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Composition laws on the Fricke surface and Markov triples

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Abstract: We determine some composition laws related to the Fricke surface and the "double" Fricke surface. This latter surface admits the squares of Markov triples as its solutions.

Key words: Fricke surface, Markov equation, Markov triples, group law on quadrics, Frobenius conjecture.

1. Introduction

It is well known that the positive integral solutions of the *Markov equation*

$$x^2 + y^2 + z^2 = 3xyz \tag{1.1}$$

can be obtained from the solution $(1, 1, 1)$ by using the *Viète transformations*

$$L : (x, y, z) \to (x, 3xy - z, y) \text{ and } R : (x, y, z) \to (y, 3yz - x, z),$$

and the permutations of these solution triples [1]. The cubic surface $\mathbf{F}$ defined over some field by (1.1) is called the *Fricke surface*. An integral solution of (1.1) is called a *Markov triple*; except $(0, 0, 0)$, these are obtained from positive integral solutions by a pair of sign changes of coordinates. The famous conjecture of Frobenius states that the largest component of a positive Markov triple determines uniquely the remaining components.

Suppose $\mathbf{H}$ is a cubic hypersurface in the projective space $\mathbb{P}^n(\mathbb{C})$ and P, Q are two generic points on $\mathbf{H}$. Then the line through P and Q intersects $\mathbf{H}$ at a third point R. Thus we may introduce the partial composition law $P \circ Q := R$ on $\mathbf{H}$. In case $\mathbf{H}$ is an elliptic curve and O is a fixed point of $\mathbf{H}$, then the product $P \star Q := O \circ (P \circ Q)$ defines an abelian group law on $\mathbf{H}$. However, in case $\dim \mathbf{H} \geq 3$, this composition is not always well-defined and does not yield a group law since it is not associative [4]. By linearity, this composition satisfies

$$\alpha \circ \beta = \gamma \iff \alpha \circ \gamma = \beta \iff \beta \circ \gamma = \iff \alpha \tag{1.2}$$

Our aim in this paper is to determine this composition law on the Fricke surface $\mathbf{F}$ in $\mathbb{C}^3$ defined by (1.1) and study it.

Sections of $\mathbf{F}$ with hyperplanes parallel to the coordinate axis give quadric curves. Any smooth quadric Q can be endowed with a genuine commutative group law, defined as follows [3]: Let $O \in Q$ be a (rational)

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point, and $A, B \in Q$. Let L be the line through O which is parallel to the line through A and B. Then $A \oplus B$ is defined to be the second intersection point of L with Q. We also determine this group law for these quadric hyperplane sections. The Frobenius conjecture on the Markov numbers can be formulated as a statement about the integral points on these quadric sections.

Now consider the equation

$$(x+y+z)^2 = 9xyz. \tag{1.3}$$

We call the surface defined by this equation the *double Fricke surface* and denote it by $\mathbf{F}^2$. Its positive integral solutions are the squared Markov triples (m^2, n^2, k^2), where (m, n, k) is a Markov triple. These integral points are generated from the triple $(1, 1, 1)$ by means of the transformations

$$(x, y, z) \to (x, 9xy - 2x - 2y - z, y), \quad (x, y, z) \to (y, 9yz - 2y - 2z - x, z).$$

and permutations of coordinates. Our second task in this paper is to determine the composition law on $\mathbf{F}^2$ as well as the quadric group laws on the associated hyperplane sections.

2. The composition law on $\mathbf{F}$.

Suppose $P = (m, n, k)$ and $Q = (a, b, c)$ are two points of $\mathbf{F}$. The line L through them has the parametrization

$$x = (a-m)t + m, \quad y = (b-n)t + n, \quad z = (c-k)t + k$$

The intersection $L \cap \mathbf{F}$ has the equation

$$[(a-m)t+m]^2 + [(b-n)t+n]^2 + [(c-k)t+k]^2 = 3[(a-m)t+m][(b-n)t+n][(c-k)t+k]$$

Since $t = 0$ and $t = 1$ are two solutions of this equation, the third solution is

$$t = \frac{3(ank + bmk + cmn) - 2(am + bn + ck) - 3mnk}{3(a-m)(b-n)(c-k)}$$

and the corresponding point (x, y, z) can be found as below:

Proposition 2.1 *If $P = (a, b, c) \in \mathbf{F}(K)$ and $Q = (m, n, k) \in \mathbf{F}(K)$ are two points of $\mathbf{F}$ defined over some field K of characteristic 0, then the composition $P \circ Q = (x, y, z) \in \mathbf{F}(K)$ is given by*

$$\begin{aligned} x &= \frac{3(ank + bcm) - 2(am + bn + ck)}{3(b-n)(c-k)} \\ y &= \frac{3(bmk + acn) - 2(am + bn + ck)}{3(a-m)(c-k)} \\ z &= \frac{3(cmn + abk) - 2(am + bn + ck)}{3(a-m)(b-n)}. \end{aligned} \tag{2.1}$$

In case $(a, b, c) = (0, 0, 0)$ or $(a, b, c) = (m, n, k)$, the composition $(a, b, c) \circ (m, n, k)$ is not defined. Otherwise, if $(a-m)(b-n)(c-k) = 0$, it is necessary to projectivize to get the answer

$$(m, n, k) \circ (m, b, c) = [m : n : k : 1] \circ [m : b : c : 1] = [0 : b-n : c-k : 0].$$

Finally, the product is not well-defined if both of the points lie on the same line at infinity of $\mathbf{F}$.

Note that in general, the $\circ$-composition of two integral Markov triples is nonintegral.

Remark. Let σ be a constant. The similarly defined composition on the surface $\mathbf{F}_\sigma : x^2+y^2+z^2=3xyz+\sigma$ has exactly the same expression as (2.1). In case $\sigma=0$, one can bring (3.5) to the form: $P\circ Q=$

$$\left(\frac{(an-bm)^2+(ak-mc)^2}{3am(b-n)(c-k)},\frac{(bk-cn)^2+(an-bm)^2}{3bn(a-m)(c-k)},\frac{(ak-mc)^2+(bk-cn)^2}{3ck(a-m)(b-n)}\right)$$

In order to carry out the analogy with the group law for elliptic curves, one may define the commutative law with identity

$$(a,b,c)\star(m,n,k):=(1,1,1)\circ((a,b,c)\circ(m,n,k)),$$

whenever the right-hand side is defined. However, $\star$ is not associative as one may easily check.

2.1. Transfer of the structures to $\mathbb{P}^2(\mathbb{Q})$

We may transfer the Viete transformations and the composition $\circ$ to $\mathbb{P}^2$ by means of the parametrization φ. The Viete transformations are given by

$$\begin{aligned} L &: [p:q:r]\to[pr:p^2+q^2:qr] \\ R &: [p:q:r]\to[qp:q^2+r^2:pr]. \end{aligned} \tag{2.2}$$

Note that these are birational transformations which are not everywhere defined. On the other hand, viewed as transformations of $\mathbb{P}^2(\mathbf{Q})$, they are well defined everywhere, except at the points $[0,0,1]$, $[0,1,0]$, and $[1,0,0]$. We may apply permutations of coordinates to (3.6) to get the involutions

$$\begin{aligned} &[p:q:r]\to[pr:qr:p^2+q^2] \\ &[p:q:r]\to[q^2+r^2:pq:pr] \\ &[p:q:r]\to[pq:p^2+r^2:rq], \end{aligned}$$

which define a birational action of $\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}$ on $\mathbb{P}^2$. We may also transfer the composition $\circ$ to $\mathbb{P}^2(\mathbf{Q})$ by means of φ as:

$$\begin{aligned} &[a:b:c],[m:n:k]\in\mathbb{P}^2(\mathbf{Q}) \implies [a:b:c]\circ[m:n:k]= \\ &\Big[\left((a^2+b^2+c^2)kn-(m^2+n^2+k^2)bc\right)\left((bm-an)^2+(cm-ak)^2\right), \\ &\left((a^2+b^2+c^2)km-(m^2+n^2+k^2)ac\right)\left((an-bm)^2+(cn-bk)^2\right), \\ &\left((a^2+b^2+c^2)mn-(m^2+n^2+k^2)ba\right)\left((ak-cm)^2+(bk-cn)^2\right).\Big] \end{aligned}$$

2.2. Hyperplane sections of F

Let H be a hyperplane in $\mathbb{P}^2(\mathbb{C})$. Then $H\cap\overline{\mathbf{F}}$ is a smooth cubic (if H generic), a nodal cubic (if H passes through the origin or H is a tangent to $\overline{\mathbf{F}}$) or a union of three lines (if H is the hyperplane at infinity) or a quadric, together with a line at infinity (if H is of the form $x=a$ or $y=b$ or $z=c$). In view of its connection to the Frobenius conjecture, we will be interested in the last case.

Suppose (m_0, n_0, k_0) is a Markov triple. Then the plane $H := \{(x, y, z) : y = n_0\}$ contains infinitely many Markov triples. The intersection $H \cap \mathbf{F}$ is a quadric Q_{n_0} with a special point (m_0, n_0, k_0), given by the equation

$$Q_{n_0} : \quad x^2 + n_0^2 + z^2 = 3xn_0z \tag{2.3}$$

We may express z explicitly in terms of x as

$$z = \frac{3xn_0 \pm \sqrt{9x^2n_0^2 - 4(n_0^2 + x^2)}}{2} = \frac{3xn_0 \pm \sqrt{x^2n_0^2 - 4(x - n_0)^2)}}{2}$$

Denote by $Q_{n_0}(\mathbf{Z})$ the set of integral points and by $Q_{n_0}(\mathbf{N})$ the set of positive integral points. We call a point of $Q_{n_0}(\mathbf{N})$ with $n_0 = \max\{m_0, n_0, k_0\}$ a *fundamental point*. Note that the famous Frobenius conjecture is the statement that every $Q_{n_0}(\mathbf{N})$ has a unique fundamental point.

The infinity of Q_{n_0}. If we consider (2.3) as a quadric in the $x - z$ plane, and projectivize it, we get the equation $x^2 + n_0^2s^2 + z^2 = 3xzn_0$. This quadric intersects the line $s = 0$ at the points $x^2 + z^2 = 3xzn_0$. Deprojectivize by setting $t = x/z \implies$

$$t^2 - 3tn_0 + 1 = 0 \implies t = \frac{3n_0 \pm \sqrt{9n_0^2 - 4}}{2}.$$

Then it is easy to check that

$$\frac{3n_0 + \sqrt{9n_0^2 - 4}}{2} = \lceil 3n_0 : 3n_0 : 3n_0 : 3n_0 : 3n_0 : \dots \rceil$$

$$\frac{3n_0 - \sqrt{9n_0^2 - 4}}{2} = \lceil 0 : 3n_0 : 3n_0 : 3n_0 : 3n_0 : 3n_0 : \dots \rceil,$$

where $\lceil n_0 : n_1 : \dots\dots \rceil = n_0 - 1/n_1 - 1/\dots.$

2.2.1. The group law on the quadric sections $H \cap \mathbf{F}$.

Let $O = (m_0, n_0, k_0)$ be a fundamental point. Suppose $P_1 := (x_1, z_1)$ and $P_2 := (x_2, z_2)$ are on Q_{n_0}. If $P_1 \neq P_2$ then the line through O and parallel to P_1P_2 has the equation

$$z = \frac{z_2 - z_1}{x_2 - x_1}(x - m_0) + k_0 = \mu(x - m_0) + k_0$$

and the intersection has the equation

$$x^2 + (\mu(x - m_0) + k_0)^2 + n_0^2 = 3xn_0\,(\mu(x - m_0) + k_0) \iff$$

Since $x = m_0$ is one solution of this equation the other solution is

$$x = \frac{{n_0}^2 + (-\mu m_0 + k_0)^2}{1 + \mu^2 - 3n_0\mu}, \quad z = k_0 - \mu\frac{2m_0 + 2\mu k_0 - 3n_0k_0 - 3m_0n_0\mu}{1 + \mu^2 - 3n_0\mu} \tag{2.4}$$

This gives the following result:

Proposition 2.2 *Let* $P_1 := (x_1, z_1)$ *and* $P_2 := (x_2, z_2)$ *are on* Q_{n_0}. *Set* $\mu := (z_2 - z_1)/(x_2 - x_1)$. *If* $P_1 \neq P_2$, *then* $P_1 \oplus P_2 = (x, z)$, *where*

$$x = \frac{\mu^2 m_0 - m_0 - 2\mu k_0 + 3n_0 k_0}{1 + \mu^2 - 3n_0\mu}, \quad z = \frac{k_0 - 2m_0\mu - \mu^2 k_0 + 3m_0 n_0 \mu^2}{1 + \mu^2 - 3n_0\mu}.$$

If $P = (x_1, z_1)$, *then* $P \oplus P = (x, z)$, *where*

$$x = \frac{x_1^2 n_0^2 + (x_1 k_0 - z_1 m_0)^2}{n_0^2 m_0}, \quad z = \frac{z_1^2 n_0^2 + (x_1 k_0 - z_1 m_0)^2}{n_0^2 k_0}.$$

It follows from this proposition that $(x_1, z_1) \oplus (z_1, x_1) = (k_0, m_0)$. Moreover, $(-m_0, -k_0)$ is an element of order 2 under $\oplus$.

Proof The case $P_1 \neq P_2$ is obtained from (2.4) by routine modifications. It remains to establish the case $P_1 = P_2$. In this case, the line tangent to the quadric is given by the equation

$$(2x_1 - 3n_0 z_1)(x - x_1) + (2z_1 - 3n_0 x_1)(z - z_1) = 0,$$

whose slope is

$$\sigma = -\frac{2x_1 - 3n_0 z_1}{2z_1 - 3n_0 x_1}$$

The line through (m_0, n_0, k_0) with this slope has the equation $z - k_0 = \sigma(x - m_0)$, and the second intersection of this line with Q_{n_0} has the equation

$$\implies x^2 + n_0^2 + (\sigma x - \sigma m_0 + k_0)^2 = 3xn_0 (\sigma x - \sigma m_0 + k_0).$$

Since m_0 is a solution to this equation, the other solution is

$$x = \frac{n_0^2 + (-\sigma m_0 + k_0)^2}{m_0(1 + \sigma^2 - 3n_0\sigma)}.$$

If we write z instead of x and m_0 instead of k_0, we get z. We obtain the desired result after some routine computations. □

2.2.2. Inverses

Proposition 2.3 *The inverse of* $P_1 := (x_1, z_1)$ *is* (x, z) *with*

$$x = \frac{(\mu^2 x_1^2 + z_1^2 - 2x_1\mu z_1 + n_0^2)(2k_0 - 3n_0 m_0)^2}{n_0^2(9n_0^2 - 4)}$$

$$z = \mu {m_0}^2 - \mu x_1 + z_1 - \frac{4\mu}{9} + \frac{\mu\,(\mu x_1 - z_1)^2\,{m_0}^2}{{n_0}^2} - \frac{4\mu\left(9\,(\mu x_1 - z_1)^2 + 4\right)}{9\,(9n_0^2 - 4)}$$

where $\mu = -\frac{2m_0 - 3n_0 k_0}{2k_0 - 3n_0 m_0}$.

Proof To find the inverse of (x_1, z_1), we start with the tangent line at $O = (m_0, n_0, k_0)$:

$$(2m_0 - 3n_0k_0)(x - m_0) + (2k_0 - 3n_0m_0)(z - k_0) = 0$$

Its slope is

$$\mu = -\frac{2m_0 - 3n_0k_0}{2k_0 - 3n_0m_0}$$

Hence the line through (x_1, z_1) and parallel to the tangent line at O has the equation $z = \mu(x - x_1) + z_1$. We are looking for the second intersection point of this line with the quadric

$$x^2 + n_0^2 + (\mu(x - x_1) + z_1)^2 = 3n_0x\,(\mu(x - x_1) + z_1)\,.$$

We know that one solution of this is x_1. Hence the other solution is

$$\frac{\mu^2x_1^2 + z_1^2 - 2x_1\mu z_1 + n_0^2}{x_1(1 + \mu^2 - 3n_0\mu)}.$$

One has

$$1 + \mu^2 - 3n_0\mu = \frac{n_0^2(9n_0^2 - 4)}{(2k_0 - 3n_0m_0)^2},$$

and we obtain desired result after some routine computations. □

2.2.3. Connections between the Viete transformations and the group law

Note that, if $(m, k) \in Q_{n_0}(\mathbf{Z})$, then so are

$$\mathsf{A}(m, k) := (m, 3mn_0 - k), \quad \mathsf{TA}(m, k) := (3mn_0 - k, m),$$

$$\mathsf{C}(m, k) := (3n_0k - m, k), \quad \mathsf{TC}(m, k) := (k, 3n_0k - m),$$

$$\mathsf{B}(m, k) = (-m, -k)$$

where $\mathsf{T}(m, k) := (k, m)$ is the transposition. Since $\mathsf{A}^2 = \mathsf{T}^2 = Id$ and $\mathsf{C} = \mathsf{TAT}$, the transformations A and T generate the infinite dihedral group. Hence, the Markov triples lying on Q_{n_0} are produced by Viète transformations, starting from a fundamental point (m_0, n_0, k_0).

Suppose $P := (x_1, z_1) \in Q_{n_0}$. Then $\mathsf{C}P = (3n_0z_1 - x_1, z_1)$. In the computation of $P \oplus \mathsf{C}P$, one has $\mu = 0$ in Proposition 2.2, so that

$$P \oplus \mathsf{C}P = (3n_0k_0 - m_0, k_0) = \mathsf{C}O$$

Similarly, $\mathsf{A}P = (x_1, 3n_0x_1 - z_1)$ and in the computation of $P \oplus \mathsf{A}P$, one has $\mu = \infty$ in Proposition 2.2, so that

$$P \oplus \mathsf{A}P = (m_0, 3n_0m_0 - k_0) = \mathsf{A}O$$

The transformation

$$\mathsf{TA} : (m, n_0, k) \to (3mn_0 - k, n_0, m)$$

can be seen as a linear transformation $(m, k) \to (3mn_0 - k, m)$, i.e.

$$\begin{pmatrix} 3n_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} = \begin{pmatrix} 3mn_0 - k, & m \end{pmatrix}$$

One has

$$\begin{pmatrix} 3n_0 & -1 \\ 1 & 0 \end{pmatrix}^r = \begin{pmatrix} b_r & -b_{r-1} \\ b_{r-1} & -b_{r-2} \end{pmatrix}$$

where b_r is the Chebyshev-like polynomial defined by the recursion

$$b_{r+2}(n_0) = 3n_0 b_{r+1}(n_0) - b_r, \quad b_0(n_0) = 1, \quad b_1(n_0) = 3n_0$$

then

$$b_2(n_0) = 9n_0^2 - 1, \quad b_3(n_0) = 27n_0^3 - 6n_0, \quad b_5(n_0) = 81n_0^4 - 27n_0^2 + 1, ...$$

so that

$$\frac{b_{r+2}(n_0)}{b_{r+1}(n_0)} = 3n_0 - \frac{1}{\frac{b_{r+1}(n_0)}{b_r(n_0)}}.$$

Hence

$$\begin{pmatrix} b_r & -b_{r-1} \\ b_{r-1} & -b_{r-2} \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} = (mb_r(n_0) - kb_{r-1}(n_0), mb_{r-1}(n_0) - kb_{r-2}(n_0)).$$

So that

$$\mathsf{TA}^r(m, n_0, k) = (mb_r(n_0) - kb_{r-1}(n_0), n_0, mb_{r-1}(n_0) - kP_{r-2}(n_0))$$
$$\mathsf{TC}^r(m, n_0, k) = (kb_r(n_0) - mb_{r-1}(n_0), n_0, kb_{r-1}(n_0) - mb_{r-2}(n_0)).$$

This yields the following lemma.

Lemma 2.4 *Let* $(\mathsf{TA})^r = \mathsf{TA}(\mathsf{TA})^{r-1}$ *and* $(\mathsf{TC})^r = \mathsf{TC}(\mathsf{TC})^{r-1}$ *denote respectively* $r-$ *times composition of* TA *and* TC. *Then it is given by*

$$\mathsf{TA}^r : (m, n_0, k) \to (mb_r(n_0) - kb_{r-1}(n_0), n_0, mb_{r-1}(n_0) - kb_{r-2}(n_0)).$$

$$\mathsf{TC}^r : (m, n_0, k) \to (kb_r(n_0) - mb_{r-1}(n_0), n_0, kb_{r-1}(n_0) - mb_{r-2}(n_0)).$$

We claim that integral points in Q_{n_0} generated with operators TA and TC form a subgroup. We have shown with Maple implementations. We know that group law on quadrics as follows: Let $O(m_0, n_0, k_0)$ in Q_{n_0} and $A, B \in Q_{n_0}$ and L be a line through O which is parallel to line through A and B. Then $A \oplus B$ is defined to be second intersection point of L with Q_{n_0}.

We can prove this claim using Lemma 2.4 and the induction method. Firstly we will show that the line parallel to TA and the lines passing through points $O(m_0, n_0, k_0)$ and TA^2 are parallel. Clearly, it is sufficient to show that their slopes are equal. For $r = 2$ slope of the tangent line at the point TA is equal slope of the line passing through $O(m_0, n_0, k_0)$ and TA^2. Equation of the tangent line at TA

$$(2(3mn_0 - k) - 3n_0 m)(x - 3mn_0 + k) + (2m - 3(3mn_0 - k)n_0)(z - m) = 0 \tag{2.5}$$

hence slope of the tangent line (2.5)

$$\mu_1 = \frac{2k - 3mn_0}{2m - 9n_0^2 m + 3n_0 k}.$$

On the other hand, equation of the line passing through $O(m_0, n_0, k_0)$ and TA^2

$$(2k - 3n_0 m)(x - m) = (z - k)(2m - 9n_0^2 m + 3n_0 k + m) \tag{2.6}$$

then slope of the equation (2.6)

$$\mu_2 = \frac{2k - 3mn_0}{2m - 9n_0^2 m + 3n_0 k}.$$

Clearly $\mu_1 = \mu_2$. For $r = n - 1$ we will assume that slope of line passing through $O(m_0, n_0, k_0)$ and TA^{n-1} is equal slope of the tangent line passing through TA^{n-2} and TA^{n-1}. Using the induction hyphothesis and simple calculations we proved slope of line passing through $O(m_0, n_0, k_0)$ and TA^n is equal slope of the tangent line passing through TA^{n-1} and TA^n. The similar process can be repeated for TC. As a result integral elements in Q_{n_0} generated with operators TA and TC form a subgroup.

3. The double Fricke surface $\mathbf{F}^2$

Recall that $\mathbf{F}^2$ is the surface given by the equation $(x+y+z)^2 = 9xyz$. Let us denote the projectivization of $\mathbf{F}^2$ inside $\mathbf{P}(\mathbb{C}^3)$ by $\overline{\mathbf{F}}^2$. It has the equation $s(x+y+z)^2 = 9xyz$. It is easy to check that $[x, y, z, s] = [0, 0, 0, 1]$ is the only singular point of $\overline{\mathbf{F}}^2$. Besides the three finite lines

$$(t, -t, 0), \quad (0, t, -t) \quad (t, 0, -t),$$

$\overline{\mathbf{F}}^2$ contains the lines $[0, y, z, 0]$, $[x, 0, z, 0]$, and $[x, y, 0, 0]$ at infinity. The rational points of this surface have the parametrization

$$(P, Q) \in (\mathbf{Q}^*)^2 \to \left(\frac{(P^2 + Q^2 + 1)^2}{9Q^2}, \frac{(P^2 + Q^2 + 1)^2}{9P^2}, \frac{(P^2 + Q^2 + 1)^2}{9P^2 Q^2} \right) \in \mathbf{F}^2(\mathbf{Q}^*)$$

as one may check.

Theorem 3.1 *Positive integral points of $\mathbf{F}^2$ are precisely those of the form (m^2, n^2, k^2), where (m, n, k) is a Markov triple. All positive integral solutions are obtained from the solution triple $(1, 1, 1)$ by use of the transformations*

$$(x, y, z) \to (x, 9xy - 2x - 2y - z, y), \quad (x, y, z) \to (y, 9yz - 2y - 2z - x, z). \tag{3.1}$$

and permutations of coordinates.

Proof Suppose that (m, n, k) is a Markov triple. Then

$$m^2 + n^2 + k^2 = 3mnk \implies (m^2 + n^2 + k^2)^2 = 9m^2 n^2 k^2,$$

i.e. (m^2, n^2, k^2) is on $\mathbf{F}^2(\mathbf{Z}_{>0})$. Now suppose $(x, y, z) \in \mathbf{F}^2(\mathbf{Z}_{>0})$. Then $(\sqrt{x}, \sqrt{y}, \sqrt{z}) \in \mathbf{F}$, since

$$(x + y + z)^2 = 9xyz \implies x + y + z = 3\sqrt{xyz} \implies (\sqrt{x})^2 + (\sqrt{y})^2 + (\sqrt{z})^2 = 3\sqrt{xyz}$$

Let (m, n, k) is solution triple. Then m is a root of the polynomial

$$f(x) = (x + n + k)^2 - 9xnk = x^2 + (2n + 2k - 9nk)x + n^2 + k^2 + 2nk.$$

The other root m_0 thus satisfies

$$m_0 = 9nk - 2n - 2k - m = \frac{(n+k)^2}{m}. \tag{3.2}$$

The first equation in (3.3) says that m_0 is integer and second that m_0 is positive. If $(m, n, k) = (1, 1, 1)$ then $(m_0, 1, 1)$ is another solution triple. Repeating with second and third element of triple, we see that for every triple (m, n, k) we get three others. □

Claim. Solution triples of $(9yz - 2y - 2z - x, y, z)$ (3.1) obtained from the fundamental solution $(1, 1, 1)$ are the triples (x^2, y^2, z^2), where (x, y, z) is a Markov triple. Solution triples of Markov equation is $(3yz - x, y, z)$.

$$((3yz - x)^2, y^2, z^2) = (9y^2z^2 - 6xyz + x^2, y^2, z^2)$$

We know that (x, y, z) is Markov triple then we can get

$$(9y^2z^2 - 2y^2 - 2z^2 - x^2, y^2, z^2).$$

Note that $\mathbf{F}^2$ has other integral points. For example, we have the following solutions produced from the "fundamental solution" $(0, 1, -1)$:

$$(0, 1, -1), (-1, -9, 1), (1, -64, -9), (-9, 100, -1), (100, -8281, -9), \ldots$$

Proposition 3.2 *Let $(a, b, c) \in \mathbf{F}^2(\mathbf{Z})$ such that a, b, c are not simultaneously positive. Then (a, b, c) is produced from a "fundamental solution" $(-n, 0, n)$ with $n \in \mathbf{Z}$ by use of the transformations (3.1) and its permutations.*

Proof Let $(a, b, c) \in \mathbf{F}^2(\mathbf{Z})$. Then a is a root of the polynomial

$$f(x) = (x + b + c)^2 - 9xbc = x^2 + (2b + 2c - 9bc)x + b^2 + c^2 + 2bc.$$

The other root a_0 thus satisfies

$$a_0 = 9bc - 2b - 2c - a = \frac{(b+c)^2}{a}. \tag{3.3}$$

Hence (a_0, b, c) is another Markov triple. Repeating with b and c, we see that for every nonsingular triple (a, b, c) we get three others.

$$\begin{aligned}
&(a_0 = 9bc - 2b - 2c - a, b, c) \qquad (3.4)\\
&(a, b_0 = 9ac - 2a - 2c - b, c)\\
&(a, b, c_0 = 9ab - 2a - 2b - c)
\end{aligned}$$

Suppose $b = 0$ is a root of the polynomial then we get from (3.4) and double Markov equation $a = -c$. On the other hand, suppose without loss of the generality, $a = -c$. Since $(a, b, c) \in \mathbf{F}^2(\mathbf{Z})$ then $b^2 = -9a^2b$. This gives $b = 0$. □

Computer experiments indicate that the Frobenius conjecture also holds for the solution tree generated from the fundamental solution $(0, n, -n)$, in the sense that if (a, b, c) belongs to this tree, then the maximum of the numbers $|a|, |b|, |c|$ determines the remaining two.

Remark. Equation 1.3 appears in Perrine [**?**] (Pg 229, Théorème 2.1) as a member of a special family (E), though it has not been given special treatment. More generally, positive integral solution n-tuples of the equation $(x_1 + \cdots + x_n)^2 = k^2 x_1 \dots x_n$ are the squares of the solution n-tuples of the *Hurwitz equation* $x_1^2 + \cdots + x_n^2 = kx_1 \dots x_n$.

3.1. The composition law on $\mathbf{F}^2$.

Suppose (m, n, k) and (a, b, c) are two rational squared Markov triples. Let L be the line through them. It has the parametrization

$$x = (a - m)t + m, \quad y = (b - n)t + n, \quad z = (c - k)t + k$$

Its intersection with the Fricke surface has the equation

$$[(a - m)t + m + (b - n)t + n + (c - k)t + k]^2 = 9[(a - m)t + m][(b - n)t + n][(c - k)t + k]$$

Since $t = 0$ and $t = 1$ are two solutions of this equation, the third solution is

$$t = \frac{9(a - m)nk + 9(b - n)mk + 9(c - k)mn - 2(a + b + c - m - n - k)(m + n + k)}{9(a - m)(b - n)(c - k)}$$

Hence the third intersection point has the coordinates

$$x = (a - m)t_0 + m, \quad y = (b - n)t_0 + n, \quad z = (c - k)t_0 + k.$$

Routine algebraic manipulations yield the following result:

Proposition 3.3 *If $P = (a, b, c) \in \mathbf{F}^2(K)$ and $Q = (m, n, k) \in \mathbf{F}^2(K)$ are two points of $\mathbf{F}^2$ defined over some field K of characteristic 0, then the composition $P \circ Q = (x, y, z) \in \mathbf{F}^2(K)$ is given by*

$$\begin{aligned} x &= \frac{9(ank + bcm) - 2(a + b + c)(m + n + k)}{9(b - n)(c - k)} \\ y &= \frac{9(bmk + acn) - 2(a + b + c)(m + n + k)}{9(a - m)(c - k)} \\ z &= \frac{9(cmn + abk) - 2(a + b + c)(m + n + k)}{9(a - m)(b - n)} \end{aligned} \tag{3.5}$$

In case $(a, b, c) = (0, 0, 0)$ or $(a, b, c) = (m, n, k)$, the composition $(a, b, c) \circ (m, n, k)$ is not defined. Otherwise, if $(a - m)(b - n)(c - k) = 0$, it is necessary to projectivize to get the answer

$$(m, n, k) \circ (m, b, c) = [m : n : k : 1] \circ [m : b : c : 1] = [0 : b - n : c - k : 0].$$

Finally, the product is not well-defined if both of the points lie on the same line at infinity of $\mathbf{F}^2$.

Consequently, we define the operation

$$(a,b,c)\bar{\circ}(m,n,k) := (x_0, y_0, z_0)$$

This is commutative though not associative, and it satisfies

$$\alpha\bar{\circ}\beta = \gamma \iff \alpha\bar{\circ}\gamma = \beta \iff \beta\bar{\circ}\gamma = \iff \alpha$$

Note that the $\bar{\circ}$-product of two integral squared Markov triples are usually a nonintegral rational squared Markov triple. Also note that this is not the same product on $\mathbf{F}(\mathbf{Q})$, i.e. if $(a,b,c),(m,n,k) \in \mathbf{F}(\mathbf{Q})$ and $(a,b,c)\circ(m,n,k) = (p,q,r)$ then

$$(a^2,b^2,c^2)\bar{\circ}(m^2,n^2,k^2) \neq (p^2,q^2,r^2)$$

Let $(a,b,c) = (2,1,1)$ and $(m,n,k) = (1,2,5)$ then

$$(p,q,r) = (2,1,1)\circ(1,2,5) = \left(\frac{15}{4}, -\frac{3}{4}, -6\right).$$

If $(a^2,b^2,c^2) = (4,1,1)$ and $(m^2,n^2,k^2) = (1,4,25)$ then

$$(4,1,1)\bar{\circ}(1,4,25) = \left(\frac{361}{72}, -\frac{7}{24}, -\frac{28}{3}\right)$$

so we get

$$(a^2,b^2,c^2)\bar{\circ}(m^2,n^2,k^2) \neq (p^2,q^2,r^2).$$

3.2. Transfer of the structures to $\mathbb{P}^2(\mathbb{Q})$

We may transfer the Viete transformations and the composition $\circ$ to $\mathbb{P}^2(\mathbb{Q})$ by means of the parametrization φ. The Viete transformations are given by

$$\begin{aligned} L &: [p:q:r] \to [pr:(p+q)^2:qr] \\ R &: [p:q:r] \to [qp:(q+r)^2:pr]. \end{aligned} \tag{3.6}$$

Note that these are birational transformations which are not everywhere defined. On the other hand, viewed as transformations of $\mathbb{P}^2(\mathbf{Q})$, they are well defined everywhere, except at the points $[0,0,1]$, $[0,1,0]$, and $[1,0,0]$. We may apply permutations of coordinates to (3.6) to get the involutions

$$[p:q:r] \to [pr:qr:(p+q)^2]$$

$$[p:q:r] \to [(q+r)^2:pq:pr]$$

$$[p:q:r] \to [pq:(p+r)^2:rq],$$

which define a birational action of $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ on $\mathbb{P}^2$. We may also transfer the composition $\circ$ to $\mathbb{P}^2(\mathbf{Q})$ by means of φ as:

$$[a:b:c],[m:n:k] \in \mathbb{P}^2(\mathbf{Q}) \implies [a:b:c]\circ[m:n:k] =$$

$$\Big[\big((a+b+c)^2kn-(m+n+k)^2bc\big)\left((ank+bcm)-2(a+b+c)(m+n+k)\right),$$
$$\big((a+b+c)^2km-(m+n+k)^2ac\big)\left((bmk+acn)-2(a+b+c)(m+n+k)\right),$$
$$\big((a+b+c)^2mn-(m+n+k)^2ba\big)\left((cmn+abk)-2(a+b+c)(m+n+k)\right).\Big]$$

3.3. Quadric sections of $\mathbf{F}^2$

Suppose (m_0,n_0,k_0) is a squared Markov triple. Then the plane $H:=\{(x,y,z): y=n_0\}$ contains infinitely many squared Markov triples. The intersection $H\cap\mathbf{F}^2$ is a quadric $Q_{n_0}\subset H$ with a special point (m_0,n_0,k_0), given by the equation

$$Q_{n_0}: \quad (x+n_0+z)^2 = 9xn_0z \tag{3.7}$$

Consider this as a quadric in the $x-z$ plane, and projectivize by s to get the equation

$$(x+n_0s+z)^2 = 9xzn_0$$

This quadric intersects the line $s=0$ at the points $(x+z)^2=9xzn_0$. Deprojectivize by setting $t=x/z$, we find the points at infinity of Q_{n_0} to be

$$t=\frac{3n_0-2\pm\sqrt{(9n_0-2)^2-4}}{2}$$

3.3.1. The group law on the quadric sections $H\cap\mathbf{F}^2$.

Proposition 3.4 *Let $P_1:=(x_1,z_1)$ and $P_2:=(x_2,z_2)$ are on Q_{n_0}. Set $\sigma=-(2x_1+2n_0+2z_1-9n_0z_1)/(2z_1+2n_0+2x_1-9n_0x_1)$. If $P_1\neq P_2$, then $P_1\oplus P_2=(x,z)$, where*

$$x=\frac{9n_0k_0(x_2-x_1)^2-(2n_0+2k_0)(x_2-x_1)(z_2-z_1)}{(z_2-z_1+x_2-x_1)^2-9n_0(x_2-x_1)(z_2-z_1)}$$
$$-\frac{(m_0+2n_0+2k_0)(x_2-x_1)^2+m_0(z_2-z_1)^2}{(z_2-z_1+x_2-x_1)^2-9n_0(x_2-x_1)(z_2-z_1)},$$

$$z=\frac{(z_2-z_1)^2(9n_0m_0-2m_0-2n_0-k_0)-2(z_2-z_1)(x_2-x_1)(m_0+n_0)+k_0}{-2n_0(n_0+x_2+z_2+x_1+z_1)+(9n_0-2)(z_2x_1+z_1x_2)-2(z_1z_2+x_1x_2)}.$$

If $P=(x_1,z_1)$, then $P\oplus P=(x,z)$, where

$$x=\frac{\left[9n_0(n_0+k_0)x_1+m_0z_1\right]^2-4(n_0+k_0+m_0)\left[x_1(x_1+n_0)(n_0+k_0)+z_1(z_1+n_0)m_0\right]}{9n_0^3m_0},$$

$$z=\frac{\left[9n_0(n_0+m_0)z_1+k_0x_1\right]^2-4(n_0+m_0+k_0)\left[z_1(z_1+n_0)(n_0+m_0)+x_1(x_1+n_0)k_0\right]}{9n_0^3k_0}.$$

Proof We take $O = (m_0, n_0, k_0)$ as the neutral element. Suppose $P_1 := (x_1, z_1)$ and $P_2 := (x_2, z_2)$ are on Q_{n_0}. If $P_1 \neq P_2$ then the line through O and parallel to P_1P_2 has the equation

$$z = \frac{z_2 - z_1}{x_2 - x_1}(x - m_0) + k_0$$

and the intersection has the equation

$$\left(x + \left(\frac{z_2 - z_1}{x_2 - x_1}(x - m_0) + k_0\right) + n_0\right)^2 = 9xn_0\left(\frac{z_2 - z_1}{x_2 - x_1}(x - m_0) + k_0\right)$$

Set $u := x - m_0$ and $\mu = z_2 - z_1/x_2 - x_1$. Then the equation becomes

$$(u + m_0 + (\mu u + k_0) + n_0)^2 = 9(u + m_0)n_0\,(\mu u + k_0)$$

and we know that $x = m_0 \iff u = 0$ is one solution of this equation.

$$(u(\mu + 1) + m_0 + n_0 + k_0)^2 = 9(u + m_0)n_0\,(\mu u + k_0)$$

$$u = \frac{-2(\mu + 1)(m_0 + n_0 + k_0) + 9n_0k_0 + 9n_0m_0\mu}{(\mu + 1)^2 - 9n_0\mu}$$

Thus $x = u + m_0$, $z = u\mu + k_0 \implies$

$$x = \frac{9n_0k_0 - 2\mu n_0 - 2\mu k_0 - m_0 - 2n_0 - 2k_0 + m_0\mu^2}{(\mu + 1)^2 - 9n_0\mu}$$

$$z = \frac{9n_0m_0\mu^2 - 2\mu^2m_0 - 2\mu^2n_0 - \mu^2k_0 - 2\mu m_0 - 2\mu n_0 + k_0}{(\mu + 1)^2 - 9n_0\mu}$$

If $z_1 = z_2 = k$ then $\mu = 0$ and $x = 9n_0k_0 - 2n_0 - 2k_0 - m_0$ $z = k_0$. Hence

$$(x_1, k) \circ (x_2, k) = (9n_0k_0 - 2n_0 - 2k_0 - m_0, k_0)$$

Note that $z_1 = z_2 = k$ happens when $(x_2, z_2) = (9n_0z_1 - 2n_0 - 2z_1 - x_1, z_1)$. Similarly, if $x_1 = x_2 = k$ then $\mu = \infty$ and $x = m_0$, $z = 9n_0m_0 - 2m_0 - 2n_0 - k_0$. Hence

$$(m, z_1) \circ (m, z_2) = (m_0, 9n_0m_0 - 2m_0 - 2n_0 - k_0).$$

Note that $x_1 = x_2 = m$ happens when $(x_2, z_2) = (x_1, 9n_0x_1 - 2n_0 - 2x_1 - z_1)$.

For general μ we have

$$x = \frac{9n_0k_0(x_2 - x_1)^2 - (2n_0 + 2k_0)(x_2 - x_1)(z_2 - z_1)}{(z_2 - z_1 + x_2 - x_1)^2 - 9n_0(x_2 - x_1)(z_2 - z_1)} - \frac{(m_0 + 2n_0 + 2k_0)(x_2 - x_1)^2 + m_0(z_2 - z_1)^2}{(z_2 - z_1 + x_2 - x_1)^2 - 9n_0(x_2 - x_1)(z_2 - z_1)}$$

For the denominator one has

$$(z_2 - z_1)^2 + 2(z_2 - z_1)(x_2 - x_1) + (x_2 - x_1)^2 - 9n_0(x_2 - x_1)(z_2 - z_1)$$
$$= -2n_0(n_0 + x_2 + z_2 + x_1 + z_1) + (9n_0 - 2)(z_2x_1 + z_1x_2) - 2(z_1z_2 + x_1x_2)$$

and

$$z = \frac{(z_2 - z_1)^2(9n_0m_0 - 2m_0 - 2n_0 - k_0) - 2(z_2 - z_1)(x_2 - x_1)(m_0 + n_0) + k_0}{-2n_0(n_0 + x_2 + z_2 + x_1 + z_1) + (9n_0 - 2)(z_2x_1 + z_1x_2) - 2(z_1z_2 + x_1x_2)}.$$

It remains to establish the case $P = Q$. In this case, the line tangent to the quadric is given by the equation

$$(2x_1 + 2n_0 + 2z_1 - 9n_0z_1)(x - x_1) + (2z_1 + 2n_0 + 2x_1 - 9n_0z_1)(z - z_1) = 0,$$

whose slope is

$$\sigma = -\frac{2x_1 + 2n_0 + 2z_1 - 9n_0z_1}{2z_1 + 2n_0 + 2x_1 - 9n_0x_1}.$$

The line through (m_0, n_0, k_0) with this slope has the equation

$$z - k_0 = -\frac{2x_1 + 2n_0 + 2z_1 - 9n_0z_1}{2z_1 + 2n_0 + 2x_1 - 9n_0x_1}(x - m_0),$$

and the second intersection of this line with Q_{n_0} has the equation

$$(x + n_0 + (\sigma(x - m_0) + k_0))^2 = 9xn_0\left(\sigma(x - m_0) + k_0\right).$$

We know that m_0 is a solution to this equation. Hence the second solution is

$$x = \frac{[(n_0 + k_0)(2z_1 + 2n_0 + 2x_1 - 9n_0x_1) + m_0(2x_1 + 2n_0 + 2z_1 - 9n_0z_1)]^2}{m_0[(9n_0z_1 - 9n_0x_1)^2 + 9n_0(2z_1 + 2n_0 + 2x_1 - 9n_0z_1)(2z_1 + 2n_0 + 2x_1 - 9n_0x_1)]}$$

We obtain desired result after some routine computations and simplification. □

3.3.2. Inverses

Proposition 3.5 *The inverse of* $P_1 := (x_1, z_1)$ *is* (x, z) *with*

$$x = \frac{\sigma^2{x_1}^2 - 2\sigma x_1z_1 + {n_0}^2 + 2n_0z_1 - 2n_0\sigma x_1 + {z_1}^2}{81n_0^4m_0}$$

$$z = -\sigma x_1 + z_1 + \frac{2\sigma\left(z_1 + \sigma x_1\right)}{81{n_0}^3m_0} + \frac{\sigma}{81{n_0}^2m_0} - \frac{\sigma\left(\sigma x_1 - z_1\right)^2}{81{n_0}^4m_0}$$

where $\sigma = -\frac{2x_1+2n_0+2z_1-9n_0z_1}{2z_1+2n_0+2x_1-9n_0x_1}$.

Proof To find the inverse of (x_1, z_1), we start with the tangent line at O:

$$(2x_1 + 2n_0 + 2z_1 - 9n_0z_1)(x - m_0) + (2z_1 + 2n_0 + 2x_1 - 9n_0x_1)(z - k_0) = 0$$

Its slope is

$$\sigma = -\frac{2x_1 + 2n_0 + 2z_1 - 9n_0z_1}{2z_1 + 2n_0 + 2x_1 - 9n_0x_1} = -\frac{m_0' - m_0}{k_0' - k_0}.$$

Hence the line through (x_1, z_1) and parallel to the tangent line at O has the equation $z = \sigma(x - x_1) + z_1$. We are looking for the second intersection point of this with the quadric:

$$(x + n_0 + \sigma(x - x_1) + z_1)^2 = 9xn_0(\sigma(x - x_1) + z_1)$$

We know that one solution of this is x_1. Hence the other solution is

$$\frac{\sigma^2 {x_1}^2 - 2\,\sigma x_1\, z_1 + {n_0}^2 + 2 n_0 z_1 - 2 n_0 \sigma x_1 + {z_1}^2}{2\sigma + 1 - 9\, n_0 \sigma + \sigma^2}.$$

Note that the denominator is independent of (x_1, z_1).

$$= (9n_0)^2 \left(-x_1^2 - z_1^2 - 2x_1 z_1 - 2n_0 z_1 - 2n_0 x_1 + 9n_0 x_1 z_1\right) = 81 n_0^4 m_0$$

□

4. Figures

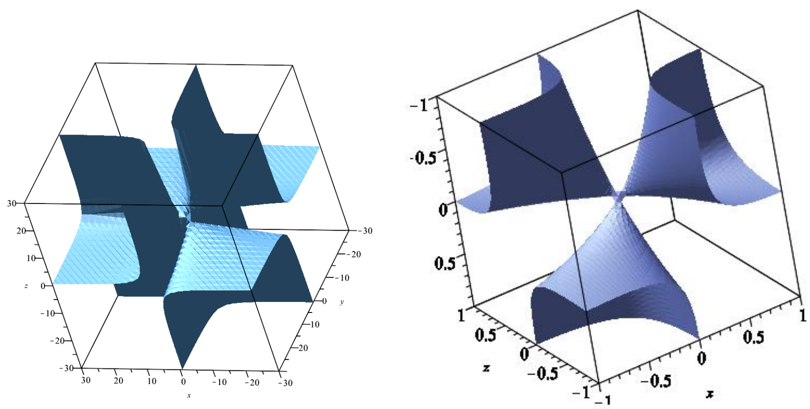

Figure 1. Fricke surface (left) and the double Fricke surface.

5. Conclusion

Effective results were obtained on the Fricke surface and double Fricke surface.

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