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Boundedness for variable fractional integral operators and their commutators on Herz–Hardy spaces with variable exponent

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Abstract: Let $E \subset \mathbb{R}^n$ be a bounded open set. In this paper, we establish the boundedness of variable fractional integral operators and their commutators on variable Herz–Hardy spaces $H\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(E)$ with three variable exponents by using the atomic decomposition.

Key words: Variable fractional integral operator, commutator, variable exponent Herz space, variable exponent Herz–Hardy space

1. Introduction

It is well known that function spaces with variable exponents have been an important topic in modern analysis, and are now of increasing applications in several areas such as harmonic analysis, approximation theory, and partial differential equations (see, for example, [3, 5–7, 14, 16, 18, 19]). Moreover, the study for Herz spaces is an important field in harmonic analysis. In 1968, Herz spaces were first introduced by Herz [10] when studying the absolute convergence of Fourier transforms. Since then, the theory of Herz-type spaces has been well developed and these spaces have been widely used in harmonic analysis and some other fields of analysis (see, for example, [15, 17]).

In particular, Izuki [11] studied the Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponent and proved the boundedness of some sublinear operators on this space. Wang and Liu [20] studied the certain Herz-Hardy space $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ or $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponent. Moreover, Almeida and Drihem [1] introduced Herz spaces with two variable exponents, and obtained the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator, and the Calderón–Zygmund singular integral operator on these space.

Let $E \subset \mathbb{R}^n$ be a bounded open set. Izuki and Noi [13] introduced variable Herz spaces $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(E)$ with three variable exponents and established the boundedness of (higher-order) commutators generated by the fractional integral operator or the Calderón–Zygmund singular integral operator on such variable exponent Herz spaces. Furthermore, Drihem and Seghiri [8] established a new equivalent norms for these function spaces. Recently, Xin and Tao [22] obtained the boundedness of Marcinkiewicz integrals with variable kernels on the homogeneous and nonhomogeneous Herz–Hardy spaces with variable exponent. Moreover, Heraiz [9] proved

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that a class of fractional integral operators and their commutators are bounded on variable Herz–Hardy spaces with three variable exponents.

In this paper, we study the boundedness of variable fractional integral operators and their commutators on the homogeneous Herz–Hardy space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$.

Let $E \subset \mathbb{R}^n$ be a bounded open set. Denote by $|E|$ the Lebesgue measure of E and by χ_E its characteristic function. For any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, denote by $Q := Q(x,r)$ a cube with the center x and the side-length r. We always assume that all cubes have sides parallel to the coordinate axes. Moreover, denote by $B := B(x,r)$ the ball in $\mathbb{R}^n$ with the center x and the radius r.

Definition 1.1 ([6]) *Let $p(\cdot): E \to (1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^p(E)$ is defined by*

$$L^{p(\cdot)}(E) := \left\{ f \text{ is measurable}: \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx < \infty \text{ for some } \eta > 0 \right\}$$

equipped with the norm

$$\|f\|_{L^{p(\cdot)}(E)} := \inf \left\{ \eta > 0 : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} dx \le 1 \right\}.$$

Moreover, the locally variable Lebesgue space $L_{\mathrm{Loc}}^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) := \{ f : f \in L^{p(\cdot)}(G) \text{ for any compact subset } G \subset E \}.$$

The weak variable exponent Lebesgue space $WL^{p(\cdot)}(E)$ consists of all Lebesgue measurable function satisfying

$$\|f\|_{WL^{p(\cdot)}(E)} := \sup_{\eta>0} \eta \left\| \chi_{\{x\in E:\,|f|>\eta\}} \right\|_{L^{p(\cdot)}(E)}.$$

It is easy to see that, if $p(\cdot) \equiv p_0$ is constant, then $L^{p(\cdot)}(E)$ is just the classical Lebesgue space $L^{p_0}(E)$.

Let $\mathcal{P}(E)$ be the set of $p(\cdot): E \to (0,\infty)$ such that

$$p^- := \operatorname{essinf}\{p(x) : x \in E\} > 1 \quad \text{and} \quad p^+ := \operatorname{esssup}\{p(x) : x \in E\} < \infty.$$

For any $p(\cdot) \in \mathcal{P}(E)$, let $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. Denote by $\mathcal{P}^0(E)$ the set of measurable function $p(\cdot): E \to (0,\infty)$ such that $0 < p^- \le p^+ < \infty$, and by $\mathcal{P}^n(E)$ the set of measurable function $p(\cdot): E \to (0,n)$ such that $0 < p^- \le p^+ < n$. Moreover, let $\mathcal{B}(E)$ be the set of $p(\cdot) \in \mathcal{P}(E)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(E)$.

Definition 1.2 ([6]) *It is said that $p(\cdot): E \to (0,\infty)$ is locally log-Hölder continuous, if there exists a constant $C > 0$ such that, for all $x, y \in E$,*

$$|p(x) - p(y)| \le \frac{C}{\log(e + 1/|x-y|)}. \tag{1.1}$$

Moreover, it is said that $p(\cdot)$ *is log-Hölder continuous at the origin, denoted by* $p(\cdot) \in LH_0(E)$, *if* $p(\cdot) \in \mathcal{P}(E)$ *satisfies*

$$|p(x) - p(0)| \leq \frac{C_0}{\log(e + 1/|x|)} \tag{1.2}$$

for all $x \in E$, *where* C_0 *is a positive constant independent of* $x \in E$. *Furthermore, it is said that* $p(\cdot)$ *is log-Hölder continuous at infinity, denoted by* $p(\cdot) \in LH_\infty(E)$, *if there exist constants* $C_\infty > 0$ *and* $p_\infty \in \mathbb{R}$ *such that*

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)} \tag{1.3}$$

for all $x \in E$. *Let* $LH(E) := LH_0(E) \cap LH_\infty(E)$.

We point out that, if $p(\cdot) \in \mathcal{P}(E) \cap LH(E)$, then $p'(\cdot) \in \mathcal{P}(E) \cap LH(E)$ and $p(\cdot) \in \mathcal{B}(E)$ (see, for example, [5, 6]).

Definition 1.3 *Let* S^{n-1} *be the unit sphere in* $\mathbb{R}^n$ $(n \geq 2)$. *Suppose that* $\beta(\cdot) \in \mathcal{P}^n(E)$ *satisfies* $0 < \beta^- \leq \beta^+ < n$, *and* $\Omega \in L^s(\mathrm{S}^{n-1})$ *with* $s > \frac{n}{n-\beta^+}$. *Then the fractional integral operator* $T_{\beta(\cdot)}$ *is defined by, for any* $f \in L^1_{\mathrm{loc}}(E)$ *and* $x \in \mathbb{R}^n$,

$$T_{\beta(\cdot)}(f)(x) := \int_E \frac{f(y)}{|x - y|^{n-\beta(x)}} dy. \tag{1.4}$$

Moreover, the fractional integral operator $T_{\Omega,\beta(\cdot)}$ *with rough kernel is defined by, for any* $f \in L^1_{\mathrm{loc}}(E)$ *and* $x \in \mathbb{R}^n$,

$$T_{\Omega,\beta(\cdot)}(f)(x) := \int_E \frac{\Omega(x - y)}{|x - y|^{n-\beta(x)}} f(y) dy. \tag{1.5}$$

Definition 1.4 *Let* $\beta(\cdot) \in \mathcal{P}^n(E) \cap LH(E)$ *and* $\Omega \in L^1(\mathrm{S}^{n-1})$. *The variable fractional maximal operator* $M_{\Omega,\beta(\cdot)}$ *with rough kernel is defined by, for any* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$M_{\Omega,\beta(\cdot)}(f)(x) := \sup_{r>0} \frac{1}{r^{n-\beta(x)}} \int_{B(x,r)\cap E} |\Omega(x - y)||f(y)| dy. \tag{1.6}$$

In particular, if $\Omega \equiv 1$, $M_{\Omega,\beta(\cdot)}(f)(x)$ *is just*

$$M_{\beta(\cdot)}(f)(x) := \sup_{r>0} \frac{1}{r^{n-\beta(x)}} \int_{B(x,r)\cap E} |f(y)| dy.$$

Then we have the following boundedness for the fractional integral operator $T_{\Omega,\beta(\cdot)}$ with rough kernel on variable Lebesgue spaces.

Theorem 1.5 *Let* $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(E) \cap LH(E)$ *and* $\beta(\cdot) \in LH(E)$ *satisfy* $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$, $0 < \beta^- \leq \beta^+ < n$, *and* $p_1^+ < \frac{n}{\beta^+}$. *Assume that* $M_{\Omega,\beta(\cdot)}$ *is as in Definition 1.4. Then there exists a constant* $C > 0$ *such that, for any* $f \in L^{p_1(\cdot)}(E)$,

$$\left\| M_{\Omega,\beta(\cdot)}(f) \right\|_{L^{p_2(\cdot)}(E)} \leq C \|f\|_{L^{p_1(\cdot)}(E)}.$$

Theorem 1.6 *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(E) \cap LH(E)$ and $\beta(\cdot) \in LH(E)$ satisfy $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$, $0 < \beta^- \leq \beta^+ < n$, and $p_1^+ < \frac{n}{\beta^+}$. Assume that $T_{\Omega,\beta(\cdot)}$ is as in Definition 1.3. Then there exists a constant $C > 0$ such that, for any $f \in L^{p_1(\cdot)}(E)$,*

$$\left\|T_{\Omega,\beta(\cdot)}(f)\right\|_{L^{p_2(\cdot)}(E)} \leq C\|f\|_{L^{p_1(\cdot)}(E)}.$$

Theorem 1.7 *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(E) \cap LH(E)$ and $\beta(\cdot) \in LH(E)$ satisfy $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$, $0 < \beta^- \leq \beta^+ < n$, and $p_1^+ < \frac{n}{\beta^+}$. Assume that $T_{\Omega,\beta(\cdot)}$ is as in Definition 1.3. Then there exists a constant $C > 0$ such that, for any $f \in L^{p_1(\cdot)}(E)$,*

$$\left\|T_{\Omega,\beta(\cdot)}(f)\right\|_{WL^{p_2(\cdot)}(E)} \leq C\|f\|_{L^{p_1(\cdot)}(E)}.$$

Definition 1.8 *The space* $\mathrm{BMO}(E)$ *is defined as*

$$\mathrm{BMO}(E) := \left\{ f \in L^1_{\mathrm{loc}}(E) : \ \|f\|_{\mathrm{BMO(E)}} := \sup_{x \in Q} \frac{1}{|Q|} \int_{\mathrm{Q}} |f(x) - f_Q| dx < \infty \right\},$$

where the supreme is taken over all cubes $Q \subset E$ and $f_Q := \frac{1}{|Q|} \int_Q f(x)dx$.

Moreover, for any given $b \in \mathrm{BMO}(E)$ and $\beta(\cdot) \in LH(E)$, the commutator $[b, T_{\beta(\cdot)}]$ is defined by, for any $x \in E$,

$$[b, T_{\beta(\cdot)}](f)(x) := b(x)T_{\beta(x)}(f)(x) - T_{\beta(x)}(bf)(x) = \int_E [b(x) - b(y)] \frac{\Omega(x-y)f(y)}{|x-y|^{n-\beta(x)}} dy.$$

Theorem 1.9 *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(E) \cap LH(E)$ and $\beta(\cdot) \in LH(E)$ satisfy $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$, $0 < \beta^- \leq \beta^+ < n$, and $p_1^+ < \frac{n}{\beta^+}$. Assume that $b \in \mathrm{BMO}(E)$. Then there exists a constant $C > 0$ such that, for any $f \in L^{p_1(\cdot)}(E)$,*

$$\left\|[b, T_{\beta(\cdot)}](f)\right\|_{WL^{p_2(\cdot)}(E)} \leq C\|b\|_{\mathrm{BMO}(E)}\|f\|_{L^{p_1(\cdot)}(E)}.$$

Let $p(\cdot), q(\cdot) \in \mathcal{P}^0(E)$. A mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\}_{v\in\mathbb{Z}}) := \sum_{v\in\mathbb{Z}} \inf \left\{ \lambda_v > 0 : \ \varrho_{p(\cdot)} \left(\frac{f_v}{\lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\},$$

where $\{f_v\}_{v\in\mathbb{Z}}$ is a sequence of measurable functions on the set E and $\{\lambda_v\}_{v\in\mathbb{Z}}$ is a sequence of nonnegative real numbers.

Moreover, the (quasi-)norm $\|\{f_v\}_{v\in\mathbb{Z}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is defined as

$$\|\{f_v\}_{v\in\mathbb{Z}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf \left\{ \mu > 0 : \ \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left\{ \frac{f_v}{\mu} \right\}_{v\in\mathbb{Z}} \right) \leq 1 \right\}.$$

Since $q^+ < \infty$, we can use a simpler expression

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\}_{v\in\mathbb{Z}}) = \sum_{v\in\mathbb{Z}} \||f_v|^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(E)}.$$

In particular, if $p(\cdot)$, $q(\cdot)$ are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$.

Now, we recall the definition of the Herz space with variable exponent (see, for example, [13]). Let $\mathbb{Z}_+ := \{0, 1, \cdots\}$ and $\mathbb{N} := \{1, \cdots\}$. For any $k \in \mathbb{Z}$, let $B_k := \{x \in \mathbb{R} : \ |x| \le 2^k\}$, $A_k := B_k \setminus B_{k-1}$, and $\chi_k = \chi_{A_k}$. Moreover, for any $k \in \mathbb{N}$, let $\tilde{\chi}_k := \chi_{A_k}$ and $\tilde{\chi}_0 := \chi_{B_0}$.

Definition 1.10 *Let $\alpha(\cdot)$, $p(\cdot)$, and $q(\cdot) \in \mathcal{P}^0(E)$. Then the inhomogeneous Herz space $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ with variable exponent is defined by*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(E) : \|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} < \infty \right\},$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} := \|f\chi_{B_0}\|_{L^{p(\cdot)}(E)} + \left\| \left\{ 2^{k\alpha(\cdot)} f\chi_k \right\}_{k\ge 1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Moreover, the homogeneous Herz space $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ with variable exponent is defined by

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(E \setminus \{0\}) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} := \left\| \left\{ 2^{k\alpha(\cdot)} f\chi_k \right\}_{k\in\mathbb{Z}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

We point out that when $\alpha(\cdot)$, $p(\cdot)$, $q(\cdot)$ are constant, then

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E) = K_p^{\alpha,q}(E),\ \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E) = \dot{K}_p^{\alpha,q}(E),$$

where $K_p^{\alpha,q}(E)$ and $\dot{K}_p^{\alpha,q}(E)$ are the classical Herz spaces.

Moreover, if $\alpha(\cdot)$, $q(\cdot) \in LH_\infty(E)$, then $K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(E)$; if $\alpha(\cdot)$, $q(\cdot) \in LH_0(E)$, then

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} \approx \left[\sum_{k=-\infty}^{-1} \|2^{k\alpha(0)} f\chi_k\|_{L^{p(\cdot)}(E)}^{q(0)} \right]^{\frac{1}{q(0)}} + \left[\sum_{k=0}^{\infty} \|2^{k\alpha_\infty} f\chi_k\|_{L^{p(\cdot)}(E)}^{q_\infty} \right]^{\frac{1}{q_\infty}};$$

see, for example, [13].

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\mathrm{supp}(\varphi) \subset B_0$ and $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$. For any $t > 0$, let $\varphi_t(\cdot) := t^{-n}\varphi(\frac{\cdot}{t})$. Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions equipped with the well-known topology determined by a countable family of norms and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space (namely, the space of all tempered distributions) equipped with the weak-$*$ topology. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the maximal function $M_\varphi(f)$ of f is defined by, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{t>0} |\varphi_t * f(x)|.$$

Now, we recall the definitions of homogeneous Herz–Hardy spaces $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$.

Definition 1.11 *Let $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot) \in \mathcal{P}^0(E)$. The homogeneous Herz–Hardy space $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $M_\varphi(f) \in \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ with the quasi-norm*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} := \|M_\varphi(f)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)}.$$

It is worth pointing out that, if $p(\cdot) \in LH(E)$, $-\frac{n}{p^+} < \alpha^- < \alpha^+ < n - \frac{n}{p^-}$, and $q(\cdot) \in \mathcal{P}^0(E)$, then $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E) = \dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ (see, for example, [9]). Moreover, if $\alpha(\cdot) = 0, p(\cdot) = q(\cdot)$, then $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ and $\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$ coincide with $L^{p(\cdot)}(E)$ (see [9]).

Theorem 1.12 *Let $\alpha(\cdot)$, $\beta(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$, $q_2(\cdot) \in LH(E)$ satisfy $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$, $0 < \beta^- \le \beta^+ < n$, and $p_1^+ < \frac{n}{\beta^+}$. If $\alpha(\cdot) \ge n(1 - \frac{1}{p_1^-})$, $q_1(0) \le q_2(0)$, and $(q_1)_\infty \le (q_2)_\infty$, then $T_{\beta(\cdot)}$ is bounded from $H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(E)$ to $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(E)$.*

Theorem 1.13 *Let $\alpha(\cdot)$, $\beta(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$, $q_2(\cdot) \in LH(E)$ satisfy $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$, $0 < \beta^- \le \beta^+ < n$, and $p_1^+ < \frac{n}{\beta^+}$. Assume that $b \in \mathrm{BMO}(E)$. If $\alpha(\cdot) \ge n(1 - \frac{1}{p_1^-})$, $q_1(0) \le q_2(0)$, and $(q_1)_\infty \le (q_2)_\infty$, then $[b, T_{\beta(\cdot)}]$ is bounded from $H\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(E)$ to $\dot{K}_{p_2(\cdot)}^{\alpha(\cdot),q_2(\cdot)}(E)$.*

This paper is organized as follows.

In Section 2, we give some auxiliary conclusions for the proofs of the main results of this paper. Moreover, in Section 3, we give out the proofs of the main results by using auxiliary conclusions presented in Section 2.

Finally, we make some conventions on notations. Throughout the whole paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \approx g$. For any given $q \in [1, \infty]$, we denote by q' its conjugate exponent, namely, $\frac{1}{q} + \frac{1}{q'} = 1$.

2. Preliminaries

In this section, we recall some useful auxiliary conclusions for the proofs of the main results of this paper.

Lemma 2.1 ([4]) *Let $\beta(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $\beta(\cdot) \in LH_0(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^n$ with $|x| < 1$,*

$$C^{-1}|x|^{\beta(0)} \le |x|^{\beta(x)} \le C|x|^{\beta(0)};$$

if $\beta(\cdot) \in LH_\infty(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that, for any $x \in \mathbb{R}^n$ with $|x| \ge 1$,

$$C^{-1}|x|^{\beta(\infty)} \le |x|^{\beta(x)} \le C|x|^{\beta(\infty)}.$$

Lemma 2.2 ([5, 14]) *If $p(\cdot) \in \mathcal{P}(E)$, then there exists a constant $C > 0$ such that, for any $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$,*

$$\int_E |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)},$$

where $r_p := 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

Lemma 2.3 ([11]) *Let* $q(\cdot) \in \mathcal{B}(E)$. *Then there exists a constant* $C > 0$ *such that, for any ball* $B \subset E$,

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{q(\cdot)}(E)} \|\chi_B\|_{L^{q'(\cdot)}(E)} \leq C.$$

Moreover, for any ball $B \subset E$ *and* $x \in B$,

$$\|\chi_B\|_{L^{q(\cdot)}(E)} \approx |B|^{\frac{1}{q(x)}}.$$

In particular, for any ball $B \subset E$ *satisfying* $|B| \leq 2^n$, $\|\chi_B\|_{L^{q(\cdot)}(E)} \approx |B|^{\frac{1}{p_\infty}}$.

Lemma 2.4 ([12]) *Let* $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that, for any ball* B *in* $\mathbb{R}^n$, *and any measurable subset* $S \subset B$,

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_1},$$

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $\delta_1, \delta_2 \in (0,1)$ *are constants independent of* B *and* S.

Lemma 2.5 ([5, 6]) *Let* $p(\cdot) \in \mathcal{P}(E)$. *Then* $\|f\|_{L^{p(\cdot)}(E)} \leq 1$ *if and only if* $\int_E |f(y)|^{p(y)} dy \leq 1$. *In particular,* $\|f\|_{L^{p(\cdot)}(E)} = 1$ *if and only if* $\int_E |f(y)|^{p(y)} dy = 1$.

Lemma 2.6 ([1]) *Let* $\alpha(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $r_1, r_2 \in (0, \infty)$. *If* $\alpha(\cdot) \in LH(\mathbb{R}^n)$, *then, for any* $x \in B(0, r_1) \setminus B(0, \frac{r_1}{2})$ *and* $y \in B(0, r_2) \setminus B(0, \frac{r_2}{2})$,

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} (\frac{r_1}{r_2})\alpha^+, & 0 < r_2 \leq \frac{r_1}{2}, \\ 1, & \frac{r_1}{2} < r_2 \leq 2r_1, \\ (\frac{r_1}{r_2})^{\alpha^-}, & r_2 > 2r_1, \end{cases}$$

with the implicit positive constant independent of x, y, r_1 *and* r_2.

Lemma 2.7 *Let* $\beta(\cdot) \in \mathcal{P}^n(E)$. *Assume that* $s > \frac{n}{n-\beta^+}$ *and* $\Omega \in L^s(\mathrm{S}^{n-1})$ *is a homogeneous function of degree zero. Then there exists a constant* $C > 0$ *such that, for any* $f \in L^1_{\mathrm{loc}}(E)$ *and* $x \in E$,

$$\left|M_{\Omega,\beta(\cdot)} f(x)\right| \leq C \|\Omega\|_{L^s(\mathrm{S}^{n-1})} \left[M_{\beta(\cdot)s'}(|f|^{s'})(x)\right]^{\frac{1}{s'}}.$$

Proof By Hölder's inequality, we have that, for any $x \in E$,

$$
\begin{aligned}
&\left|M_{\Omega,\beta(\cdot)}(f)(x)\right| \\
&= \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta(x)}{n}}} \int_{B(x,r)\cap E} |\Omega(x-y)||f(y)|dy \\
&\le \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta(x)}{n}}} \left(\int_{B(x,r)\cap E} |\Omega(x-y)|^s dy\right)^{1/s} \left(\int_{B(x,r)\cap E} |f(y)|^{s'} dy\right)^{1/s'} \\
&\le C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \left(\sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta(x)s'}{n}}} \int_{B(x,r)\cap E} |f(y)|^{s'} dy\right)^{1/s'} \\
&= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} \left[M_{\beta(\cdot)s'}(|f|^{s'})(x)\right]^{\frac{1}{s'}}.
\end{aligned}
$$

□

Lemma 2.8 ([2]) *Let $\beta(\cdot) \in \mathcal{P}^n(E)$. Suppose that p_1, $p_2(\cdot) \in LH(E)$ satisfy $p_1^+ < \frac{n}{\beta^+}$ and $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n}$. Then there exists a constant $C > 0$ such that, for any $f \in L^{p_1(\cdot)}(E)$,*

$$\|M_{\beta(\cdot)}(f)\|_{L^{p_2(\cdot)}(E)} \le C\|f\|_{L^{p_1(\cdot)}(E)}.$$

Lemma 2.9 *Let $\beta(\cdot) \in \mathcal{P}^n(E)$ satisfy $0 < \beta^- \le \beta^+ < n$ and $\epsilon \in (\max\{\beta^-, n-\beta^+\})$. Then there exists a positive constant $C := C_{(\beta^-,n,\epsilon)}$ such that, for any $f \in L^1_{\mathrm{loc}}(E)$ and $x \in E$,*

$$\left|T_{\Omega,\beta(\cdot)}(f)(x)\right| \le C\left[M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)\right]^{1/2}\left[M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)\right]^{1/2},$$

where

$$M_{\Omega,\beta(\cdot)}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta(x)}{n}}} \int_{B(x,r)\cap E} |\Omega(x-y)||f(y)|dy.$$

Proof We prove this lemma via borrow some ideas from [21]. Let $\delta \in (0,\infty)$ be a given constant. Then, for any $x \in E$,

$$
\begin{aligned}
T_{\Omega,\beta(\cdot)}(f)(x) &= \int_{\{y\in E:\ |x-y|<\delta\}} \frac{\Omega(x-y)f(y)}{|x-y|^{n-\beta(x)}}dy + \int_{\{y\in E:\ |x-y|\ge\delta\}} \frac{\Omega(x-y)f(y)}{|x-y|^{n-\beta(x)}}dy \\
&=: I_1 + I_2.
\end{aligned}
$$

For any $i \in \mathbb{Z}_+$, let $A_i := \{y \in E:\ 2^{-i-1}\delta \le |x-y| < 2^{-i}\delta\}$. Then we have

$$
\begin{aligned}
|I_1| &\le \sum_{i=0}^{\infty} \int_{A_i} \frac{|\Omega(x-y)||f(y)|}{|x-y|^{n-\beta(x)}}dy \\
&\le \sum_{i=0}^{\infty} \frac{1}{(2^{-i-1}\delta)^{n-\beta(x)+\epsilon}} \int_{B_i} |\Omega(x-y)||f(y)|dy,
\end{aligned}
$$

where $B_i := \{y \in E : \ |x-y| < 2^{-i}\delta\}$. Thus, $I_1 \le C\delta^{\epsilon} M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)$. Similarly,

$$|I_2| \le C\sum_{i=1}^{\infty} \frac{1}{(2^{i-1}\delta)^{n-\beta(x)}} \int_{B_i} |\Omega(x-y)||f(y)|dy \\ \le C\delta^{-\epsilon} M_{\Omega,\beta(\cdot)+\epsilon}(f)(x).$$

Take δ satisfy $\delta^{\epsilon} = [M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)/M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{1/2}$. Then we find that

$$|T_{\Omega,\beta(\cdot)}f(x)| \le C[\delta^{\epsilon} M_{\Omega,\beta(\cdot)-\epsilon}(f)(x) + \delta^{-\epsilon} M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)] \\ \le C[M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{1/2}[M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{1/2}.$$

This finishes the proof of this lemma. □

Lemma 2.10 ([12]) *Let $q(\cdot) \in \mathcal{P}(E)$ and m be a given positive integer. Then, for any $b \in \mathrm{BMO}(E)$, and any $i, j \in \mathbb{Z}$ with $j > i$,*

$$C^{-1}\|b\|^m_{\mathrm{BMO}(E)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(E)}} \|(b-b_B)^m \chi_B\|_{L^{q(\cdot)}(E)} \le C\|b\|^m_{\mathrm{BMO}(E)},$$

$$\|(b-b_{B_i})^m \chi_{B_j}\|_{L^{q(\cdot)}(E)} \le C(j-i)^m \|b\|^m_{\mathrm{BMO}(E)} \|\chi_{B_j}\|_{L^{q(\cdot)}(E)}.$$

Similarly to the above Lemma 2.9, the conclusion about commutators is given without proof.

Lemma 2.11 *Let $\beta(\cdot) \in \mathcal{P}^n(E)$, $m \in \mathbb{N}$, and $\epsilon \in (\max\{\beta^-, n-\beta^+\})$. Then there exists a positive constant $C = C_{(\beta^-,n,\epsilon)}$ such that, for any $f \in L^1_{\mathrm{loc}}(E)$, $b \in L^1_{\mathrm{loc}}(E)$ and $x \in E$,*

$$\left|T^{m,b}_{\Omega,\beta(\cdot)}(f)(x)\right| \le C[M^{m,b}_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{1/2}[M^{m,b}_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{1/2}$$

where

$$M^{m,b}_{\Omega,\beta(\cdot)}(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta(x)}{n}}} \int_{B(x,r)\cap E} |\Omega(x-y)||b(x)-b(y)|^m |f(y)|dy.$$

Definition 2.12 ([9]) *Let $\alpha(\cdot) \in L^{\infty}(E)$, $p(\cdot) \in \mathcal{P}(E), q(\cdot) \in \mathcal{P}^0(E)$, and $s \in \mathbb{Z}_+$. A function a is said to be a* central $(\alpha(\cdot), q(\cdot))$-atom, *if*

(1) $\mathrm{supp}(a) \subset \overline{B(0,r)} \cap E = \{x \in E : |x| \le r\}$ *with* $r > 0$;

(2) $\|a\|_{L^{p(\cdot)}} \le |\overline{B(0,r)} \cap E|^{-\frac{\alpha(0)}{n}}$ *with* $0 < r < 1$;

(3) $\|a\|_{L^{p(\cdot)}} \le |\overline{B(0,r)} \cap E|^{-\frac{\alpha_\infty}{n}}$ *with* $r \ge 1$;

(4) $\int_E x^{\eta} a(x)dx = 0, \ |\eta| \le s$.

A function a is said to be a central $(\alpha(\cdot), q(\cdot))$-atom of restricted type, *if a satisfies the conditions* (3), (4), *and* $\mathrm{supp}(a) \subset B(0,r) \cap E$ *with* $r \ge 1$.

Lemma 2.13 ([9]) *Let* $\alpha(\cdot)$, $q(\cdot) \in LH(E)$ *and* $p(\cdot) \in \mathcal{P}(E) \cap LH(E)$. *Then, for any given* $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$, *the following conclusion is true:*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k,$$

where the series converges in the sense of distributions, for any $k \in \mathbb{Z}$, $\lambda_k \geq 0$ *and* a_k *is a central* $(\alpha(\cdot), p(\cdot))$-*atom with* $\operatorname{supp}(a) \subset B_k \cap E$, *and*

$$\left[\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right]^{\frac{1}{q(0)}} + \left[\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right]^{\frac{1}{q_\infty}} \leq C\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)}.$$

Conversely, if

$$\alpha(\cdot) \geq n\left(1 - \frac{1}{p^-}\right) \quad \text{and} \quad s \geq \left[\alpha^+ + n\left(\frac{1}{p^-} - 1\right)\right],$$

then $f \in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)$, *and*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(E)} \approx \inf\left\{\left[\sum_{k=-\infty}^{-1} |\lambda_k|^{q(0)}\right]^{\frac{1}{q(0)}} + \left[\sum_{k=0}^{\infty} |\lambda_k|^{q_\infty}\right]^{\frac{1}{q_\infty}}\right\},$$

where the infimum is taken over all the decompositions of f.

Lemma 2.14 ([9]) *Let* $0 < a < 1$ *and* $0 < q \leq \infty$. *Suppose that* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *is a sequence of positive real numbers such that*

$$\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\|_{\ell^q} =: I < \infty.$$

Then the sequences $\{\delta_k : \delta_k = \sum_{j\leq k} a^{k-j}\varepsilon_j\}_{k\in\mathbb{Z}}$ *and* $\{\eta_k : \eta_k = \sum_{j\geq k} a^{j-k}\varepsilon_j\}_{k\in\mathbb{Z}}$ *belong to* ℓ^q, *and*

$$\|\{\delta_k\}_{k\in\mathbb{Z}}\|_{\ell^q} + \|\{\eta_k\}_{k\in\mathbb{Z}}\|_{\ell^q} \leq CI,$$

where $C > 0$ *is a constant depending only on* a *and* q.

3. Proofs of the main results

In this section, we give the proofs of main results of this paper presented in Section 1.

Proof [Proof of Theorem 1.5] By Lemma 2.7, we have that, for any $\lambda > 0$,

$$\int_E \left(\frac{M_{\Omega,\beta(\cdot)}(f)(x)}{\lambda}\right)^{p_2(x)} dx \leq C\int_E \left(\frac{M_{\beta(\cdot)s'}(|f|^{s'})(x)}{\lambda}\right)^{p_2(x)/s'} dx.$$

Moreover, since

$$\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\beta(\cdot)}{n},$$

it follows that

$$\frac{s'}{p_1(\cdot)} - \frac{s'}{p_2(\cdot)} = \frac{\beta(\cdot)s'}{n},$$

and

$$\left\{\lambda > 0: \int_E \left(\frac{M_{\Omega,\beta(\cdot)}(f)(x)}{\lambda}\right)^{p_2(x)} dx \le 1\right\}$$
$$\supset \left\{\lambda > 0 : C\int_E \left(\frac{M_{\beta(\cdot)s'}(|f(x)|^{s'})}{\lambda}\right)^{p_2(x)/s'} dx \le 1\right\},$$

which implies that

$$\inf\left\{\lambda > 0: \int_E \left(\frac{M_{\Omega,\beta(\cdot)} f(x)}{\lambda}\right)^{p_2(x)} dx \le 1\right\}$$
$$\le \inf\left\{\lambda > 0: C\int_E \left(\frac{M_{\beta(\cdot)s'}(|f(x)|^{s'})}{\lambda}\right)^{p_2(x)} dx \le 1\right\}.$$

By this and Lemma 2.8, we have

$$\|M_{\Omega,\beta(\cdot)}(f)\|_{L^{p_2(\cdot)}(E)} \le C\left\|M_{\beta(\cdot)s'}(|f|^{s'})\right\|_{L^{p_2(\cdot)/s'}(E)}$$
$$\le C\||f|^{s'}\|_{L^{p_1(\cdot)/s'}(E)} \le C\|f\|_{L^{p_1(\cdot)}(E)}.$$

This finishes the proof of Theorem 1.5. □

Proof [Proof of Theorem 1.6] Let $f \in L^{p_1(\cdot)}(E)$. Without loss of generality, we may assume $\|f\|_{L^{p_1(\cdot)}(E)} = 1$. Based on $p_2^+ < \infty$ and Lemma 2.5, to finish the proof of Theorem 1.6, it suffices to prove that $\int_E |T_{\Omega,\beta(\cdot)}(f)|^{p_2(x)} dx \le C$.

Fix $\epsilon \in (0, \max\{\beta^-, n-\beta^+\})$ be such that $\frac{p_1^-}{\epsilon p_2^+ + 1} > 1$. For any $x \in E$, define $r(x) := \frac{2}{\epsilon p_2(x)+1}$. By $r^- > 1$, we know that $(r')^+ < \infty$,

$$\frac{1}{p_1(\cdot)} - \frac{2}{r(\cdot)p_2(\cdot)} = \frac{\beta(\cdot)-\epsilon}{n},$$

and

$$\frac{1}{p_1(\cdot)} - \frac{2}{r'(\cdot)p_2(\cdot)} = \frac{\beta(\cdot)+\epsilon}{n}.$$

By Lemma 2.9, we conclude that

$$\int_E |T_{\Omega,\beta(\cdot)}(f)(x)|^{p_2(x)} dx \le C\int_E [M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{\frac{p_2(x)}{2}} [M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{\frac{p_2(x)}{2}} dx.$$

Moreover, from Lemma 2.2, we deduce that

$$\int_E |T_{\Omega,\beta(\cdot)}(f)(x)|^{p_2(x)} dx \le C\left\|[M_{\Omega,\beta(\cdot)-\epsilon}(f)]^{\frac{p_2(\cdot)}{2}}\right\|_{L^{r(\cdot)}(E)} \left\|[M_{\Omega,\beta(\cdot)+\epsilon}(f)]^{\frac{p_2(\cdot)}{2}}\right\|_{L^{r'(\cdot)}(E)}.$$

Since that, for all $x \in E$ and $\lambda > 1$, $\lambda^{\frac{2}{p_2(x)}} \geq \lambda^{\frac{2}{p_2^+}}$, it follows that, for any $\lambda > 1$,

$$\int_E \left(\frac{[M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{\frac{p_2(x)}{2}}}{\lambda} \right)^{r(x)} dx = \int_E \left(\frac{M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)}{\lambda^{\frac{2}{p_2(x)}}} \right)^{\frac{r(x)p_2(x)}{2}} dx$$
$$\leq \int_E \left(\frac{M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)}{\lambda^{\frac{2}{(p_2)^+}}} \right)^{\frac{r(x)p_2(x)}{2}} dx.$$

By this and Theorem 1.5, we know that

$$\left\| [M_{\Omega,\beta(\cdot)-\epsilon}(f)]^{\frac{p_2(\cdot)}{2}} \right\|_{L^{r(\cdot)}(E)} \leq \left\| M_{\Omega,\beta(\cdot)-\epsilon}(f) \right\|^{p_2^+/2}_{L^{r(\cdot)p_2(\cdot)/2}(E)} \leq C\|f\|^{p_2^+/2}_{L^{p_1(\cdot)}(E)} \leq C.$$

Similarly, we have that, for any $\lambda > 1$,

$$\int_E \left(\frac{[M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{\frac{p_2(x)}{2}}}{\lambda} \right)^{r(x)} dx = \int_E \left(\frac{M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)}{\lambda^{\frac{2}{p_2(x)}}} \right)^{\frac{r(x)p_2(x)}{2}} dx$$
$$\leq \int_E \left(\frac{M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)}{\lambda^{\frac{2}{(p_2)^+}}} \right)^{\frac{r(x)p_2(x)}{2}} dx,$$

which, together with Theorem 1.5, implies that

$$\left\| [M_{\Omega,\beta(\cdot)+\epsilon}(f)]^{\frac{p_2(\cdot)}{2}} \right\|_{L^{r(\cdot)}(E)} \leq \left\| M_{\Omega,\beta(\cdot)+\epsilon}(f) \right\|^{p_2^+/2}_{L^{r(\cdot)p_2(\cdot)/2}(E)} \leq C\|f\|^{p_2^+/2}_{L^{p_1(\cdot)}(E)} \leq C.$$

Thus, we obtain that

$$\int_E |T_{\Omega,\beta(\cdot)}(f)|^{p_2(x)} dx \leq C.$$

This finishes the proof of Theorem 1.6. □

Proof [Proof of Theorem 1.7] Let $f \in L^{p_1(\cdot)}(E)$. Without loss of generality, we may assume $\|f\|_{L^{p_1(\cdot)}(E)} = 1$. Based on Lemma 2.5, to prove this theorem, it suffices to show that, for any $\eta \in (0, \infty)$,

$$\left\| \eta \chi_{\{x \in E:\, |T_{\Omega,\beta(\cdot)}(f)| > \eta\}} \right\|_{L^{p_2(\cdot)}(E)} = \int_{\{x \in E:\, |T_{\Omega,\beta(\cdot)}(f)| > \eta\}} \eta^{p_2(x)} dx \leq C.$$

Let $\epsilon \in (0, \max\{\alpha^-, n - \alpha^+\})$ be such that $\frac{p_1^-}{\epsilon p_2^+ + 1} > 1$. Define $r(x) := \frac{2}{\epsilon p_2(x) + 1}$ for any $x \in E$. Since $r^- > 1$ and $(r'(\cdot))^+ < \infty$, it follows, from Lemma 2.9 and Young's inequality, that, for any $x \in E$,

$$|T_{\Omega,\beta(\cdot)}(f)(x)| \leq C[M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{1/2}[M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{1/2}$$
$$\leq \frac{[CM_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{r(x)/2}}{r(x)} + \frac{[CM_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{r'(x)/2}}{r'(x)}$$
$$\leq C^{r^+/2}[M_{\Omega,\beta(\cdot)-\epsilon}(f)(x)]^{r(x)/2} + C^{r^-/2}[M_{\Omega,\beta(\cdot)+\epsilon}(f)(x)]^{r(x)/2}.$$

Therefore,

$$\int_{\{x\in E:\ |T_{\Omega,\beta(\cdot)}(f)|>\eta\}} \eta^{p_2(x)}dx \leq \int_{\{x\in E:\ [M_{\Omega,\beta(\cdot)-\epsilon}(f)]^{r(x)/2}>\eta/2C\}} \eta^{p_2(x)}dx + \int_{\{x\in E:\ [M_{\Omega,\beta(\cdot)+\epsilon}(f)]^{r'(x)/2}>\eta/2C\}} \eta^{p_2(x)}dx.$$

Repeating the argument used in the proof of [5, Theorems 1.9], we can obtain that

$$\int_{\{x\in E:\ [M_{\Omega,\beta(\cdot)-\epsilon}(f)]^{r(x)/2}>\eta/2C\}} \eta^{p_2(x)}dx \leq C\int_{\{x\in E:\ [M_{\Omega,\beta(\cdot)+\epsilon}(f)]^{r'(x)/2}>\eta/2C\}} \eta^{p_2(x)}dx.$$

Then, we have

$$\int_{\{x\in E:\ |T_{\Omega,\beta(\cdot)}(f)|>\eta\}} \eta^{p_2(x)}dx \leq C.$$

Thus

$$\sup_{\eta>0}\eta\left\|\chi_{\{|T_{\Omega,\beta(\cdot)}(f)|>\eta\}}\right\|_{L^{p_2}(E)} \leq C\|f\|_{L^{p_1(\cdot)}(E)}.$$

□

Proof [Proof of Theorem 1.9] Applying Lemma 2.11 and an argument similar to that used in the proof of Theorem 1.7, we can prove Theorem 1.9. We omit the details here. □

Proof [Proof of Theorem 1.12] Let $f\in H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)$. We only need to prove

$$\|T_{\beta(\cdot)}(f)\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)} \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)}.$$

By Lemma 2.13, we know that $f=\sum\limits_{i=-\infty}^{+\infty}\lambda_i a_i$, where $\{\lambda_i\}_{i\in\mathbb{Z}}\subset[0,\infty)$ and for any $i\in\mathbb{Z}$, a_i is an $(\alpha(\cdot),\, p_1(\cdot))$-

atom with $\operatorname{supp}(a_i) \subset B_i \cap E$. Then, we have

$$\begin{aligned}
\|T_{\beta(\cdot)}(f)\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)} \le C &\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\|T_{\beta(\cdot)}(f)\chi_k\|_{p_2(\cdot)}^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&+ C\left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty(q_2)_\infty}\|T_{\beta(\cdot)}(f)\chi_k\|_{p_2(\cdot)}^{(q_2)_\infty}\right\}^{\frac{1}{(q_2)_\infty}} \\
\lesssim &\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|\|T_{\beta(\cdot)}(a_i)\chi_k\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&+ \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=k-1}^{\infty}|\lambda_i|\|T_{\beta(\cdot)}(a_i)\chi_k\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&+ \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha_\infty(q_2)_\infty}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|\|T_{\beta(\cdot)}(a_i)\chi_k\|_{p_2(\cdot)}\right]^{(q_2)_\infty}\right\}^{\frac{1}{(q_2)_\infty}} \\
&+ \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha_\infty(q_2)_\infty}\left[\sum_{i=k-1}^{\infty}|\lambda_i|\|T_{\beta(\cdot)}(a_i)\chi_k\|_{p_2(\cdot)}\right]^{(q_2)_\infty}\right\}^{\frac{1}{(q_2)_\infty}} \\
=: &\, J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We first estimate J_1. If $x \in A_k$, $y \in B_i$, and $i \le k-2$, then $|x-y| \approx |x| \approx 2^k$. By this and the s-order vanishing moments of a_i with $s \ge [\alpha^+ + n(\frac{1}{p_1^-} - 1)]$, we find that

$$|T_{\beta(\cdot)}(a_i)(x)| \le C\int_{B_i\cap E}\frac{|a_i(y)||y|^{s+1}}{|x|^{n-\beta(x)+s+1}}dy.$$

Moreover, by Lemma 2.1, we know that, when $k<0$ and $|x-y|<1$, then $\beta(x) \sim \beta(0)$, which implies that

$$\begin{aligned}
|T_{\beta(\cdot)}(a_i)(x)| &\le C\int_{B_i\cap E}\frac{|a_i(y)||y|^{s+1}}{|x|^{n-\beta(0)+s+1}}dy \\
&\le C2^{-k[n-\beta(0)+s+1]+i(s+1)}\int_{B_i\cap E}|a_i(y)|dy.
\end{aligned}$$

From this and Hölder's inequality, it follows that

$$|T_{\beta(\cdot)}(a_i)(x)| \le C2^{-k[n-\beta(0)+s+1]+i(s+1)}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}(E)}\|a_i\|_{L^{p_1(\cdot)}(E)}.$$

Moreover, it is easy to see that

$$|T_{\beta(\cdot)}(\chi_{B_k})(x)| \ge \int_{B_k\cap E}\frac{dy}{|x-y|^{n-\beta(x)}}\chi_{B_k}(x) \ge C2^{k\beta(\cdot)}\chi_{B_k}(x).$$

By the above two estimates and lemma, we have that

$$\begin{aligned}\left\|T_{\beta(\cdot)}(\chi_{B_k})\right\|_{L^{p_2(\cdot)}(E)} &\le C2^{-k[n-\beta(0)+s+1]+i(s+1)}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}(E)}\|a_i\|_{L^{p_1(\cdot)}(B_i)}\|\chi_k\|_{L^{p_2(\cdot)}(B_k)}\\ &\le C2^{-k[n+s+1]+i(s+1)}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}(E)}\|a_i\|_{L^{p_1(\cdot)}(E)}\|T_{\beta(\cdot)}(\chi_{B_k})\|_{L^{p_2(\cdot)}(E)}\\ &\le C2^{-k[n+s+1]+i(s+1)}\|\chi_{B_i}\|_{L^{p_1'(\cdot)}(E)}\|a_i\|_{L^{p_1(\cdot)}(E)}\|\chi_{B_k}\|_{L^{p_2(\cdot)}(E)}.\end{aligned}$$

From Lemma 2.3, it follows that

$$\begin{aligned}J_1 &= \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|\|T_{\beta(\cdot)}(a_i)\chi_k\|_{L^{p_2(\cdot)}(E)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}\\ &\le C\left\{\sum_{k=-\infty}^{-1}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|2^{(i-k)[n+s+1-(\alpha+\frac{n}{p_1})(0)]}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}.\end{aligned}$$

Since

$$s+1-\left[\alpha^+ + n\left(\frac{1}{p^-}-1\right)\right] > 0,$$

it follows, from Lemma 2.14, that

$$J_1 \le C\left[\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right]^{\frac{1}{q_2(0)}} \le C\left[\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right]^{\frac{1}{q_1(0)}} \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

We now estimate J_2. By Theorem 1.6, we know that

$$\begin{aligned}J_2 &= \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=k-1}^{\infty}|\lambda_i|\|T_{\beta(\cdot)}(a_i)\chi_k\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}\\ &\le \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=k-1}^{\infty}|\lambda_i|\|a_i\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}\\ &\le \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=k-1}^{-1}|\lambda_i|\|a_i\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}\\ &\quad + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=0}^{\infty}|\lambda_i|\|a_i\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}\\ &\le C\left\{\sum_{k=-\infty}^{-1}\left[\sum_{i=k-1}^{-1}|\lambda_i|2^{(k-i)\alpha(0)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}\\ &\quad + C\left\{\sum_{k=-\infty}^{-1}\left[\sum_{i=0}^{\infty}|\lambda_i|2^{(k-i)\alpha^- + k(\alpha(0)-\alpha^-)+i(\alpha^- - \alpha_\infty)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}.\end{aligned}$$

Since $\alpha^- \le \min\{\alpha(0), \alpha_\infty\}$, it follows that

$$k[\alpha(0) - \alpha^-] + i(\alpha^- - \alpha_\infty) \le 0,$$

which, together with Theorem 1.6, implies that

$$J_2 \le C\left[\sum_{k=-\infty}^{-1} |\lambda_k|^{q_2(0)}\right]^{\frac{1}{q_2(0)}} \le C\left[\sum_{k=-\infty}^{-1} |\lambda_k|^{q_1(0)}\right]^{\frac{1}{q_1(0)}} \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

When $k \ge 0$, $|x-y| \ge 1$, then $\beta(x) \sim \beta(\infty)$. By this and Theorem 1.6, we find that

$$\begin{aligned}\left|T_{\beta(\cdot)}(a_i)(x)\right| &\le C\int_{B_i\cap E} \frac{|a_i(y)||y|^{s+1}}{|x|^{n-\beta(\infty)+s+1}}dy \\ &\le C2^{-k(n-\beta(\infty)+s+1)+i(s+1)}\int_{B_i\cap E}|a_i(y)|dy.\end{aligned}$$

Using this estimate and repeating the above proof, we know that

$$J_2 \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}$$

in this case. Thus, we have $J_2 \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}$.

Estimating J_3 and J_4 by using the same technique as for J_1 and J_2, we find that

$$J_3 \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)} \text{ and } J_4 \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

By the estimates for J_1, J_2, J_3, and J_4, we know that

$$\left\|T_{\beta(\cdot)}(f)\right\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)} \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

This finishes the proof of Theorem 1.12. □

Proof [Proof of Theorem 1.13] Let $f \in H\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)$. It suffices to show that

$$\left\|[b, T_{\beta(\cdot)}](f)\right\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)} \le C\|f\|_{H\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)}.$$

By Lemma 2.13, $f = \sum_{i=-\infty}^{+\infty} \lambda_i a_i$, where $\{\lambda_i\}_{i\in\mathbb{Z}} \subset [0,\infty)$ and for any $i \in \mathbb{Z}$, a_i is an $(\alpha(\cdot), p_1(\cdot))$-atom with

$\operatorname{supp}(a_i) \subset B_i \cap E$. Thus, we have

$$\begin{aligned}
&\left\|[b,T_{\beta(\cdot)}](f)\right\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)} \\
&\leq C\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\|[b,T_{\beta(\cdot)}](f)\chi_k\|_{p_2(\cdot)}^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&\quad + C\left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty(q_2)_\infty}\|[b,T_{\beta(\cdot)}](f)\chi_k\|_{p_2(\cdot)}^{(q_2)_\infty}\right\}^{\frac{1}{(q_2)_\infty}} \\
&\lesssim \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|\|[b,T_{\beta(\cdot)}](a_i)\chi_k\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&\quad + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)}\left[\sum_{i=k-1}^{\infty}|\lambda_i|\|[b,T_{\beta(\cdot)}](a_i)\chi_k\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&\quad + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha_\infty(q_2)_\infty}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|\|[b,T_{\beta(\cdot)}](a_i)\chi_k\|_{p_2(\cdot)}\right]^{(q_2)_\infty}\right\}^{\frac{1}{(q_2)_\infty}} \\
&\quad + \left\{\sum_{k=-\infty}^{-1} 2^{k\alpha_\infty(q_2)_\infty}\left[\sum_{i=k-1}^{\infty}|\lambda_i|\|[b,T_{\beta(\cdot)}](a_i)\chi_k\|_{p_2(\cdot)}\right]^{(q_2)_\infty}\right\}^{\frac{1}{(q_2)_\infty}} \\
&=: H_1+H_2+H_3+H_4.
\end{aligned}$$

We first estimate H_1. By using the same estimate as for J_1, we have

$$\left|[b,T_{\beta(\cdot)}](a_i)(x)\right| \leq \int_{B_i\cap E}|b(x)-b(y)|\frac{|a_i(y)||y|^{s+1}}{|x|^{n-\beta(x)+s+1}}dy.$$

Moreover, by Lemma 2.1, we know that, when $k<0$ and $|x-y|<1$, then $\beta(x)\sim\beta(0)$, which implies that

$$\begin{aligned}
&\left|[b,T_{\beta(\cdot)}](a_i)(x)\right| \\
&\leq C\int_{B_i\cap E}|b(x)-b(y)|\frac{|a_i(y)||y|^{s+1}}{|x|^{n-\beta(0)+s+1}}dy \\
&\leq C2^{-k[n-\beta(0)+s+1]+i(s+1)}\int_{B_i\cap E}|b(x)-b(y)||a_i(y)|dy \\
&\leq C2^{-k[n-\beta(0)+s+1]+i(s+1)}\left[|b(x)-b_{B_i}|\int_{B_i\cap E}|a_i(y)|dy+\int_{B_i\cap E}|b_{B_i}-b(y)||a_i(y)|dy\right].
\end{aligned}$$

From this and Hölder's inequality, it follows that

$$\begin{aligned}
\left|[b,T_{\beta(\cdot)}](a_i)(x)\right| &\leq C2^{-k[n-\beta(0)+s+1]+i(s+1)}\|a_i\|_{L^{p_1(\cdot)}(E)} \\
&\quad\times\left[|b(x)-b_{B_i}|\|\chi_{B_i}\|_{L^{p_1'(\cdot)}(E)}+\||b_{B_i}-b|\chi_{B_i}\|_{L^{p_1'(\cdot)}(E)}\right].
\end{aligned}$$

By the above estimates and Lemma 2.10, we know that

$$\begin{aligned}
&\left\|[b,T_{\beta(\cdot)}](a_i)\chi_k\right\|_{L^{p_2(\cdot)}(E)} \\
&\quad\leq C2^{-k[n-\beta(0)+s+1]+i(s+1)}\|a_i\|_{p_1(\cdot)}\left[\||b-b_{B_i}|\chi_k\|_{p_2(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)}\right. \\
&\qquad\left.+\ \||b_{B_i}-b|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\right] \\
&\quad\leq C2^{-k[n-\beta(0)+s+1]+i(s+1)}\|a_i\|_{p_1(\cdot)}\left[(k-i)\|b\|_{\mathrm{BMO}(E)}\|\chi_{B_k}\|_{p_2(\cdot)}\|\chi_{B_i}\|_{p_1'(\cdot)}\right. \\
&\qquad\left.+\ \|b\|_{\mathrm{BMO}(E)}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)}\right] \\
&\quad\leq C(k-i)2^{-k[n-\beta(0)+s+1]+i(s+1)}\|a_i\|_{p_1(\cdot)}\|a_i\|_{p_1(\cdot)}\|b\|_{\mathrm{BMO}(E)}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_2(\cdot)} \\
&\quad\leq C(k-i)2^{-k[n-\beta(0)+s+1]+i(s+1)}\|a_i\|_{p_1(\cdot)}\|a_i\|_{p_1(\cdot)}\|b\|_{\mathrm{BMO}(E)}\|\chi_{B_i}\|_{p_1'(\cdot)}\|T_{\beta(\cdot)}\chi_k\|_{p_2(\cdot)} \\
&\quad\leq C(k-i)2^{-k[n-\beta(0)+s+1]+i(s+1)}\|a_i\|_{p_1(\cdot)}\|a_i\|_{p_1(\cdot)}\|b\|_{\mathrm{BMO}(E)}\|\chi_{B_i}\|_{p_1'(\cdot)}\|\chi_k\|_{p_1(\cdot)}.
\end{aligned}$$

From Lemma 2.3, we deduce that

$$\begin{aligned}
H_1 &= \left\{\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_2(0)}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|\|[b,T_{\beta(\cdot)}](a_i)\chi_k\|_{p_2(\cdot)}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}} \\
&\leq C\|b\|_{\mathrm{BMO}(E)}\left\{\sum_{k=-\infty}^{-1}\left[\sum_{i=-\infty}^{k-2}|\lambda_i|(k-i)2^{(i-k)(n+s+1-(\alpha+\frac{n}{p_1})(0))}\right]^{q_2(0)}\right\}^{\frac{1}{q_2(0)}}.
\end{aligned}$$

Since

$$s+1-\left[\alpha^+ + n\left(\frac{1}{p^-}-1\right)\right]>0,$$

it follows, from Lemma 2.14, that

$$\begin{aligned}
H_1 &\leq C\|b\|_{\mathrm{BMO}(E)}\left[\sum_{k=-\infty}^{-1}|\lambda_k|^{q_2(0)}\right]^{\frac{1}{q_2(0)}} \\
&\leq C\left[\sum_{k=-\infty}^{-1}|\lambda_k|^{q_1(0)}\right]^{\frac{1}{q_1(0)}} \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.
\end{aligned}$$

We now estimate H_2. By Lemma 2.10, we know that

$$H_2 = \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left[\sum_{i=k-1}^{\infty} |\lambda_i| \|[b, T_{\beta(\cdot)}](a_i)\chi_k\|_{p_2(\cdot)} \right]^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}$$

$$\leq C\|b\|_{\mathrm{BMO}(E)} \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left[\sum_{i=k-1}^{\infty} |\lambda_i| \|a_i\|_{p_2(\cdot)} \right]^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}$$

$$\leq C\|b\|_{\mathrm{BMO}(E)} \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left[\sum_{i=k-1}^{-1} |\lambda_i| \|a_i\|_{p_2(\cdot)} \right]^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}$$

$$+ C\|b\|_{\mathrm{BMO}(E)} \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_2(0)} \left[\sum_{i=0}^{\infty} |\lambda_i| \|a_i\|_{p_2(\cdot)} \right]^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}$$

$$\leq C\|b\|_{\mathrm{BMO}(E)} \left\{ \sum_{k=-\infty}^{-1} \left[\sum_{i=k-1}^{-1} |\lambda_i| 2^{(k-i)\alpha(0)} \right]^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}$$

$$+ C\|b\|_{\mathrm{BMO}(E)} \left\{ \sum_{k=-\infty}^{-1} \left[\sum_{i=0}^{\infty} |\lambda_i| 2^{(k-i)\alpha^- + k[\alpha(0)-\alpha^-] + i(\alpha^- - \alpha_\infty)} \right]^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}.$$

Since $\alpha^- \leq \min\{\alpha(0), \alpha_\infty\}$, it follows that

$$k[\alpha(0) - \alpha^-] + i(\alpha^- - \alpha_\infty) \leq 0,$$

which, together with Lemma 2.10, implies that

$$H_2 \leq C \left[\sum_{k=-\infty}^{-1} |\lambda_k|^{q_2(0)} \right]^{\frac{1}{q_2(0)}} \leq C \left[\sum_{k=-\infty}^{-1} |\lambda_k|^{q_1(0)} \right]^{\frac{1}{q_1(0)}} \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

When $k \geq 0$, $|x-y| \geq 1$, then $\beta(x) \sim \beta(\infty)$. By this and Theorem 1.6, we have

$$\left|[b, T_{\beta(\cdot)}](a_i)(x)\right| \leq C \int_{B_i \cap E} |b(x) - b(y)| \frac{|a_i(y)||y|^{s+1}}{|x|^{n-\beta(\infty)+s+1}} dy$$

$$\leq C 2^{-k[n-\beta(\infty)+s+1]+i(s+1)} \int_{B_i \cap E} |b(x) - b(y)||a_i(y)| dy.$$

Using this estimate and repeating the above proof, we know that $H_2 \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}$ in this case. Thus, we know that $H_2 \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}$.

Estimating H_3 and H_4 by using the same technique as for H_1 and H_2, we know that

$$H_3 \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)} \quad \text{and} \quad H_4 \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

From the above estimates for H_1, H_2, H_3, and H_4, we deduce that

$$\big\|[b,T_{\beta(\cdot)}](f)\big\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_2(\cdot)}(E)} \leq C\|f\|_{H\dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(E)}.$$

This finishes the proof of Theorem 1.13. □

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