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# Minimal generators of annihilators of even neat elements in the exterior algebra

**Dedicated to the memory of Professor Cemal Koç**

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**Abstract:** This paper deals with an exterior algebra of a vector space whose base field is of positive characteristic. In this work, a minimal set of generators forming the annihilator of even neat elements of such an exterior algebra is exhibited. The annihilator of some special type of even neat elements is determined to prove the conjecture established in [3]. Moreover, a vector space basis for the annihilators under consideration is calculated.

**Key words:** Exterior algebra, Frobenius algebra, symmetric algebra, minimal generators, annihilator

## 1. Introduction

Let $V$ be a finite dimensional vector space over a field $F$, and let $E(V)$ be the exterior algebra on $V$. An element $\xi \in E(V)$ is called a decomposable $m-$vector if $\xi = x_1 \wedge x_2 \wedge \cdots \wedge x_m$ for some $x_1, x_2, \cdots, x_m \in V$. A sum $\mu = \xi_1 + \xi_2 + \cdots + \xi_s$ of decomposable elements of $E(V)$ is said to be **neat** if $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_s \neq 0$. This neat element $\mu$ is called **even** if each term $\xi_k = x_{k1} \wedge x_{k2} \wedge \cdots \wedge x_{kn_k}$ for each $k \in \{1, 2, \ldots, s\}$ is $n_k-$vector with even $n_k$.

In this paper we want

**(i)** to prove the annihilator of even neat element $\mu = \xi_1 + \xi_2 + \cdots + \xi_s$ is generated as an ideal by products of the form

$$(\xi_{i_1} - \xi_{j_1}) \cdots (\xi_{i_r} - \xi_{j_r}) u_{k_1} \ldots u_{k_t}$$

where $u_{k_l} \in M_{k_l}$ and $\{i_1, \ldots, i_r; j_1, \ldots, j_r; k_1, \ldots, k_t\} = \{1, 2, \ldots, s\}$ with $2r + t = s$ when $Char(F) = p > \frac{s+1}{2}$.

**(ii)** to determine minimal generators of the annihilator of $\mu$ for all characteristics,

**(iii)** to describe the vector space structure of both the principal ideal $(\mu)$ and its annihilator $Ann(\mu)$ in $E(V)$ by using stack-sortable polynomials introduced in [1] and the results of [2].

**Remark 1.1** *The first aim is achieved in [3] under the assumption the characteristic of the field is zero.*

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To begin with we give the notations that will be used throughout the text.

$S = \{1, \ldots, s\}$

$\Gamma$ : the ideal of the polynomial ring $F[x_1, \ldots, x_s]$ generated by $x_1^2, \ldots, x_s^2$.

$A = F[x_1, \ldots, x_s]/\Gamma = F[\xi_1, \ldots, \xi_s]$ with $\xi_i = x_i + \Gamma$.

$A_k$ : the homogeneous component of $A$ consisting of homogeneous elements of degree $k$.

$\mu = \xi_1 + \cdots + \xi_s$

$\mathcal{M}_t$ : the set of $t$-$th$ degree monomials $M_K = \xi_{k_1} \cdots \xi_{k_t}$ where $K = \{k_1, \ldots, k_t\} \subset \{1, \ldots, s\}$.

$\mathcal{G}_r$ : the set of elements of the form

$$\gamma_{I,J} = (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r})$$

where $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_r\}$ are disjoint subsets of $S$.

$\mathcal{G}_S$ : the set of all products of the form $\gamma_{I,J} M_K$ with $I \cup J \cup K = S$ and $|I| + |J| + |K| = |S|$.

$\mathcal{P}_S$ : the set of stack-sortable polynomials

$$(\zeta_{\sigma(1)} - \eta_1) \cdots (\zeta_{\sigma(d)} - \eta_d) \text{ with } \sigma \text{ is a stack-sortable permutation,}$$

where $\eta_k = \xi_{2r}$, $\zeta_k = \xi_{2r-1}$ and $\eta_d = 0$ when $s$ is odd, [2].

In [3, Theorem 14] it is proved that if $Char(F) = 0$, then $Ann(\mathcal{G}_S) = (\mu)$, equivalently $Ann(\mu) = (\mathcal{G}_S)$. Also, the nonzero characteristic cases, i.e. $Char(F) = p$ are discussed in [3, Proposition 15]. In the case $Char(F) = p$, for $\eta = \xi_1 \cdots \xi_m u_{2m} u_{2m+1} \cdots u_s$ with $u_i \in M_i$, we have

**(a)** $\eta \in Ann(\mathcal{G}_S)$, and

**(b)** $\eta \in Ann(\mu)$ if and only if $m < p$.

As a corollary of Proposition 15, it is proved that if $Char(F) = p > \frac{s+1}{2}$ then any element of the form $\eta = \xi_1 \cdots \xi_m u_{2m} u_{2m+1} \cdots u_s$ is in $(\mu)$. Hence when $Char(F) = p > \frac{s+1}{2}$ we have $Ann(\mathcal{G}_S) = (\mu)$. On the other hand it can be constructed several examples that $Ann(\mathcal{G}_S) \neq (\mu)$ in the case $Char(F) = p \leq \frac{s+1}{2}$.
For example, consider $\mu = \xi_1 + \xi_2 + \xi_3$. Then $Ann(\mathcal{G}_S)$ is generated by $(\xi_1 - \xi_2)u_3$, $(\xi_1 - \xi_3)u_2$, and $(\xi_2 - \xi_3)u_1$. Observe that $\xi_1\xi_2 \in Ann(\mathcal{G}_S)$ but $\xi_1\xi_2 \notin (\mu)$ in the case that $Char(F) = 2$.

In this paper to reach our aims, in section 2 we show that the algebra $A = F[\xi_1, \ldots, \xi_s]$ has a nondegenerate bilinear form which is also symmetric. That is $A$ is a Frobenius algebra as well as a symmetric algebra. By [3], the exterior algebra $E(V)$ is a Frobenius algebra. Hence, the following equations hold

$$\dim(E) = \dim(E\mu) + \dim Ann_E(\mu) \text{ and } \dim(A) = \dim(A\mu) + \dim Ann_A(\mu).$$

In section 3, we determine generators of $Ann(\mathcal{G}_S)$ and show that $Ann(\mathcal{G}_S) = A\mu$ for positive characteristic greater than $\frac{s+1}{2}$ in Theorem 3.5. In the last section, regarding $A$ as a subalgebra of an exterior algebra $E(V)$ we determine the annihilator of $\mu$ in $E$. By using techniques of [2] and linearly independence of $\{\theta(p^\tau(\overline{\xi};\overline{\eta})) \mid \tau \in St_m^{(2m-s)}\}$ proved in [2, Theorem 3] we determine the minimal generators of $\mu$ for all characteristics.

## 2. Frobenius algebra structure of multilinear polynomials

Recall that a finite dimensional algebra is called a *Frobenius algebra* if there is a *nondegenerate bilinear form* $B$ satisfying the associativity condition $B(ab,c)=B(a,bc)$ for all elements $a,b,$ and $c$ of the algebra. Further, if $B$ is symmetric the algebra is called a *symmetric algebra.* The exterior algebra is an important example of Frobenius algebras [3]. First of all we note that $A$ is a symmetric algebra.

**Lemma 2.1** *$A=F[\xi_1,\dots,\xi_s]$ is a symmetric algebra, i.e. it has a nondegenerate symmetric bilinear form*

$$B:A\times A\to F \quad \text{such that } B(ab,c)=B(a,bc) \quad \text{for all } \ a,b,c\in A$$

**Proof** Let $\varphi$ be the linear form on $A$ sending each $a\in A$ to its leading coefficient i.e. the coefficient of the monomial $\xi_1\dots\xi_s$ in the expression for $a$. Then

$$B(a,b)=\varphi(ab)$$

provides a bilinear form whose matrix relative to the standard basis of monomials is a permutation matrix since

$$B(\xi_{i_1}\cdots\xi_{i_k},\xi_{j_1}\cdots\xi_{j_l})=\begin{cases}1 & \text{if} \quad \{i_1,\cdots,i_{k,},j_1,\cdots,j_l\}=S\\ 0 & \text{otherwise}\end{cases}.$$

The last part follows at once from the associativity in $A$. □

## 3. Annihilators of principal ideals of the exterior algebra

In this section we will describe generators of $Ann_A(\mu)$ where $Char(F)>\frac{s+1}{2}$. First we give the following lemmas that are used in several proofs in this paper.

**Lemma 3.1** *Let $\eta\in\mathcal{G}_S$. Then $\mu\eta=0$.*

**Proof** Since $\eta\in\mathcal{G}_S$, we write

$$\eta=\xi_{k_1}\cdots\xi_{k_t}(\xi_{i_1}-\xi_{j_1})(\xi_{i_2}-\xi_{j_2})\cdots(\xi_{i_r}-\xi_{j_r})$$

where $\{i_1,\dots,i_r;j_1,\dots,j_r;k_1,\dots,k_t\}=\{1,\dots,s\}=S.$

$$\begin{aligned}\mu\eta &= (\xi_1+\cdots+\xi_s)\xi_{k_1}\cdots\xi_{k_t}(\xi_{i_1}-\xi_{j_1})\cdots(\xi_{i_r}-\xi_{j_r})\\ &= ((\xi_{i_1}+\xi_{j_1})+\cdots+(\xi_{i_r}+\xi_{j_r}))\xi_{k_1}\cdots\xi_{k_t}(\xi_{i_1}-\xi_{j_1})\cdots(\xi_{i_r}-\xi_{j_r})\end{aligned}$$

since $\xi_{k_u}\xi_{k_u}=0$ for all $u=1,2,\dots t$.
Also, for all $u=1,2,\dots r$, $(\xi_{i_u}+\xi_{j_u})(\xi_{i_u}-\xi_{j_u})=\xi_{i_u}\xi_{j_u}-\xi_{j_u}\xi_{i_u}=0$. Then we get $\mu\eta=0$. □

**Lemma 3.2** *Let $\omega\in A_r$ such that $\omega=\omega_0+\omega_1\xi_s$ with $\omega_0,\omega_1\in F[\xi_1,\dots,\xi_{s-1}]$ where*

$$\omega_0=\omega_0'(\xi_1+\cdots+\xi_{s-1}) \quad \textit{and} \quad \omega_1-\omega_0'=\omega_1'(\xi_1+\cdots+\xi_{s-1})$$

*then $\omega=\omega'\mu$ for some $\omega'\in F[\xi_1,\dots,\xi_s]$.*

**Proof** Suppose $\omega \in A_r$ as defined in the hypothesis. Then

$$\begin{aligned}\omega &= \omega_0'(\xi_1+\cdots+\xi_{s-1})+\omega_1'(\xi_1+\cdots+\xi_{s-1})\xi_s+\omega_0'\xi_s\\ &= \omega_0'(\xi_1+\cdots+\xi_s)+\omega_1'(\xi_1+\cdots+\xi_{s-1}+\xi_s)\xi_s\\ &= (\omega_0'+\omega_1'\xi_s)(\xi_1+\cdots+\xi_s)=\omega'\mu\end{aligned}$$

where $\omega' \in F[\xi_1,\ldots,\xi_s]$. □

**Lemma 3.3** *If $Char(F) > r > \frac{s}{2}$, then $\mathcal{M}_r \subset A\mu$.*

**Proof** Since $s < 2r$, it follows from $s - r < r$ that

$$(\xi_{j_1}+\cdots+\xi_{j_{s-r}})^r = 0$$

for any $J = \{j_1,\ldots,j_{s-r}\} \subset S$. Therefore for each $I = \{i_1,\ldots,i_r\}$ by considering its complement $J = \{j_1,\ldots,j_{s-r}\}$ in $S$, we obtain

$$\begin{aligned}\xi_{i_1}\cdots\xi_{i_r} &= \frac{1}{r!}(\xi_{i_1}+\cdots+\xi_{i_r})^r\\ &= \frac{1}{r!}(\xi_{i_1}+\cdots+\xi_{i_r})^r-(-1)^r\frac{1}{r!}(\xi_{j_1}+\cdots+\xi_{j_{s-r}})^r\end{aligned}$$

Take $a = \xi_{i_1}+\cdots+\xi_{i_r}$ and $b = (-1)(\xi_{j_1}+\cdots+\xi_{j_{s-r}})$. Then

$$\begin{aligned}\xi_{i_1}\cdots\xi_{i_r} &= \frac{1}{r!}(a^r-b^r)\\ &= \frac{1}{r!}(a-b)(\underbrace{a^{r-1}+a^{r-2}b+\cdots+b^{r-1}}_{\beta})\\ &= \frac{1}{r!}(\xi_{i_1}+\cdots+\xi_{i_r}+\xi_{j_1}+\cdots+\xi_{j_{s-r}})\beta\\ &= \frac{1}{r!}(\xi_1+\cdots+\xi_s)\beta\end{aligned}$$

where $I \cup J = S$ for some $\beta \in A$. □

**Proposition 3.4** *If $\omega \in A_r$ annihilates all elements of the form*

$$\xi_{k_1}\cdots\xi_{k_t}(\xi_{i_1}-\xi_{j_1})(\xi_{i_2}-\xi_{j_2})\cdots(\xi_{i_r}-\xi_{j_r})$$

*then $\omega \in A_{r-1}\mu$.*

**Proof** We use induction on the pairs $(s,r)$ where $2r \le s$ with respect to lexicographic ordering. The case $(s,1)$ is trivial. Suppose the assertion is true for all $(s',r') < (s,r)$. Let

$$\omega\xi_{k_1}\cdots\xi_{k_t}(\xi_{i_1}-\xi_{j_1})(\xi_{i_2}-\xi_{j_2})\cdots(\xi_{i_r}-\xi_{j_r}) = 0 \;\text{ when } I\cup J\cup K = S.$$

There are two cases to consider.

**Case 1:** $s > 2r$. In this case by writing $\omega = \omega_0 + \omega_1 \xi_s$ with $\omega_0, \omega_1 \in F[\xi_1, \ldots, \xi_{s-1}]$ we obtain

$$\omega \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) = 0.$$

We may assume WLOG that $\xi_{k_t} = \xi_s$. Then

$$\begin{aligned}
\omega \xi_{k_1} \cdots \xi_{k_{t-1}} \xi_s (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) &= 0 \\
(\omega_0 + \omega_1 \xi_s) \xi_{k_1} \cdots \xi_{k_{t-1}} \xi_s (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) &= 0 \\
\omega_0 \xi_{k_1} \cdots \xi_{k_{t-1}} \xi_s (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) &= 0 \\
\omega_0 \xi_{k_1} \cdots \xi_{k_{t-1}} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) &= 0
\end{aligned}$$

when $I \cup J \cup K = S - \{s\}$. Application of the induction hypothesis to the pair of $(s-1, r)$ gives

$$\begin{aligned}
\omega_0 &= \alpha_0(\xi_1 + \cdots + \xi_{s-1}) \quad \text{where } \alpha_0 \in F[\xi_1, \ldots, \xi_{s-1}] \\
\omega &= \alpha_0(\xi_1 + \cdots + \xi_{s-1}) + \omega_1 \xi_s \\
&= \alpha_0(\xi_1 + \cdots + \xi_s) + (\omega_1 - \alpha_0)\xi_s \\
&= \alpha_0 \mu + (\omega_1 - \alpha_0)\xi_s.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\omega \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) &= 0 \\
(\alpha_0 \mu + (\omega_1 - \alpha_0)\xi_s) \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) &= 0
\end{aligned}$$

when $I \cup J \cup K = S$ and by Lemma 3.1 we get,

$$(\omega_1 - \alpha_0)\xi_s \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_r} - \xi_{j_r}) = 0.$$

Now, we may assume WLOG $\xi_{j_r} = \xi_s$ and so

$$(\omega_1 - \alpha_0)\xi_s \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_{r-1}} - \xi_{j_{r-1}})(\xi_{i_r} - \xi_s) = 0$$

when $I \cup J \cup K = S$. Hence

$$\begin{aligned}
(\omega_1 - \alpha_0)\xi_s \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_{r-1}} - \xi_{j_{r-1}})\xi_{i_r} &= 0 \\
(\omega_1 - \alpha_0) \xi_{k_1} \cdots \xi_{k_t} (\xi_{i_1} - \xi_{j_1})(\xi_{i_2} - \xi_{j_2}) \cdots (\xi_{i_{r-1}} - \xi_{j_{r-1}})\xi_{i_r} &= 0
\end{aligned}$$

when $I \cup J \cup K = S - \{s\}$, and the induction hypothesis yields

$$\omega_1 - \alpha_0 = \alpha_1(\xi_1 + \cdots + \xi_{s-1})$$

and hence by Lemma 3.2 $\omega = \omega' \mu$ for some $\omega' \in F[\xi_1, \ldots, \xi_s]$ as asserted.

**Case 2:** $s = 2r$. Again we write $\omega = \omega_0 + \omega_1 \xi_s$ with $\omega_0$ and $\omega_1$ in $F[\xi_1, \ldots, \xi_{s-1}]$ of degrees $r$ and $r-1$, respectively. Since $s - 1 < 2r$, by Lemma 3.3, $\omega_0 = \alpha_0(\xi_1 + \cdots + \xi_{s-1})$ and therefore

$$\begin{aligned}
\omega &= \omega_0 + \omega_1 \xi_s \\
&= \alpha_0 \mu + (\omega_1 - \alpha_0)\xi_s.
\end{aligned}$$

with $\alpha_0$ and $\omega_1-\alpha_0$ of degree $r-1$ in $F[\xi_1,\ldots,\xi_{s-1}]$ as in Case 1. Thus, we have

$$\begin{aligned}(\omega_1-\alpha_0)\xi_s(\xi_{i_1}-\xi_{j_1})(\xi_{i_2}-\xi_{j_2})\cdots(\xi_{i_{r-1}}-\xi_{j_{r-1}})\xi_{i_r}\xi_{k_1}\cdots\xi_{k_t} &= 0\\ (\omega_1-\alpha_0)(\xi_{i_1}-\xi_{j_1})(\xi_{i_2}-\xi_{j_2})\cdots(\xi_{i_{r-1}}-\xi_{j_{r-1}})\xi_{i_r}\xi_{k_1}\cdots\xi_{k_t} &= 0\end{aligned}$$

and it yields that

$$\omega_1-\alpha_0=\beta_0(\xi_1+\cdots+\xi_{s-1})$$

by induction applied to the pair $(s-1,r-1)$. By Lemma 3.2, $\omega=\omega'\mu$ and the proof is completed. □

Now we can establish the following theorem.

**Theorem 3.5** *Using the notation given in the introduction, we have* $Ann(\mathcal{G}_S)=A\mu$ *and hence*

$$A\mathcal{G}_S=Ann(A\mu)\quad \textit{and}\ \dim(A\mathcal{G}_S)+\dim(A\mu)=\dim(A)=2^s$$

**Proof** The inclusion $A\mu\subset Ann(\mathcal{G}_S)$ is obvious. It remains to prove $Ann(\mathcal{G}_S)\subset A\mu$. Thus, take any $\omega\in Ann(\mathcal{G}_S)$ of degree $r$ and show that $\omega\in A\mu$. If $r>\frac{s}{2}$ the assertion follows from Lemma 3.3. In the case $r\le\frac{s}{2}$, letting $t=s-2r$ we can write

$$\omega\xi_{k_1}\cdots\xi_{k_t}(\xi_{i_1}-\xi_{j_1})(\xi_{i_2}-\xi_{j_2})\cdots(\xi_{i_r}-\xi_{j_r})=0\ \text{ when } I\cup J\cup K=S$$

and the result follows from Proposition 3.4. □

As a final remark for this section, we exhibit a basis for $Ann(A\mu)$. Since every element in $\mathcal{G}_S$ is a linear combination of stack-sortable polynomials (see [1]), we achieve the equality

$$Ann(A\mu)=A\mathcal{P}_S.$$

Note that $\mathcal{I}=Ann(A\mu)$ is a graded ideal, say

$$\mathcal{I}=\mathcal{I}_d\oplus\cdots\oplus\mathcal{I}_s$$

where $d$ is the integral part of $\dfrac{s+1}{2}$. As stated in [2, Theorem 3 and 4] the set $\{\theta(p^\tau(\overline{\xi};\overline{\eta}))\mid\tau\in St_m^{(2m-s)}\}$ is a basis for $\mathcal{I}_m$ and that $\dim(\mathcal{I})=\dbinom{s}{s-d}$.

## 4. An F-basis for annihilators of principal ideals

In this section, we regard $A$ as a subalgebra of an exterior algebra $E(V)$. In addition to the notations given in the introduction, we also introduce the following notations:

$V=V_1\oplus\cdots\oplus V_s$ : an $F-$space of dimension $n$ with a direct sum decomposition

$X_k=\{x_{k1},\ldots,x_{kn_k}\}$ : a basis for $V_k$

$X=\bigcup_{k=1}^{s}X_k$, a basis for $V$

$E = E(V)$ : the exterior algebra on $V$

$\xi_k = x_{k1} \dots x_{kn_k}$ and $n_t$ 's are all even

$\mu = \xi_1 + \dots + \xi_s$

$A = F[\xi_1, \dots, \xi_s]$ as a subalgebra of $E$

$p_k$ : a product of elements of $X_k$ different from 1 and $\xi_k$

$P_K =$ The set of products of the form $p_{k_1} \dots p_{k_t}$ for $K = \{k_1, \dots, k_t\} \subset S$

$\mathcal{G}'_K = \mathcal{G}_{S-K} P_K$

$\mathcal{A} := A\mathcal{G}'_K$

Now we can state our main result.

**Theorem 4.1** *The annihilator of $\mu$ in $E$ is the ideal $\mathcal{A}$ generated by elements of the form*

$$(\xi_{i_1} - \xi_{j_1}) \cdots (\xi_{i_r} - \xi_{j_r}) u_{k_1} \cdots u_{k_t}$$

*where $u_k \in \{x_{k1}, \dots, x_{kn_k}\}$ and $\{i_1, \dots, i_r; j_1, \dots, j_r; k_1, \dots, k_t\} = \{1, \dots, s\}$ when $Char(F) > \frac{s+1}{2}$.*

*Further the elements $(\xi_{i_1} - \xi_{j_1}) \cdots (\xi_{i_r} - \xi_{j_r}) u_{k_1} \cdots u_{k_t}$ for which*

$$(\xi_{i_1} - \xi_{j_1}) \cdots (\xi_{i_r} - \xi_{j_r}) \xi_{k_1} \dots \xi_{k_t} \in \{\theta(p^\tau(\overline{\xi}; \overline{\eta})) \mid \tau \in St_m^{(2m-s)}\}$$

*(see [2], Section 3) form a minimal generating set.*

**Proof** As proved in [3], $E$ is a Frobenius algebra. By Lemma 2.1, $A = F[\xi_1, \dots, \xi_s]$ is also Frobenius algebra. Therefore

$$\dim(E) = \dim(E\mu) + \dim Ann_E(\mu) \text{ and } \dim(A) = \dim(A\mu) + \dim Ann_A(\mu)$$

Since $\mathcal{A} \subset Ann_E(\mu)$ is obvious, for converse inclusion, it is sufficient to show that

$$\dim(\mathcal{A}) = \dim(E) - \dim(E\mu).$$

Hence using the direct sum decomposition

$$E = \bigoplus_{1 \le l_1 < \cdots < l_k \le n} A p_{l_1} \cdots p_{l_k}$$

where each $p_k$ is a product of the $x_{kj}$, factors of $\xi_k$, we have

$$\begin{aligned} E\mu &= \bigoplus_{1 \le l_1 < \cdots < l_k \le s} A\mu p_{l_1} \cdots p_{l_k} \\ &= \bigoplus_{1 \le l_1 < \cdots < l_k \le s} A(\mu - \xi_{l_1} - \xi_{l_2} - \cdots - \xi_{l_k}) p_{l_1} \cdots p_{l_k} . \end{aligned}$$

As $\mathcal{A}$ is spanned by elements of the form

$$(\xi_{i_1} - \xi_{j_1}) \cdots (\xi_{i_r} - \xi_{j_r}) \xi_{k_1} \cdots \xi_{k_t} p_{l_1} \cdots p_{l_m} ,$$

it follows that

$$\mathcal{A} = \bigoplus_{1 \le l_1 < \cdots < l_k \le s} \left( \bigoplus_{p_{l_1} \cdots p_{l_k} \in P_L} A\mathcal{G}_{S-L} p_{l_1} \cdots p_{l_k} \right)$$

where we assume $p_l \neq 1$ and $\xi_l$. Thus, elements of $\mathcal{A}$ are linear combinations of the products

$$(\xi_{i_1} - \xi_{j_1}) \cdots (\xi_{i_m} - \xi_{j_m}) \xi_{k_1} \cdots \xi_{k_t} p_{l_1} \cdots p_{l_q}$$

where the set of indices is equal to $S$ and $p_l \neq 1$ and $\xi_l$. Consequently, elements of $E\mu$ are linear combinations of

$$(\xi_{i_1} + \cdots \xi_{i_m}) \xi_{k_1} \cdots \xi_{k_t} p_{l_1} \cdots p_{l_q}$$

with the set of indices equal to $S$ again.

Thus, by using Theorem 3.5

$$\begin{aligned}
\mathcal{A} &= \bigoplus_{1 \le l_1 < \cdots < l_k \le s} \left( \bigoplus_{p_{l_1} \cdots p_{l_k} \in P_L} (A\mathcal{G}_{L'}) p_{l_1} \cdots p_{l_k} \right) \\
&= \bigoplus_{1 \le l_1 < \cdots < l_k \le s} \left( \bigoplus_{p_{l_1} \cdots p_{l_k} \in P_L} (A_{L'}\mathcal{G}_{L'}) p_{l_1} \cdots p_{l_k} \right) \\
&= \bigoplus_{1 \le l_1 < \cdots < l_k \le s} \left( \bigoplus_{p_{l_1} \cdots p_{l_k} \in P_L} Ann_{A_{L'}}(A_{L'}\mu_{L'}) p_{l_1} \cdots p_{l_k} \right) \\
&= \bigoplus_{L \subset S} Ann_{A_{L'}}(A_{L'}\mu_{L'}) P_L
\end{aligned}$$

and

$$\begin{aligned}
E\mu &= \bigoplus_{1 \le l_1 < \cdots < l_k \le n} \left( \bigoplus_{p_{l_1} \cdots p_{l_k} \in P_L} A\mu p_{l_1} \cdots p_{l_k} \right) \\
&= \bigoplus_{1 \le l_1 < \cdots < l_k \le n} \left( \bigoplus_{p_{l_1} \cdots p_{l_k} \in P_L} A_{L'}\mu_{L'} p_{l_1} \cdots p_{l_k} \right) \\
&= \bigoplus_{L \subset S} (A_{L'}\mu_{L'}) P_L .
\end{aligned}$$

This yields

$$\dim \mathcal{A} = \sum_{L\subset S} \dim(Ann_{A_{L'}}(A_{L'}\mu_{L'}))|P_L| \qquad \text{and}$$

$$\dim E\mu = \sum_{L\subset S} \dim(A_{L'}\mu_{L'})|P_L| \; .$$

By Theorem 3.5, we have

$$\dim Ann_{A_{L'}}(A_{L'}\mu_{L'}) + \dim(A_{L'}\mu_{L'}) = \dim A_{L'} = 2^{|L'|}.$$

Also, we note that the polynomials

$$s_k = \sum_{\substack{L\subset S\\ |L|=k}} z_{l_1}\cdots z_{l_k}$$

are elementary symmetric polynomials in $z_1,\dots,z_s$ and therefore

$$\sum_{k=0}^{s} s_k z^{s-k} = (z+z_1)\cdots(z+z_s).$$

By letting $z_l := 2^{n_l}-2$, we conclude that

$$\begin{aligned}
\dim(E\mu)+\dim\mathcal{A} &= \sum_{L\subset S} 2^{|L'|}(2^{n_{l_1}}-2)\cdots(2^{n_{l_k}}-2)\\
&= \sum_{k=0}^{s}\sum_{\substack{L\subset S\\ |L|=k}} 2^{|L'|}(2^{n_{l_1}}-2)\cdots(2^{n_{l_k}}-2)\\
&= \sum_{k=0}^{s} 2^{s-k}\sum_{\substack{L\subset S\\ |L|=k}} z_{l_1}\cdots z_{l_k}\\
&= \sum_{k=0}^{s} 2^{s-k}s_k = (2+z_1)\cdots(2+z_s)\\
&= 2^{n_1+\cdots+n_s} = 2^n = \dim(E)
\end{aligned}$$

which means that $\mathcal{A} = Ann_E(\mu)$.

The last part of the theorem can be obtained by using the techniques from [2] and linearly independence of $\{\theta(p^\tau(\overline{\xi};\overline{\eta})) \mid \tau \in St_m^{(2m-s)}\}$ is proved in [2, Theorem 3]. □

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