big] \circ \Big[(c,s) \odot (c',s')\Big]
\end{aligned}$$

whenever compositions are defined. Let 0 be the identity element of S. Since

$$(c, s) = (\widehat{c} \cdot \varepsilon d_0(c), 0 + s) = (\widehat{c}, 0) \odot (\varepsilon d_0(c), s),$$

all morphisms of $\theta(S, \omega)$ satify the Schreier condition. Other details are straightforward and so is omitted. □

Corollary 3.6 *Let $\mathcal{M}$ be a Schreier internal groupoid in* Mon*. Then, the categories Cov(*Sig*)/$\mathcal{M}$ and Act(*Sig*)/$\mathcal{M}$ are naturally equivalent.*

Proof Consider the functor $\theta\colon Act(\text{Sig})/\mathcal{M} \to Cov(\text{Sig})/\mathcal{M}$ and the object (S, ω) of $Act(\text{Sig})/\mathcal{M}$. Then, $\theta(S, \omega) = (S, M \ltimes S, p)$ is an object of $Cov(\text{Sig})/\mathcal{M}$ according the proof of Theorem 3.5. Here, $(S, M \ltimes S)$ is a Schreier internal groupoid in Mon with the inverse map $\eta(c, s) = (c^{-1}, c \centerdot s)$ where c^{-1} is the inverse morphism of c in the groupoid $\mathcal{M}$. □

Let ω be an action of a Schreier internal category $\mathcal{M}$ in Mon on a monoid $(S, +)$. Considering the crossed semimodule (A, B, ∂) corresponding to $\mathcal{M}$, an action of S on A is defined by $s \star c$ such that $(s \star c) \cdot 1_{\omega(s)} = 1_{\omega(s)} \cdot c$. Since

$$(\omega(s) \bullet c) \cdot 1_{\omega(s)} = 1_{\omega(s)} \cdot c = (s \star c) \cdot 1_{\omega(s)} \ ,$$

under the Schreier condition we obtain that

$$s \star c = \omega(s) \bullet c. \tag{3.2}$$

This means that the action of S on A is defined via ω.

Theorem 3.7 *By the action defined above, (A, S, φ) is a crossed semimodule where $\varphi\colon A \to S$ is given by $\varphi(c) = c \centerdot 0$ such that $\omega\varphi = \partial$, for $c \in A, s \in S$.*

Proof Due to (3.1), we write

$$\varphi(c \cdot c') = (c \cdot c') \centerdot 0 = (c \cdot c') \centerdot (0 + 0) = (c \centerdot 0) + (c' \centerdot 0) = \varphi(c) + \varphi(c')$$

for $c, c' \in A$, and so φ is a homomorphism of monoids. Now, we prove that the conditions [CSM 1] and [CSM 2] are satisfied.

[CSM 1] Using the interchange law (3.1) and the condition [AC 2] we write

$$\begin{aligned}
\varphi(s \star c) + s &= \Big((s \star c) \centerdot 0\Big) + \Big(1_{\omega(s)} \centerdot s\Big) \\
&= \Big((s \star c) \cdot 1_{\omega(s)}\Big) \centerdot \Big(0 + s\Big) \\
&= \Big(1_{\omega(s)} \cdot c\Big) \centerdot \Big(s + 0\Big) \\
&= \Big(1_{\omega(s)} \centerdot s\Big) + \Big(c \centerdot 0\Big) \\
&= s + \varphi(c).
\end{aligned}$$

[CSM 2] Since $d_1(c) = \partial(c) = \omega\varphi(c)$, by the interchange law (2.1) we get

$$\begin{aligned}(\varphi(c) \star c_1) \cdot c &= \Big((\varphi(c) \star c_1) \circ 1_e\Big) \cdot \Big(1_{\omega\varphi(c)} \circ c\Big) \\ &= \Big((\varphi(c) \star c_1) \cdot 1_{\omega\varphi(c)}\Big) \circ \Big(1_e \cdot c\Big) \\ &= \Big(1_{\omega\varphi(c)} \cdot c_1\Big) \circ \Big(c \cdot 1_e\Big) \\ &= \Big(1_{\omega\varphi(c)} \circ c\Big) \cdot \Big(c_1 \circ 1_e\Big) \\ &= c \cdot c_1 .\end{aligned}$$

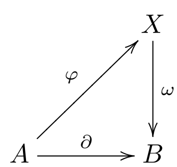
□

Corollary 3.8 *Let ω be an action of a Schreier internal groupoid $\mathcal{M}$ in* Mon *on a monoid $(S,+)$. Considering the object (A,B,∂) of* Csm* *corresponding to $\mathcal{M}$, (A,S,φ) is an object of* Csm* *by the Theorem 3.7.*

4. Liftings of crossed semimodules

In this section we introduce the notion of liftings crossed semimodule, which extends the notion of liftings crossed modules [24]. Then, we show that liftings of a certain crossed semimodule are naturally equivalent to actions of Schreier internal category in Mon, where the Schreier internal category corresponds to the crossed semimodule.

Definition 4.1 *Let (A,B,∂) be a crossed semimodule, X be a monoid and $\omega\colon X \to B$ be a homomorphism of monoids. If (A,X,φ) is a crossed semimodule with the action of X on A via ω such that the following diagram is commutative, then (A,X,φ) is called a lifting of (A,B,∂) over ω and denoted by (φ,X,ω).*

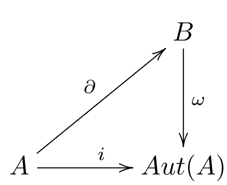


Example 4.2 *Every crossed semimodule (A,B,∂) lifts to itself via 1_B.*

Example 4.3 *The crossed semimodule (A,S,φ), which is obtained in Theorem 3.7. is a lifting of the crossed semimodule (A,B,∂) corresponding to the Schreier internal category $\mathcal{M}$.*

Remark 4.4 *If (A,X,φ) is a lifting of (A,B,∂) over ω, then $\langle 1_A,\omega\rangle$ is a morphism of crossed semimodules.*

Example 4.5 *Let A be a group. A crossed semimodule (A,B,∂) is a lifting of the automorphism crossed module $(A,Aut(A),i)$ over the action of B on A is defined by $\omega\colon B \to Aut(A)$, $\omega_b(a) = b \bullet a$.*



Proposition 4.6 *Let* (A, B, ∂) *be a crossed semimodule and* (A, X, φ) *is a lifting of* (A, B, ∂) *over* ω. *Given monoids* B', X' *and isomorphisms* $f\colon B \to B'$, $g\colon X \to X'$, *we obtain crossed semimodules* (A, B', ∂') *and* (A, X', φ') *where* $\partial' = f\partial$, $\varphi' = g\varphi$, *then* (A, X', φ') *is a lifting of* (A, B', ∂') *over* $\omega' = f\omega g$.

Proof First, we prove that (A, B', ∂') is a crossed semimodule. Action of B' on A is defined by $b' \bullet a = f^{-1}(b') \bullet a$ where $b' \in B'$ and $a \in A$. For $a_1 \in A$

[CSM 1] $\partial'(b' \bullet a)b' = f\partial(f^{-1}(b') \bullet a)ff^{-1}(b') = f\Big(\partial(f^{-1}(b') \bullet a)f^{-1}(b')\Big) = f(f^{-1}(b')\partial(a)) = b'\partial'(a)$

[CSM 2] $(\partial'(a) \bullet a_1)a = (f\partial(a) \bullet a_1)a = (f^{-1}f\partial(a) \bullet a_1)a = (\partial(a) \bullet a_1)a = aa_1$.

Using the same method, it can be easily proved that (A, X', φ') is a crossed semimodule.

Since $\omega\varphi = \partial$, we have

$$\omega'\varphi' = (f\omega g^{-1})(g\varphi) = f\omega\varphi = \partial\varphi = \partial'.$$

□

Proposition 4.7 *Let* (A, B, ∂) *be a crossed semimodule,* (A, X, φ) *be a lifting of* (A, B, ∂) *over* ω *and* (A, X', φ') *be a lifting of* (A, X, φ) *over* ω'. *Then,* (A, X', φ') *is a lifting of* (A, B, ∂) *over* $\omega\omega'$.

Proposition 4.8 *Let* (A, B, ∂) *be a crossed semimodule,* X *be a monoid and* $\omega\colon X \to B$ *be a homomorphism of monoids. If* (A, X, φ) *is a pre-crossed semimodule with the action of* X *on* A *via* ω *such that the following diagram is commutative, then* (A, X, φ) *is a lifting of* (A, B, ∂) *over* ω.

Proof For $a, a_1 \in A$, since

$$(\varphi(a) \star a_1)a = (\omega\varphi(a) \bullet a_1)a = (\partial(a) \bullet a_1)a = aa_1,$$

the condition [CSM 2] is satisfied. Hence, (A, X, φ) is a crossed semimodule. □

Proposition 4.9 *Let* (A, B, ∂) *be a crossed semimodule,* $(X, +)$ *be a monoid and* $\omega\colon X \to B$ *and* $\varphi\colon A \to X$ *be homomorphisms of monoids such that* $\omega\varphi = \partial$. *Then,* (A, X, φ) *is a crossed semimodule with the action defined via* ω, *if and only if the map*

$$\overline{\varphi}\colon X \ltimes A \to X, \quad \overline{\varphi}(x, a) = \varphi(a) + x$$

is a homomorphism of monoids where $X \ltimes A$ *is the semi-direct product of monoids.*

Proof Assume that $\overline{\varphi}$ is a homomorphism of monoids. We only prove the condition [CSM 1].

$$\varphi(x \star a) + x = \overline{\varphi}(x, x \star a) = \overline{\varphi}((x, e_A)(0, a)) = \overline{\varphi}(x, e_A)\overline{\varphi}(0, a) = \varphi(e_A) + x + \varphi(a) + 0 = x + \varphi(a)$$

Now suppose that (A, X, φ) is a crossed semimodule. By [CSM 1] we have

$$\begin{aligned}
\overline{\varphi}((x, a)(x', a')) &= \overline{\varphi}(x + x', a(x \star a')) \\
&= \varphi(a(x \star a')) + x + x' \\
&= \varphi(a) + \varphi(x \star a') + x + x' \\
&= \varphi(a) + x + \varphi(a') + x' \\
&= \overline{\varphi}(x, a) + \overline{\varphi}(x', a').
\end{aligned}$$

□

Proposition 4.10 *Let (A,B,∂) be a crossed semimodule, X be a monoid and $\omega\colon X\to B$, $\varphi\colon A\to X$ be homomorphisms of monoids such that $\omega\varphi=\partial$. If ω is injective, then (A,X,φ) is a lifting of (A,B,∂) over ω.*

Proof According to above Proposition, we only need to prove that $\overline{\varphi}\colon X\ltimes A\to X$, $\overline{\varphi}(x,a)=\varphi(a)+x$ is a homomorphism of monoids, i.e., $\varphi(x\bullet a)+x=x+\varphi(a)$, for $x\in X, a\in A$.

$$\omega(\varphi(x\star a)+x)=\omega\varphi(\omega(x)\star a)\omega(x)=\partial(\omega(x)\star a)\omega(x)=\omega(x)\partial(a)$$

and

$$\omega(x+\varphi(a))=\omega(x)\omega\varphi(a)=\omega(x)\partial(a).$$

Since ω is injective, we get $\varphi(x\bullet a)+x=x+\varphi(a)$. □

Definition 4.11 *Let (A,B,∂) be a crossed semimodule, (A,X,φ) and (A,X',φ') be liftings of (A,B,∂) over ω and ω', respectively. A homomorphism $f\colon X\to X'$ is called morphism of liftings of ∂ if $f\varphi=\varphi'$ and $\omega' f=\omega$. Hence, the liftings of (A,B,∂) and their morphisms form a category denoted by* LCSM$/(A,B,\partial)$.

Note that $\lambda=\langle 1_A,f\rangle$ is a morphism of crossed semimodules (A,X,φ) and (A,X',φ'), since

$$1_A(x\bullet a)=x\bullet a=\omega(x)\bullet a=\omega' f(x)\bullet a=f(x)\bullet a.$$

Theorem 4.12 *Let (A,B,∂) be a crossed semimodule which is obtained from a Schreier internal category $\mathcal{M}$ in* MON. *Then, the categories* LCSM$/(A,B,\partial)$ *and Act(*SIC*)$/\mathcal{M}$ are equivalent.*

Proof Let we define a functor $\delta\colon$ *Act*(SIC)$/\mathcal{M}\to$ LCSM$/(A,B,\partial)$, which is an equivalence of categories such that $\delta(S,\omega)=(\varphi,S,\omega)$ where ω is an action of $\mathcal{M}$ on S and φ is defined by $\varphi(c)=c\centerdot 0$ for $c\in A$. By (3.2), the action of S on A is defined via ω. Due to [AC 1], $\omega\varphi=\partial$. Hence, by the Theorem 3.7, (A,S,φ) is a lifting of (A,B,∂) over ω.

Let $f\colon S\to S'$ be a morphism of *Act*(SIC)$/\mathcal{M}$. Since

$$f\varphi(c)=f(c\centerdot 0_X)=c\centerdot f(0_X)=c\centerdot 0_{X'}=\varphi'(c),$$

then $\delta(f)=f$ is a morphism of LCSM$/(A,B,\partial)$.

Now we define a functor $\gamma\colon$ LCSM$/(A,B,\partial)\to$ *Act*(SIC)$/\mathcal{M}$ as a weak inverse of δ such that $\gamma(\varphi,X,\omega)=(X,\omega)$ where the action of $\mathcal{M}=(B,B\ltimes A)$ on X is defined by

$$\centerdot\colon (B\ltimes A)_{d_0}\times_\omega X\to X,\quad (b,a)\centerdot x=\varphi(a)+x$$

where $d_0(b,a)=b=\omega(x)$. We will verify that $\centerdot$ is an action.

[AC 1] $\omega((b,a)\centerdot x)=\omega(\varphi(a)+x)=\partial(a)\omega(x)=\partial(a)b=d_1(b,a)$, where $b=\omega(x)$.

[AC 2] $1_{\omega(x)}\centerdot x=(\omega(x),e_A)\centerdot x=\varphi(e_A)+x=0+x=x$

[AC 3] $\Big((b_1,a_1)\circ(b,a)\Big)\centerdot x=(b,a_1a)\centerdot x=\varphi(a_1a)+x=\varphi(a_1)+\varphi(a)+x=(b_1,a_1)\centerdot\Big((b,a)\centerdot x\Big)$

whenever composition and actions are defined. Now we prove the interchange rule (3.1). Let $b=\omega(x)$ and $b'=\omega(x')$. Then, $(b,a)\centerdot x$ and $(b',a')\centerdot x'$ are defined. Hence, we have

$$\begin{aligned}\Big((b,a)(b',a')\Big)\centerdot(x+x') &= \Big(bb',a(b\bullet a')\Big)\centerdot(x+x')\\ &= \varphi(a(b\bullet a'))+x+x'\\ &= \varphi(a)+\varphi(\omega(x)\bullet a')+x+x'\\ &= \varphi(a)+\varphi(x\bullet a')+x+x'\\ &= \varphi(a)+x+\varphi(a')+x'\\ &= \Big((b,a)\centerdot x\Big)+\Big((b',a')\centerdot x'\Big).\end{aligned}$$

Let $f\colon X\to X'$ be a morphism of liftings of (A,B,∂). Since $f\varphi=\varphi'$, we write

$$f((b,a)\centerdot x)=f(\varphi(a)+x)=f\varphi(a)+f(x)=\varphi'(a)+f(x)=(b,a)\centerdot f(x),$$

then $\gamma(f)=f$ is a morphism of $Act(\textsc{Sic})/\mathcal{M}$.

To define a natural equivalence $\eta\colon 1_{Act(\textsc{Sic})/\mathcal{M}}\to\gamma\delta$, a map $\eta_S\colon(S,\omega)\to\gamma\delta(S,\omega)$ is defined to be identity on elements of S. Now, we must prove that $\eta_{\mathcal{M}}(c)\centerdot s=c\centerdot s$ where $d_0(c)=\omega(s)$.

$$\eta_{\mathcal{M}}(c)\centerdot s=(d_0(c),\widehat{c})\centerdot s=\varphi(\widehat{c})+s=(\widehat{c}\centerdot 0)+s=(\widehat{c}\centerdot 0)+(1_{\omega(s)}\centerdot s)=(\widehat{c}\cdot 1_{d_0(c)})\centerdot(0+s)=c\centerdot s$$

A natural equivalence $\beta\colon 1_{\mathrm{L}\textsc{Csm}/(A,B,\partial)}\to\delta\gamma$ is given such that $\beta(\varphi,X,\omega)=(\varphi',X,\omega)$. Since

$$\varphi'(e_B,a)=(e_B,a)\centerdot 0=\varphi(a)+0=\varphi(a),$$

, then we write $\delta\gamma\simeq 1_{\mathrm{L}\textsc{Csm}/(A,B,\partial)}$. Other details are straightforward and so are omitted. □

Corollary 4.13 *Let (A,B,∂) be an object of* C*SM* which is obtained from a Schreier internal groupoid $\mathcal{M}$ in* M*ON. Then, the categories L*C*SM*/(A,B,∂) and Act(*S*IG)/$\mathcal{M}$ are naturally equivalent.*

Definition 4.14 *Let $\lambda=\langle\lambda_1,\lambda_2\rangle\colon(\widetilde{A},\widetilde{B},\widetilde{\partial})\to(A,B,\partial)$ be a morphism of crossed semimodules and $\lambda_1\colon\widetilde{A}\to A$ is an isomorphism. Then, λ is called a covering morphism and $(\widetilde{A},\widetilde{B},\widetilde{\partial})$ is called a covering crossed semimodule of (A,B,∂).*

Let *Cov*(CSM)$/(A,B,\partial)$ denote the category of covering morphisms of the crossed semimodule (A,B,∂). Hence, we can give the following theorem.

Theorem 4.15 *The categories L*C*SM$/(A,B,\partial)$ and Cov(*C*SM)$/(A,B,\partial)$ are naturally equivalent where (A,B,∂) is a crossed semimodule.*

Proof Given a lifting (φ, X, ω) of (A, B, ∂), clearly $\langle 1_A, \omega\rangle$ is a covering morphism and (A, X, φ) is a covering crossed semimodule of (A, B, ∂). Let $f\colon X \to X'$ be a morphism of liftings of ∂ over ω. Then $\langle 1_A, f\rangle$ is a morphism of $Cov(\text{Csm})/(A, B, \partial)$.

Conversely, given a covering crossed semimodule $(\widetilde{A}, \widetilde{B}, \widetilde{\partial})$ of (A, B, ∂) with a covering morphism $\lambda = \langle \lambda_1, \lambda_2\rangle$, then $(\varphi, \widetilde{B}, \omega)$ is a lifting of (A, B, ∂) where $\varphi = \widetilde{\partial}\lambda_1^{-1}$, $\omega = \lambda_2$ and the action of $\widetilde{B}$ on A is defined by $\widetilde{b} \star a = \lambda_2(b) \bullet a$. We will verify that $(\varphi, \widetilde{B}, \omega)$ is a crossed semimodule. Let $a, a_1 \in A$ and $\widetilde{b} \in \widetilde{B}$.

[CSM 1] $\varphi(\widetilde{b} \star a)\widetilde{b} = \widetilde{\partial}\lambda_1^{-1}\Big(\lambda_2(\widetilde{b}) \bullet \lambda_1\lambda_1^{-1}(a)\Big)\widetilde{b} = \widetilde{\partial}\lambda_1^{-1}\lambda_1\Big(\widetilde{b} \bullet \lambda_1^{-1}(a)\Big)\widetilde{b} = \widetilde{b}\widetilde{\partial}\lambda_1^{-1}(a) = \widetilde{b}\varphi(a)$

[CSM 2] $(\varphi(a) \star a_1)a = (\lambda_2\widetilde{\partial}\lambda_1^{-1}(a) \bullet a_1)a = (\partial\lambda_1\lambda_1^{-1}(a) \bullet a_1)a = (\partial(a) \bullet a_1)a = aa_1\,,$

Let $\kappa = \langle \kappa_1, \kappa_2\rangle\colon (\widetilde{A}, \widetilde{B}, \widetilde{\partial}) \to (\overline{A}, \overline{B}, \overline{\partial})$ be a morphisms of coverings of the crossed semimodule (A, B, ∂) where $\mu = \langle \mu_1, \mu_2\rangle\colon (\overline{A}, \overline{B}, \overline{\partial}) \to (A, B, \partial)$ is a morphism of crossed semimodules. Since

$$\kappa_2\widetilde{\partial}\lambda_1^{-1} = \overline{\partial}\kappa_1\lambda_1^{-1} = \overline{\partial}\mu_1^{-1},$$

then $\kappa_2\colon (\widetilde{B}, \widetilde{\partial}\lambda_1^{-1}) \to (\overline{B}, \overline{\partial}\mu_1^{-1})$ is a morphism of liftings.

Clearly, these constructions above are functorial. Other details of the proof are straightforward and so are omitted. □

5. Simplicial monoids of Moore length 1

In this section, we prove an equivalence between the category Csm* and the category of simplicial monoids in Mon whose Moore length is 1 under some conditions. Hence we obtain an extension of the equivalence between crossed modules over groups and simplicial groups of Moore length 1.

Consider a simplicial monoid $\mathcal{M}$ which consists of a family of monoids $\{M_n\}$ together with face and degeneracy maps as follows:

$$M_0 \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} \xrightarrow{s_0} M_1 \;\; \begin{array}{c}\xleftarrow{\partial_0} \\ \xrightarrow{s_0} \\ \xleftarrow{\partial_1} \\ \xrightarrow{s_1} \\ \xleftarrow{\partial_2}\end{array} \;\; M_2 \ldots M_{n-1} \;\; \begin{array}{c}\xleftarrow{\partial_0} \\ \xrightarrow{s_0} \\ \xleftarrow{\partial_1} \\ \vdots \\ \xrightarrow{s_{n-1}} \\ \xleftarrow{\partial_n}\end{array} \;\; M_n \ldots$$

satisfying following usual simplicial identities given in [16, 23].

$$\partial_i\partial_j = \partial_{j-1}\partial_i \quad , \quad if \quad i < j$$

$$\partial_i s_j = \begin{cases} s_{j-1}\partial_i & , \quad i < j \\ 1 & , \quad i \in \{j, j+1\} \\ s_j\partial_{i-1} & , \quad i > j+1 \end{cases}$$

$$s_j s_i = s_i s_{j-1} \quad , \quad if \quad i < j$$

A simplicial monoid morphism $f\colon \mathcal{M} \to \mathcal{M}'$ is a family of monoid homomorphisms $\{f_n\colon M_n \to M'_n\}$ such that $\partial'_i f_n = f_{n-1}\partial_i$ and $f_n s_j = s'_j f_{n-1}$, for i, j and k. The category of simplicial monoids will be denoted by SimpMon.

For a simplicial monoid $\mathcal{M}$ in Mon, the Moore complex of length one can be pictured as follows:

$$M_0 \underset{\partial_1}{\overset{\partial_0}{\underset{\xrightarrow{s_0}}{\leftleftarrows}}} Ker\partial_0 \longleftarrow \{e\}\ldots\{e\} \longleftarrow \{e\}\ldots$$

Let $\text{Simp}^*_{\leq 1}(\text{Mon})$ be the category of simplicial monoids whose Moore complex has length one satisfying following two conditions.

[SM 1] $Ker\partial_0$ is a group.

[SM 2] For each $m \in M_1$ there exist a unique $\widetilde{m} \in Ker\partial_0$ such that $m = \widetilde{m}s_0\partial_0(m)$.

Theorem 5.1 *The category* Csm^* *is naturally equivalent to the category* $\text{Simp}^*_{\leq 1}(\text{Mon})$.

Proof Let $\mathcal{M}$ be an object of $\text{Simp}^*_{\leq 1}(\text{Mon})$. We define a functor

$$\gamma\colon \text{Simp}^*_{\leq 1}(\text{Mon}) \to \text{Csm}^*$$

such that $\gamma(\mathcal{M}) = (A, B, \partial)$, $A = Ker\partial_0$, $B = M_0$ and $\partial = \partial_1|_{Ker\partial_0}$ where an action of B on A is defined such that

$$(x \bullet y)s_0(x) = s_0(x)y$$

for $x \in B, y \in A$. Clearly, under the condition [SM 2], $\bullet$ is an action of monoids. We verify that the conditions [CSM 1] and [CSM 2] are satisfied. Let $x \in B$ and $y, y_1 \in A$.

[CSM 1] $\partial(x \bullet y)x = \partial_1(x \bullet y)\partial_1 s_0(x) = \partial_1\Big((x \bullet y)s_0(x)\Big) = \partial_1\Big(s_0(x)y\Big) = \partial_1 s_0(x)\partial_1(y) = x\partial(y)$

[CSM 2] Since $\partial_0\Big((\partial(y) \bullet y_1)yy_1^{-1}y^{-1}\Big) = e$ and

$$\partial_1\Big((\partial(y) \bullet y_1)yy_1^{-1}y^{-1}\Big) = \partial_1(\partial_1(y) \bullet y_1)\partial_1(y)\partial_1(y_1^{-1})\partial_1(y^{-1}) = \partial_1(y)\partial_1(y_1)\partial_1(y_1^{-1})\partial_1(y^{-1}) = e,$$

then $(\partial(y) \bullet y_1)yy_1^{-1}y^{-1} \in Ker\partial_0 \cap Ker\partial_1 = \{e\}$ and so $(\partial(y) \bullet y_1)y = yy_1$.

Given a morphism $\{f_0, f_1\}$ of $\text{Simp}^*_{\leq 1}(\text{Mon})$, $\langle f_1, f_0\rangle$ is a morphism of Csm^*. Since $f_1 s_0(x) \in A'$ and

$$f_1(x \bullet y)f_1 s_0(x) = f_1((x \bullet y)s_0(x)) = f_1(s_0(x)y) = s'_0 f_0(x)f_1(y) = (f_0(x) \bullet f_1(y))f_1 s_0(x),$$

the condition [SM 2] allows us to write $f_1(x \bullet y) = f_0(x) \bullet f_1(y)$.

Now, we define a functor $\psi\colon \text{Csm}^* \to \text{Simp}^*_{\leq 1}(\text{Mon})$ as a weak inverse of γ. Given an object (A, B, ∂) of Csm^*, we can obtain an object $\psi(A, B, \partial)$ of $\text{Simp}^*_{\leq 1}(\text{Mon})$ as in the following way. Consider the semi-direct product $B \ltimes A$ with the multiplication $(b, a)(b', a') = (bb', a(b \bullet a'))$. The face and degeneracy maps are defined as follows:

$$\partial_0\colon B \ltimes A \to B \;\; , \;\; \partial_0(b, a) = b \;\; , \;\; \partial_1\colon B \ltimes A \to B \;\; , \;\; \partial_1(b, a) = \partial(a)b \;\; , \;\; s_0\colon B \to B \ltimes A \;\; , \;\; s_0(b) = (b, e).$$

Clearly $Ker\partial_0 = \{(e,a)|a \in A\}$ is a group. Since $(b,a) = (e,a)(b,e)$, all pairs of $B \ltimes A$ satisfy the condition [SM 2]. Let $\langle \lambda_1, \lambda_2 \rangle$ be a morphism of CSM*. Then, $\{\lambda_1, \lambda_2 \times \lambda_1\}$ is a morphism of $\mathrm{SIMP}^*_{\leq 1}(\mathrm{MON})$.

To define a natural equivalence $\eta\colon 1_{\mathrm{SIMP}^*_{\leq 1}(\mathrm{MON})} \to \psi\gamma$, a map $\eta_{\mathcal{M}}$ is defined to be identity on M_0 and is defined by $y \mapsto (e,y)$ on $Ker\partial_0$. In order to define a natural transformation $\mu\colon 1_{\mathrm{CSM}^*} \to \gamma\psi$, $\mu_{(A,B,\partial)}$ is given by $a \mapsto (e,a)$ on A. Other details are straightforward and so are omitted. □

Hence, we obtain the following corollary as a result of this equivalence.

Corollary 5.2 *The following data are naturally equivalent:*

1. *the category* CSM* *of crossed semimodules*
2. *the category* SIG *of Schreier internal groupoids in* MON
3. *the category* $\mathrm{CAT}^{\mathbf{1}}$*-*MON *of* $cat^{\mathbf{1}}$*-monoids (see [29])*
4. *the category* $\mathrm{SIMP}^*_{\leq 1}(\mathrm{MON})$ *of simplicial monoids whose Moore complex has length one*

Then, we can write the following result as a restriction of this corollary.

Corollary 5.3 *The following data are naturally equivalent:*

1. *the category of crossed modules over groups*
2. *the category of group-groupoids (2-groups)*
3. *the category of* $cat^{\mathbf{1}}$*-groups*
4. *the category of simplicial groups whose Moore complex has length one*

6. Conclusion

In [1], the notion of liftings crossed modules in an arbitrary category of groups with operations introduced. Using the results of papers [22, 26], it would be interesting to explore similar notions for liftings crossed semimodules in an arbitrary category of monoid with operations and so it could be possible to obtain concrete examples in the cases of monoids, semirings, and distributive lattices.

Acknowledgment

We would like to thank the referee for useful remarks, which help us to improve the paper and the editors for the editorial process.

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