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Small genus-4 Lefschetz fibrations on simply-connected 4-manifolds

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Abstract: We consider simply connected 4-manifolds admitting Lefschetz fibrations over the 2-sphere. We explicitly construct nonhyperelliptic and hyperelliptic Lefschetz fibrations of genus 4 on simply-connected 4-manifolds which are exotic symplectic 4-manifolds in the homeomorphism classes of $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$, respectively. From these, we provide upper bounds for the minimal number of singular fibers of such fibrations. In addition, we prove that this number is equal to 18 for $g=3$ when such fibrations are hyperelliptic. Moreover, we discuss these numbers for higher genera.

Key words: Symplectic 4-manifolds, mapping class groups, Lefschetz fibrations, exotic manifolds

1. Introduction

Due to the pioneering works of Donaldson [13] and Gomp [21], Lefschetz pencils and Lefschetz fibrations play an important role in studying 4-manifold topology. Donaldson proved that every symplectic 4-manifold, up to blow-ups, corresponds to a Lefschetz fibration with a finite number of singularities of a prescribed type, which provides a way to study combinatorially via a positive factorization of its monodromy if exists. Conversely, Gompf showed that any 4-manifold admitting a genus-g Lefschetz fibration is a symplectic 4-manifold if $g \geq 2$.

Every nontrivial Lefschetz fibration admits certain singular fibers associated to its monodromy. The number of its singular fibers provides us important information about its total spaces such as the Euler characteristic, the signature, and so on. Since it has been known that the number of singular fibers in a Lefschetz fibration cannot be arbitrary, determining the minimal number of singular fibers in a Lefschetz fibration is an interesting problem to be investigated.

Let $N(g,h)$ be the minimal number of singular fibers in all nontrivial relatively minimal genus-g Lefschetz fibrations of over the oriented closed surface of genus-h. For $h \geq 1$, the exact value of the number $N(g,h)$ is almost known (except $N(g,1)$ for $g \geq 3$ and $N_{2,2}$) [23, 25, 26, 31, 37]. For $h=0$, it is known that $N(2,0)=7$ by Xiao's construction [38] and also by the existence of a relation among seven positive Dehn twists in the mapping class group of genus-2 surface with one boundary component obtained by Baykur and Korkmaz [6]. However, the exact value of this number is not known for $h=0$ and $g \geq 3$. The best known estimates for $N(g,0)$: $N(g,0) \leq 2g+4$ if g is even and $N(g,0) \leq 2g+10$ if g is odd [10, 11, 24]. If we consider hyperelliptic Lefschetz fibrations, i.e. their vanishing cycles are invariant under a hyperelliptic involution ι (see for instance

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Figure 5), then some results about this number are known. Let $M(g,h)$ denote the minimal number of singular fibers in all nontrivial genus- g hyperelliptic Lefschetz fibrations over the 2-sphere. Baykur and Korkmaz [7] constructed a hyperelliptic genus-3 Lefschetz fibration over the 2-sphere with 12 singular fibers and then they proved that $M(3,0)=12$. In [3], the author proved the following result:

(1) $M(4,0)=12$ and $M(6,0)=16$,

(2) $M(8,0)=19$ or 20 and $M(10,0)=23$ or 24,

(3) $M(5,0)\geq 15$, $M(7,0)\geq 17$, and $M(9,0)\geq 24$.

Moreover, when the total spaces of such fibrations are complex surfaces, she proved that it is equal to $2g+4$ if $g\geq 4$ and even and it has a lower bound $2g+6$ if $g\geq 7$ and odd. Thus, the exact value of the number $M(g,0)$ is not known (except for $g=3,4$ and 6). Therefore, this question is also open for hyperelliptic Lefschetz fibrations.

Lefschetz fibration structures on various smooth 4-manifolds with small numbers of singular fibers may provide us the existence of symplectic structures on 4-manifolds in the homeomorphism classes of simply-connected 4-manifolds with very small topology, which has been an interesting topic containing several construction techniques (e.g., [1, 2, 4, 12, 16–20, 28, 34–36]). Recently, some authors have studied Lefschetz fibration structures to produce exotic 4-manifolds (which are homeomorphic but not diffeomorphic to standard ones). Since it is natural to relate small (as in small second homology) exotic 4-manifolds to small (as in small number of positive Dehn twists) Lefschetz fibrations, it is of interest to find the minimal number of singular fibers in Lefschetz fibrations on simply connected 4-manifolds.

Let N_g be the minimal number of singular fibers in all genus- g Lefschetz fibrations on a simply-connected 4-manifold over the 2-sphere having at least one singular fiber. It is known that the minimal number of singular fibers in all torus Lefschetz fibrations is 12. One can conclude that $N_1=12$ by the existence of torus Lefschetz fibrations with 12 singular fibers on the elliptic surface $E(1)=\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. A genus-2 Lefschetz fibration with 14 singular fibers on a symplectic 4-manifold which is an exotic copy of $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$ was constructed by Baykur and Korkmaz [6]. By [6, Theorem 2], one can obtain that $N_2=14$.

Let us define M_g as the minimal number of singular fibers in all hyperelliptic Lefschetz fibrations on a simply-connected 4-manifolds over $\mathbb{S}^2$ having at least one singular fiber. By the same argument above, $M_1=12$ and $M_2=14$.

The purpose of the present paper is to estimate the numbers of N_g and M_g. In this direction, in Section 2, we first give some preliminary information and results. In Section 3, we explicitly construct two genus-4 Lefscherz fibrations on simply connected 4-manifolds. To do this, we first mention the positive factorizaton W for a genus-3 hyperelliptic Lefschetz fibration given by Baykur [5]*. Then we construct a genus-4 nonhyperelliptic Lefschetz fibration (X_1,f_1) using the Baykur's monodromy and the monodromy that gives the smallest genus-2 Lefschetz fibration. Here, we use the breeding technique to construct this positive factorization (see [5, 7] for more applications of this technique). We also prove that the 4-manifold X_1 is an exotic copy of $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$ (Theorem 3.2). Similarly, we produce another monodromy which gives a genus-4 hyperelliptic Lefschetz fibration (X_2,f_2) using the monodromy of generalized Matsumoto's fibration for $g=4$ and again the monodromy of the smallest genus-2 Lefschetz fibration. We prove that the 4-manifold X_2 is an exotic $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ (Theorem 3.3). In the last section, we examine the numbers N_g and M_g for $g\leq 4$. We first give a different proof for the result $N_2=M_2=14$ given in [6, Theorem 2] (Proposition 4.4(a)). Moreover,

* https://people.math.umass.edu/~baykur/docs/Genus3LPsFINAL.pdf

we prove that $M_3 = 18$ (Proposition 4.4(b)). Similarly, using the existence of genus-4 nonhyperelliptic and hyperelliptic Lefschetz fibrations (X_1, f_1) and (X_2, f_2) constructed in Section 3, we conclude that $N_4 \leq 23$ and $M_4 \leq 24$. We then give better estimates for the number M_4 (Proposition 4.4(c)). Finally, we discuss the numbers N_g and M_g for higher genus.

2. Preliminaries

This section presents the necessary background and the known results used in our proofs.

2.1. Mapping class groups

Let us denote a compact connected oriented smooth surface of genus g with $n \geq 0$ boundary components by Σ_g^n. Let Mod_g^n denote the mapping class group of Σ_g^n, i.e., the group of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g^n fixing all points on the boundary. We assume all isotopies are identity on the boundary. When $n = 0$, we will denote Mod_g^n and Σ_g^n by Mod_g and Σ_g, respectively. Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if g and h are two diffeomorphisms, then the composition gh means that we apply h first and then g.

Now, let us remind the following basic properties of Dehn twists. Let a and b be simple closed curves on Σ_g^n and $f \in \mathrm{Mod}_g^n$.

- **Commutativity:** If a and b are disjoint, then $t_a t_b = t_b t_a$.
- **Conjugation:** If $f(a) = b$, then $f t_a f^{-1} = t_b$.

(Here, t_a denotes the positive Dehn twist about a simple closed curve a.)

2.2. Lefschetz fibrations

We remind some basic definitions and facts about Lefschetz fibrations. Throughout the paper we denote the 2-sphere by $\mathbb{S}^2$. Let M be a closed connected oriented smooth 4-manifold. A Lefschetz fibration on M is a smooth surjective map if it has only finitely many critical points $\{p_1, p_2, \ldots, p_n\}$ such that around each of which it is expressed in the form of $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates compatible with the orientations of M and $\mathbb{S}^2$ (In general, the base of a Lefschetz fibration can be a closed orientable surface of genus $h \geq 0$, but throughout this paper, we only consider $\mathbb{S}^2$). The genus-g of a regular fiber is defined to be the genus of the fibration. The inverse image of a critical value is called a singular fiber. We assume that each singular fiber contains only one critical point, which can be obtained by a small perturbation. Each singular fiber is obtained by collapsing a simple closed curve, called vanishing cycle, on a nearby regular fiber to a point. If the vanishing cycle is nonseparating (respectively separating), then the corresponding singular fiber is called irreducible (respectively reducible). Throughout the paper, we also assume that all Lefschetz fibrations are nontrivial and relatively minimal, i.e. they have at least one singular fiber and no fiber containing a (-1)-sphere.

A Lefschetz fibration can be described via its monodromy, which is an element in the mapping class group Mod_g. The monodromy of a Lefschetz fibration $f : M \to \mathbb{S}^2$ is given by a positive factorization

$$t_{a_1} t_{a_2} \cdots t_{a_m} = 1$$

in Mod_g up to Hurwitz moves (exchanging subwords $t_{a_i}t_{a_{i+1}} = t_{a_{i+1}}t_{t^{-1}_{a_{i+1}}(a_i)}$) and global conjugations (changing each t_{a_i} with $t_{\varphi(a_i)}$ for some $\varphi \in \mathrm{Mod}_g$), where a_i's are vanishing cycles of the singular fibers. A map $\sigma : M \to \mathbb{S}^2$ is a section if $f \circ \sigma = id_{\mathbb{S}^2}$. If a positive relation $t_{a_1}t_{a_2}\cdots t_{a_m} = 1$ in Mod_g has a lifting to Mod_g^k so that

$$t_{\tilde{a}_1}t_{\tilde{a}_2}\cdots t_{\tilde{a}_m} = t_{\delta_1}^{n_1}t_{\delta_2}^{n_2}\cdots t_{\delta_k}^{n_k},$$

where each n_i is integer and δ_i is a boundary curve, then the Lefschetz fibration $f : M \to \mathbb{S}^2$ admits k disjoint sections $S_1, \ldots, S_k$, where S_j is of self-intersection $-n_j$ and vice versa [8]. We say that two Lefschetz fibrations $f_1 : M_1 \to \mathbb{S}^2$ and $f_2 : M_2 \to \mathbb{S}^2$ are isomorphic if there exist orientation preserving diffeomorphisms $G : M_1 \to M_2$ and $g : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f_2 \circ G = g \circ f_1$.

The hyperelliptic mapping class group HMod_g of Σ_g is defined as the subgroup of Mod_g that is the centralizer of a hyperelliptic involution $\iota : \Sigma_g \to \Sigma_g$. A Lefschetz fibration is said to be hyperelliptic if its vanishing cycles are invariant under the hyperelliptic involution ι up to isotopy.

For a genus-g Lefschetz fibration $f : M \to \mathbb{S}^2$, the Euler characteristic, $e(M)$, of the 4-manifold M can be computed as

$$e(M) = 4 - 4g + n + s,$$

where n and s are the numbers of nonseparating and separating vanishing cycles, respectively. Also we define the following invariant associated to the 4-manifold M:

$$\chi_h(M) = \frac{e(M) + \sigma(M)}{4},$$

where $\sigma(M)$ is the signature of M. Let us note that if M is a complex surface, $\chi_h(M)$ is the holomorphic Euler characteristic.

It follows from the theory of Lefschetz fibrations that if a Lefschetz fibration $f : M \to \mathbb{S}^2$ with a regular fiber Σ_g and the monodromy $t_{\alpha_1}t_{\alpha_2}\cdots t_{\alpha_m} = 1$ admits a section, then the fundamental group $\pi_1(M)$ of M is isomorphic to the group $\pi_1(\Sigma_g)$ divided by the normal closure of the vanishing cycles (cf [21]), that is,

$$\pi_1(M) \cong \pi_1(\Sigma_g)/\langle \alpha_1, \alpha_2, \ldots \alpha_m \rangle$$

The signature $\sigma(M)$ of M, which is another invariant of the Lefschetz fibration $f : M \to \mathbb{S}^2$ can be computed using several techniques. For instance, Endo and Nagami [15] gave a useful method which uses the signatures of the relations involved in its monodromy. For an integer-valued function I_g on the set of relators of Mod_g (see [15] for its definition and properties), the following theorem holds:

Theorem 2.1 *[15] Let $f : M \to \mathbb{S}^2$ be a genus-g Lefschetz fibration with the monodromy $t_{c_1}t_{c_2}\cdots t_{c_n} = 1$. Then the signature of M is*

$$\sigma(M) = I_g(c_1c_2\cdots c_n).$$

This method allows us to compute the signature of a Lefschetz fibration $f : M \to \mathbb{S}^2$ as the sum of basic relations in its monodromy. Let us recall some signatures that we will need later. For the proof, see [15].

- $I_g(a) = -1$, where a is the isotopy class of a separating curve.

- $I_g((B_0B_1\cdots B_gC)^2)=-4$ if g is even.

Here, the word $(B_0B_1\cdots B_gC)^2$ is the relator coming from Matsumoto's relation which is explained later (see 2.4).

One can also use Ozbagci's algorithm [32] to compute the signature of a Lefschetz fibration $f:M\to\mathbb{S}^2$. For hyperelliptic Lefschetz fibrations, we have the following useful lemma:

Lemma 2.2 *[14, 29, 30] Let $f\colon M\to\mathbb{S}^2$ be a genus-g hyperelliptic Lefschetz fibration. Let n and $s=\sum_{h=1}^{[g/2]}s_h$ denote the numbers of nonseparating and separating vanishing cycles of this fibration, respectively, where s_h is the number of separating vanishing cycles that separate the genus-g surface into two surfaces one of which has genus h. Then the signature of M is*

$$\sigma(M) \quad = \quad -\frac{g+1}{2g+1}n+\sum_{h=1}^{[g/2]}\left(\frac{4h(g-h)}{2g+1}-1\right)s_h.$$

Remark 2.3 *One can easily obtain that $\sigma(M)\le n-s-2$ using $b_1(M)\le 2g-1$ by the handlebody decomposition of nontrivial Lefschetz fibrations $f:M\to\mathbb{S}^2$ and the fact that such fibrations have at least one non-separating vanishing cycle. If the Lefschetz fibration $f:M\to\mathbb{S}^2$ is hyperelliptic, then Ozbagci [32] proved that $\sigma(M)\le n-s-4$. Later, for every Lefschetz fibration $f:M\to\mathbb{S}^2$, Cadavid [10] improved the upper bound of signature $\sigma(M)$ showing that*

$$\sigma(M)\le n-s-2(2g-b_1(M)). \tag{2.1}$$

When the 4-manifold X is simply-connected, the above inequality becomes

$$\sigma(X)\le n-s-4g. \tag{2.2}$$

For a closed orientable surface of genus $g\ge 1$, the first homology group $H_1(\mathrm{HMod}_g;\mathbb{Z})$ of the hyperelliptic mapping class group HMod_g has the following isomorphism:

$$H_1(\mathrm{HMod}_g;\mathbb{Z})\cong\begin{cases}\mathbb{Z}/4(2g+1), & \text{if } g \text{ is odd},\\ \mathbb{Z}/2(2g+1), & \text{if } g \text{ is even},\end{cases}$$

which can be proven by the presentation of the hyperelliptic mapping class group HMod_g [9]. In the hyperelliptic mapping class group HMod_g, all Dehn twists about nonseparating simple closed curves are nontrivial and each of them maps to the same generator in $H_1(\mathrm{HMod}_g;\mathbb{Z})$ under the natural map $\mathrm{HMod}_g\to H_1(\mathrm{HMod}_g;\mathbb{Z})$. Thus, the number of twists of a factorization of identity in HMod_g consisting of positive Dehn twists about nonseparating simple closed curves is divisible by $2(2g+1)$ (respectively $4(2g+1)$) if g is even (respectively odd). We say that a separating simple closed curve on Σ_g is of *type* h if it separates Σ_g into two subsurfaces of genera h and $g-h$. It is known that each separating simple closed curve of type h can be written as a product of $2h(4h+2)$ positive Dehn twists about nonseparating simple closed curves. Therefore, we have the following lemma which gives a relation between the number of nonseparating vanishing cycles and that of separating vanishing cycles in a genus-g hyperelliptic Lefschetz fibration:

Lemma 2.4 *[3] Let n (or s) be the number of nonseparating (resp. separating) vanishing cycles in a genus-g hyperelliptic Lefschetz fibration over $\mathbb{S}^2$. Then, we have*

$$n + \sum_{h=1}^{[g/2]} 2h(4h+2)s_h \equiv \begin{cases} 0 \pmod{4(2g+1)}, & \text{if } g \text{ is odd,} \\ 0 \pmod{2(2g+1)}, & \text{if } g \text{ is even,} \end{cases}$$

where s_h is the number of separating vanishing cycles of type h with $s = \sum_{h=1}^{[g/2]} s_h$.

2.3. The smallest genus two Lefschetz fibration

In [6], Baykur and Korkmaz obtained the following relation in Mod_2^1:

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_C t_{x_4} = t_\delta,$$

which can be rewritten as

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C = t_\delta,$$

where $t_c(x_4) = B_2$. Set $T = t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C$. They also showed that the positive factorization $T = t_\delta$ realizes the smallest genus-2 Lefschetz fibrations whose total space is diffeomorphic to $(\mathbb{T}^2 \times \mathbb{S}^2)\#3\overline{\mathbb{C}P^2}$. Stipsicz and Yun [37] obtained the following lifting of T:

$$T = t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C = t_{\delta_1} t_{\delta_2} \tag{2.3}$$

in Mod_2^2, where the curves x_i, B_2, C, d, e, δ_1, and δ_2 are as depicted in Figure 1. The relation (2.3) and also a further lift to Mod_2^3 is given by Baykur [5].

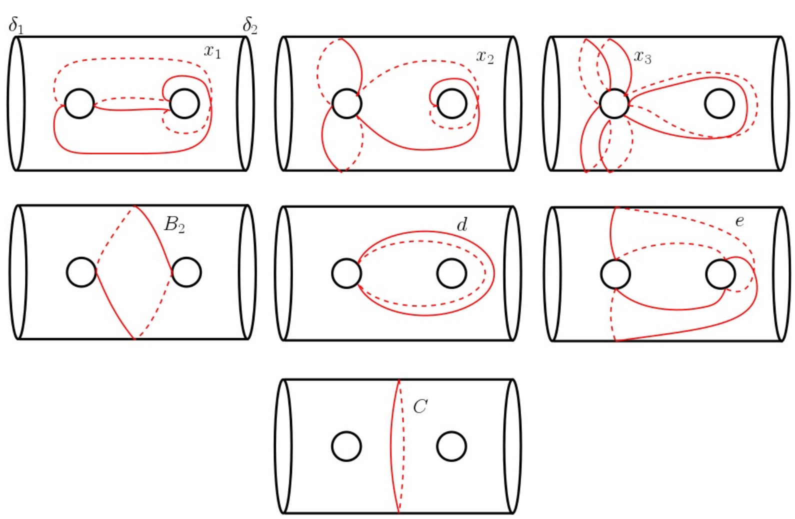

Figure 1. The curves x_i, B_2, C, d, and e on the surface Σ_2^2.

2.4. Generalized Matsumoto's relation

A relation with eight positive Dehn twists was discovered by Matsumoto [30], which is the global monodromy of a Lefschetz fibration on $(\mathbb{T}^2 \times \mathbb{S}^2)\#4\overline{\mathbb{C}P^2}$. It is later generalized to higher genus surfaces by Korkmaz [24], independently by Cadavid [10] and recently by a different proof [11]. A lift of this relation to Mod_g^1 was first discovered by Ozbagci and Stipsicz [33] and another lift to Mod_g^2 by Korkmaz [27]. However, we will use the following lift to Mod_g^2 proved by Hamada [22]:

$$V_g = \begin{cases} (t_{B_0}t_{B_1}\cdots t_{B_g}t_C)^2 & \text{if} \quad g=2k, \\ (t_{B_0}t_{B_1}\cdots t_{B_g}t_a^2t_b^2)^2 & \text{if} \quad g=2k+1, \end{cases}$$

where δ_i are the boundary parallel curves, and the curves B_i and C are as shown in Figure 2.

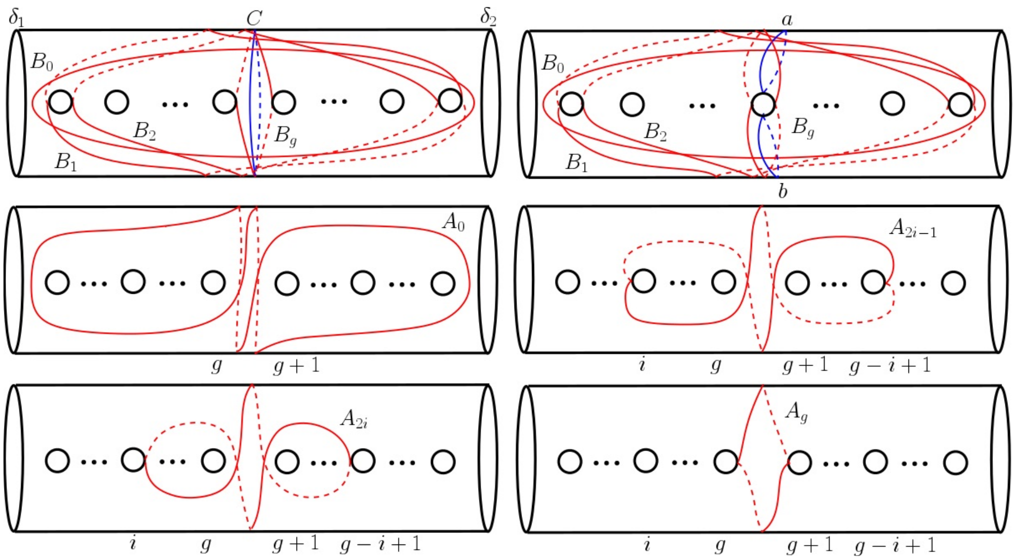

Figure 2. The curves B_i, A_i, C, a, and b on the surface Σ_g^2.

One may rewrite the generalized Matsumoto's relation for $g=2k$ as follows:

$$\begin{aligned}(t_{B_0}t_{B_1}\cdots t_{B_g}t_C)^2 &= (t_{B_0}t_{B_1}\cdots t_{B_g}t_C)(t_{B_0}t_{B_1}\cdots t_{B_g}t_C) \\ &= (t_Ct_{t_C^{-1}(B_0)}t_{t_C^{-1}(B_1)}\cdots t_{t_C^{-1}(B_g)})(t_{B_0}t_{B_1}\cdots t_{B_g}t_C) = t_{\delta_1}t_{\delta_2}.\end{aligned}$$

Since the Dehn twist t_C commutes with t_{δ_1} and t_{δ_2}, we get

$$V_g = t_C^2t_{A_0}t_{A_1}\cdots t_{A_g}t_{B_0}t_{B_1}\cdots t_{B_g} = t_{\delta_1}t_{\delta_2} \tag{2.4}$$

in Mod_g^2, where each $A_i = t_C^{-1}(B_i)$ is shown in Figure 2. Note that the total space of the genus-g Lefschetz fibration is diffeomorphic to $(\Sigma_k\times\mathbb{S}^2)\#4\overline{\mathbb{C}P^2}$ (respectively $(\Sigma_k\times\mathbb{S}^2)\#8\overline{\mathbb{C}P^2}$) if $g=2k$ (respectively $g=2k+1$).

3. Small Lefschetz fibrations of fiber genus 4 on simply connected 4-manifolds

In this section, our aim is to construct small genus-4 Lefschetz fibrations on simply connected 4-manifolds. In order to make our construction, we will first derive a positive factorization of $t_{\delta_1}t_{\delta_2}$ in Mod_2^2 with $(n,s)=(4,3)$, which will be one of our building blocks. For completeness of our contructions, using the breeding technique [5, 7], we also give the positive factorization W of $t_{\delta_1}t_{\delta_2}$ in Mod_3^2 with $(n,s)=(12,6)$ constructed by Baykur [5].

Afterwards, we derive a nonhyperelliptic genus-4 Lefschetz fibration on an exotic copy of $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$ by breeding the factorization W with the Matsumoto's genus-2 factorization V_2 given in (2.4). Moreover, we construct a hyperelliptic genus-4 Lefschetz fibration on an exotic copy of $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ using again the breeding technique to the generalized Matsumoto's factorization V_4 given in (2.4) and the factorizations which give the smallest genus-2 Lefschetz fibration.

Consider the curves x'_1, x'_2, x'_3, B_2, C, d', and e' on the genus-2 surface Σ_2^2 shown in Figures 1 and 3. Note that the closed surface Σ_2 is embedded in $\mathbb{R}^3$ in such a way that it is invariant under the involution υ shown in Figure 3. One can observe that these curves can be obtained by applying the involution υ to the curves contained in factorization (2.3). Then the relation in the following lemma is obtained from the relation (2.3) by rotating the surface Σ_2^2. A proof of this relation using Alexander's method can be found in the appendix.

Lemma 3.1 *There is a relation*

$$t_{e'}t_{x'_1}t_{x'_2}t_{x'_3}t_{d'}t_{B_2}t_C = t_{\delta_1}t_{\delta_2}. \tag{3.1}$$

in Mod_2^2*, where the curves* x'_i*,* C*,* d'*, and* e' *are as in Figure 3 and* B_2 *is as in Figure 1.*

We remark that any genus-2 Lefschetz fibration prescribed by a monodromy with $(n,s)=(4,3)$ has total space diffeomorphic to $(\mathbb{T}^2\times\mathbb{S}^2)\#3\overline{\mathbb{C}P^2}$. So the corresponding genus-2 Lefschetz fibration to our monodromy is diffeomorphic to the smallest genus-2 Lefschetz fibration constructed by Baykur and Korkmaz [6].

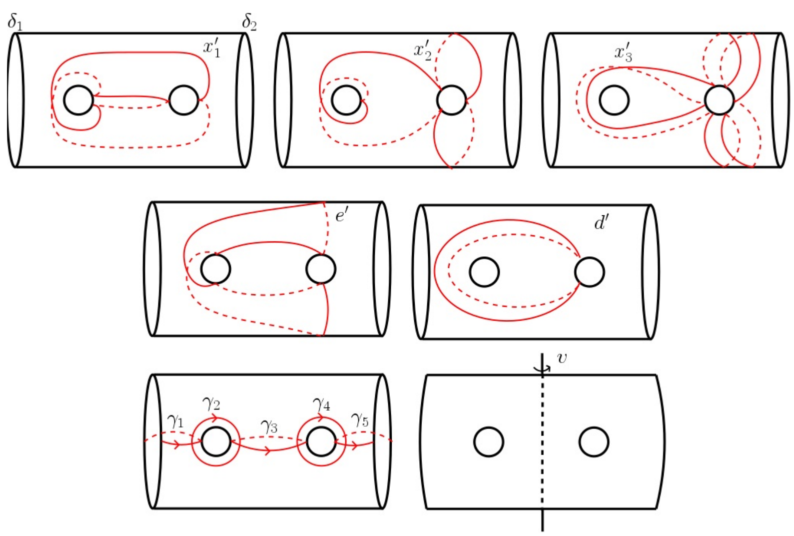

Figure 3. The curves x'_i, d', e', γ_j's on the surface Σ_2^2 and the involution υ on the surface Σ_2.

One of the building blocks of our monodromy construction is the monodromy of a genus-3 Lefschetz fibration on an exotic copy of $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$ given by Baykur [5]. For completeness, we give the construction of this monodromy in detail.

Set $T'=t_{e'}t_{x'_1}t_{x'_2}t_{x'_3}t_{d'}t_{B_2}t_C$ or $T'=t_{e'}P't_C$ so that $T'=t_{e'}P't_C=t_{\delta_1}t_{\delta_2}$ in Mod_2^2. Since t_C and the factorization $t_{e'}P'$ commute, we have

$$T'=t_{e'}P't_C=t_Ct_{e'}P'=t_{\delta_1}t_{\delta_2} \tag{3.2}$$

Similarly, let us write the positive factorization T in (2.3) as $T = t_e P t_C$, where $P = t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2}$ so that $T = t_e P t_C = t_{\delta_1} t_{\delta_2}$ in Mod_2^2. By the commutativity of t_C and $t_e P$, we have

$$T = t_e P t_C = t_C t_e P = t_{\delta_1} t_{\delta_2}. \tag{3.3}$$

Let us embed the relation (3.3) into Mod_3^2 so that the boundary parallel curve δ_2 in Figure 1 is mapped to the curve C' in Figure 4. Hence, we get the following relation:

$$T = t_e P t_C = t_C t_e P = t_{\delta_1} t_{C'}. \tag{3.4}$$

We also embed the relation (3.2) into Mod_3^2 so that the curve δ_1 in Figure 1 is mapped to the curve C in Figure 4. In this case, the curves B_2 and C appearing in the factorization T' are mapped to the curves B_2' and C' in Figure 4 so that the factorization $P' = t_{x_1'} t_{x_2'} t_{x_3'} t_{d'} t_{B_2'}$. We thus have

$$T' = t_{e'} P' t_{C'} = t_{C'} t_{e'} P' = t_C t_{\delta_2}. \tag{3.5}$$

Combining the relations (3.4) and (3.5), the following relation in Mod_3^2 holds:

$$TT' = (t_C t_e P)(t_{e'} P' t_{C'}) = t_{\delta_1} t_{C'} t_C t_{\delta_2}.$$

By the fact that t_C commutes with $t_e P t_{e'} P' t_{C'}$ and the curves δ_1, δ_2, C and C' are all disjoint, the relation can be written as

$$TT' = t_e P t_{e'} P' t_{C'} t_C = t_{\delta_1} t_{\delta_2} t_{C'} t_C,$$

which gives the following relation

$$t_e P t_{e'} P' t_{C'} t_C t_C^{-1} t_{C'}^{-1} = t_{\delta_1} t_{\delta_2}.$$

Finally, we obtain the following identity in Mod_3^2:

$$t_e P t_{e'} P' = t_{\delta_1} t_{\delta_2}. \tag{3.6}$$

Consider the diffeomorphism $\varphi = t_{a_3}^3 t_{b_2} t_{c_1}$, where the curves are shown in Figure 5. One can easily verify that $\varphi(C) = e$. Then by conjugation of T' by φ, we obtain the following factorization of $t_{\delta_1} t_{\delta_2}$ in Mod_3^2:

$$(T')^{\varphi} = t_{\varphi(C')} t_{\varphi(e')} (P'^{\varphi}) = t_{\varphi(C)} t_{\varphi(\delta_2)},$$

which can be written as

$$(T')^{\varphi} = t_{C'} t_{\varphi(e')} t_{\varphi(x_1')} t_{\varphi(x_2')} t_{\varphi(x_3')} t_{\varphi(d')} t_{\varphi(B_2')} = t_e t_{\delta_2},$$

since $\varphi(C') = C'$ and $\varphi(\delta_2) = \delta_2$, where $(T')^{\varphi}$ denotes the conjugate factorization. Let us denote $(P')^{\varphi}$ by $\overline{P}$ and denote $\varphi(\alpha') = \overline{\alpha}$ for every curve α' appearing in the factorization $t_{e'} P'$. Therefore, we have the following relation:

$$(T')^{\varphi} = t_{C'} t_{\overline{e}} \overline{P} = t_{C'} t_{\overline{e}} t_{\overline{x_1}} t_{\overline{x_2}} t_{\overline{x_3}} t_{\overline{d}} t_{\overline{B_2}} = t_e t_{\delta_2}, \tag{3.7}$$

where the curves $\overline{e}$, $\overline{x_i}$'s, $\overline{d}$, and $\overline{B_2}$ are shown in Figure 5. Therefore, the relation (3.7) together with the relation (3.6) give rise to the following relation in Mod_3^2:

$$(t_e P t_{e'} P')(t_{C'} t_{\overline{e}} \overline{P}) = (t_{\delta_1} t_{\delta_2})(t_e t_{\delta_2}),$$

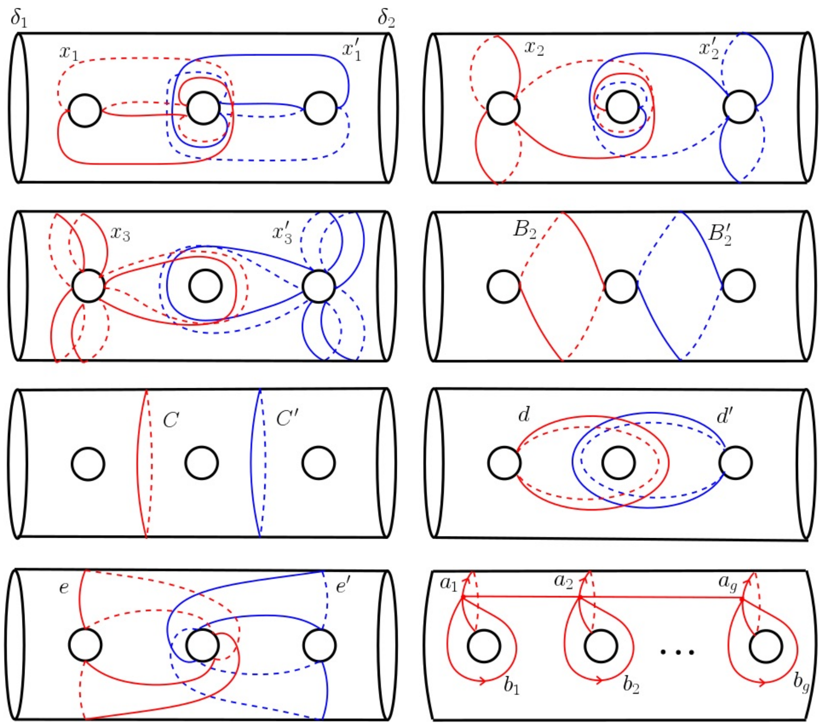

Figure 4. The curves coming from the factorizations T and T' and the generators of $\pi_1(\Sigma_g)$.

which implies that

$$t_e^{-1} t_e P t_{e'} P' t_{C'} t_{\overline{e}} \overline{P} = t_{\delta_1} t_{\delta_2}^2$$

by commutativity of the curve t_e and $t_{\delta_1} t_{\delta_2}$. Finally, by canceling the t_e factors, we get the desired equation in Mod_3^2:

$$P t_{e'} P' t_{C'} t_{\overline{e}} \overline{P} = t_{\delta_1} t_{\delta_2}^2.$$

This relation is the monodromy factorization for our genus-3 Lefschetz fibration. Observe that it admits two sections: one is of (-1) self-intersection and the other is of (-2). Capping off the boundary components, the factorization $P t_{e'} P' t_{C'} t_{\overline{e}} \overline{P}$ gives the following factorization of the identity in Mod_3:

$$t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_{e'} t_{x_1'} t_{x_2'} t_{x_3'} t_{d'} t_{B_2'} t_{C'} t_{\overline{e}} t_{\overline{x_1}} t_{\overline{x_2}} t_{\overline{x_3}} t_{\overline{d}} t_{\overline{B_2}} = 1. \tag{3.8}$$

Let W be the positive factorization given in (3.8) and let (X, f) be the corresponding genus-3 Lefschetz fibration which admits 18 singular fibers with $(n, s) = (12, 6)$. It is proved by Baykur in [5] that the 4-manifold X is an exotic $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$.

Since the Lefschetz fibration (X_3, f_3) is hyperelliptic, using the signature formula for hyperelliptic fibrations in Lemma 2.2, the signature $\sigma(X_3)$ of X_3 is given by

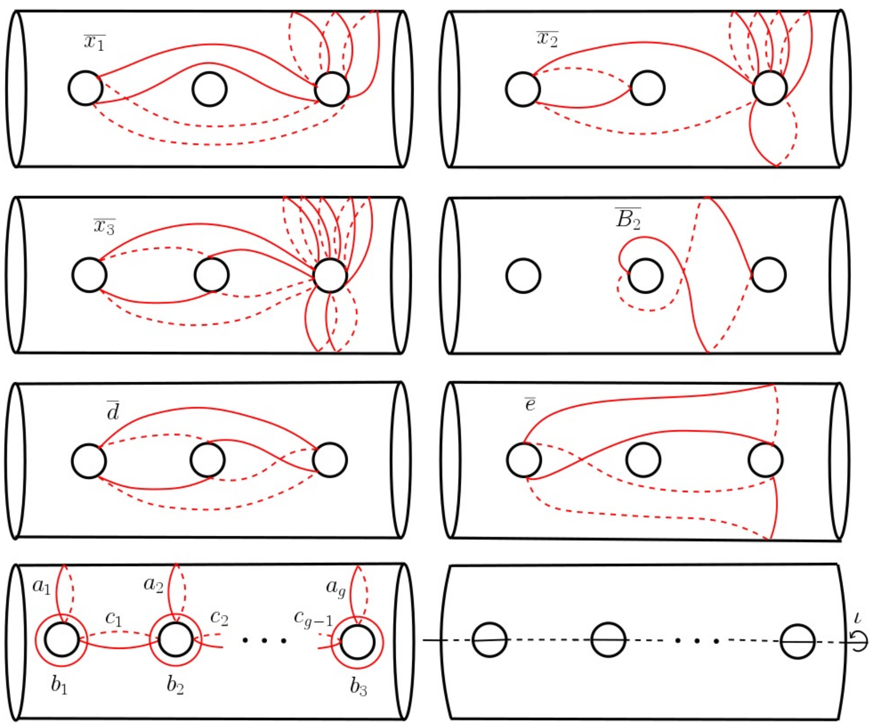

Figure 5. The curves $\overline{x_i}'s$, $\overline{B_2}$, $\overline{d}$, $\overline{e}$ and the hyperelliptic involution ι.

3.1. Constructing a small nonhyperelliptic genus-4 Lefschetz fibration on a simply-connected 4-manifold.

Consider the relation (3.8) in Mod_3^2, which can be rewritten as

$$\begin{aligned} W &= Pt_{e'}P't_{C'}t_{\overline{e}}\overline{P} \\ &= t_{\overline{e}}\overline{P}Pt_{e'}P't_{C'} \\ &= t_{\overline{e}}t_{\overline{x_1}}t_{\overline{x_2}}t_{\overline{x_3}}t_{\overline{d}}t_{\overline{B_2}}t_{x_1}t_{x_2}t_{x_3}t_dt_{B_2}t_{e'}t_{x_1'}t_{x_2'}t_{x_3'}t_{d'}t_{B_2'}t_{C'} \\ &= t_{\delta_1}t_{\delta_2}^2, \end{aligned}$$

since the factorization $t_{\overline{e}}\overline{P}$ commutes with t_{δ_1} and t_{δ_2}. We embed this relation into the surface Σ_4^2 so that the boundary parallel curve δ_2 is mapped to C'', where the curves are as in Figure 6. We also consider the Matsumoto's relation (2.4) in Mod_2^2

$$V_2 = t_C^2t_{A_0}t_{A_1}t_{A_2}t_{B_0}t_{B_1}t_{B_2} = t_{\delta_1}t_{\delta_2},$$

and embed it into Σ_4^2 in such a way that the curves $\delta_1 = \partial\Sigma_2^2, A_i, B_i$, and C are mapped to the curves C', A_i'', B_i'' and C'' for $i = 0, 1, 2$, respectively, where the curves are depicted in Figure 6. Thus, we get

$$V_2W = \big(t_{C''}^2t_{A_0''}t_{A_1''}t_{A_2''}t_{B_0''}t_{B_1''}t_{B_2''}\big)\big(t_{\overline{e}}\overline{P}Pt_{e'}P't_{C'}\big) = \big(t_{C'}t_{\delta_2}\big)\big(t_{\delta_1}t_{C''}^2\big),$$

which gives the relation

$$t_{C''}^{-2}\big(t_{C''}^2t_{A_0''}t_{A_1''}t_{A_2''}t_{B_0''}t_{B_1''}t_{B_2''}\big)\big(t_{\overline{e}}\overline{P}Pt_{e'}P't_{C'}\big)t_{C'}^{-1} = t_{\delta_1}t_{\delta_2},$$

by the commutativity of Dehn twists $t_{C'}$, $t_{C''}$, t_{δ_1} and t_{δ_2}. Therefore, we reach the following relation in Mod_4^2:

$$t_{A_0''}t_{A_1''}t_{A_2''}t_{B_0''}t_{B_1''}t_{B_2''}t_{\overline{e}}\overline{P}Pt_{e'}P' = t_{\delta_1}t_{\delta_2},$$

which gives the monodromy factorization for our first genus-4 Lefschetz fibration. Note that it admits two sections of self-intersection (-1). By capping of the boundary components, we get the following factorization of identity in Mod_4:

$$t_{A_0''}t_{A_1''}t_{A_2''}t_{B_0''}t_{B_1''}t_{B_2''}t_{\overline{e}}t_{\overline{x_1}}t_{\overline{x_2}}t_{\overline{x_3}}t_{\overline{d}}t_{\overline{B_2}}t_{x_1}t_{x_2}t_{x_3}t_dt_{B_2}t_{e'}t_{x_1'}t_{x_2'}t_{x_3'}t_{d'}t_{B_2'} = 1. \tag{3.9}$$

Let us denote the positive factorization (3.9) by W_1 and the corresponding genus-4 Lefschetz fibration by (X_1, f_1). It admits 23 singular fibers with $(n, s) = (18, 5)$. Observe that the Lefschetz fibration (X_1, f_1) is nonhyperelliptic. Since it admits a section, the fundamental group $\pi_1(X_1)$ of the 4-manifold X_1 is isomorphic to the quotient of $\pi_1(\Sigma_4)$ by the normal subgroup generated by the vanishing cycles of the Lefschetz fibration (X_1, f_1).

We now show that the 4-manifold X_1 is simply connected.

Consider the generators a_i, b_i of $\pi(\Sigma_4)$ depicted in Figure 4 for $g = 4$. Thus, $\pi_1(X_1)$ has a presentation with generators $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ and with defining relations

$$b_4^{-1}b_3^{-1}b_2^{-1}b_1^{-1}(a_1b_1a_1^{-1})(a_2b_2a_2^{-1})(a_3b_3a_3^{-1})(a_4b_4a_4^{-1}) = 1. \tag{3.10}$$

$$x_i = x_i' = \overline{x_i} = d = d' = \overline{d} = B_2 = B_2' = \overline{B_2} = e' = \overline{e} = A_{i-1}'' = B_{i-1}'' = 1, \quad i = 1, 2, 3,$$

where the curves shown in Figure 6. One can get that $\pi_1(X_1)$ has the following relations (among many others):

$$\begin{aligned}
x_1 &= b_1b_2a_2^{-1}a_1b_2a_2^{-1}a_1 = 1, && (3.11)\\
x_1' &= b_2a_2^{-1}a_3b_2a_2^{-1}a_3b_3 = 1, && (3.12)\\
\overline{x_1} &= a_1^{-1}a_2b_2a_2^{-1}a_3b_3a_3^3a_1^{-1}a_2b_2a_2^{-1}a_3 = 1, && (3.13)\\
x_2 &= a_1^2b_1b_2^2a_2^{-1}a_1 = 1, && (3.14)\\
B_2 &= a_2^{-1}[a_1, b_1^{-1}]a_1^{-1} = 1, && (3.15)\\
B_2' &= a_3^{-1}a_2b_2^{-1}a_2^{-1}[a_1, b_1^{-1}]b_2a_2^{-1} = 1, && (3.16)\\
\overline{B_2} &= a_3^{-1}a_2b_2^{-1}a_2^{-1}[a_1, b_1^{-1}]b_2^2a_2^{-1} = 1, && (3.17)\\
B_0^{''} &= b_3b_4 = 1, && (3.18)\\
B_1^{''} &= a_4^{-1}b_4^{-1}b_3^{-1}a_3^{-1} = 1, && (3.19)\\
B_2^{''} &= b_4a_4^{-1}b_4^{-1}a_3^{-1} = 1. && (3.20)
\end{aligned}$$

The relations (3.16) and (3.17) imply that $b_2 = 1$. It follows from the relations (3.18) and (3.19) that we get $b_3 = b_4^{-1}$ and $a_3 = a_4^{-1}$, which also implies that $[a_4, b_4] = 1$ by (3.20). Using these relations, one can obtain $[a_1, b_1] = 1$ by the relation (3.10). Moreover, by considering the relations (3.15) and (3.16), we get $a_1 = a_2^{-1} = a_3 = a_4^{-1}$. Thus, the relations (3.11) and (3.12) imply that $b_1 = b_3 = b_4^{-1} = a_1^{-4}$.

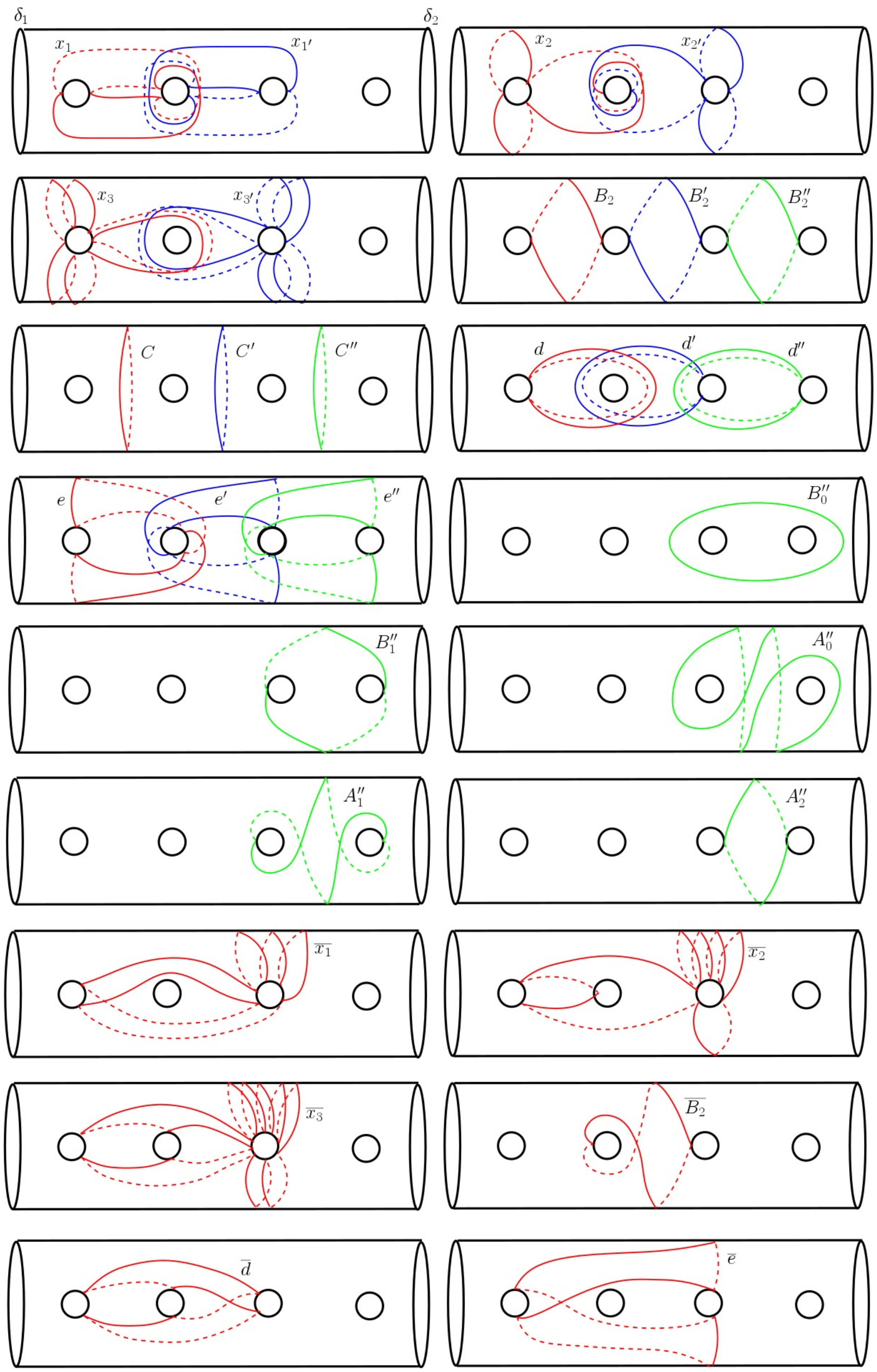

Figure 6. The curves on the surface Σ_4^2.

Hence, all relations show that $\pi_1(X_1)$ is generated by a single element, say a_1. However, the relation (3.13) becomes $b_3 = a_1^{-3}$. Therefore, we conclude that $a_1 = 1$ by the fact that $b_3 = a_1^{-4}$. Consequently, the fundamental group $\pi_1(X_1)$ is trivial.

Theorem 3.2 *The 4-manifold X_1 is an exotic copy of $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$.*

Proof The Euler characteristic $e(X_1)$ of X_1, is given by

$$\begin{aligned} e(X_1) &= 4-4g+n+s \\ &= 4-4(4)+18+5=11 \end{aligned}$$

for $(n,s)=(18,5)$. We compute the signature $\sigma(X_1)$ of X_1 using Endo and Nagami's method. We obtain the monodromy of X_1 by breeding the Matsumoto's relation with the monodromy W, and then by cancelling the Dehn twists $t^2_{C''}$ and $t_{C'}$. Theorem 2.1 implies that

$$\begin{aligned} \sigma(X_1) &= I_2(C''^2A''_0A''_1A''_2B''_0B''_1B''_2)+I_3(\overline{e}\,\overline{x_1}\,\overline{x_2}\,\overline{x_3}\,\overline{d}\,\overline{B_2}x_1x_2x_3dB_2e'x'_1x'_2x_3d'B'_2C')+I_2((C'')^{-2})+I_3((C')^{-1}) \\ &= I_2(C''^2A''_0A''_1A''_2B''_0B''_1B''_2)+I_3(\overline{e}\,\overline{x_1}\,\overline{x_2}\,\overline{x_3}\,\overline{d}\,\overline{B_2}x_1x_2x_3dB_2e'x'_1x'_2x_3d'B'_2C')-2I_2(C'')-I_3(C') \\ &= (-4)+(-6)-2(-1)-(-1)=-7, \end{aligned}$$

where the numbers $I_2(C''^2A''_0A''_1A''_2B''_0B''_1B''_2)=-4$ and $I_3(\overline{e}\,\overline{x_1}\,\overline{x_2}\,\overline{x_3}\,\overline{d}\,\overline{B_2}x_1x_2x_3dB_2e'x'_1x'_2x_3d'B'_2C')=-6$ coming from the genus-2 Matsumoto's relation and the monodromy W whose corresponding genus-3 Lefschetz fibration (X,f) has signature -6 proved by Baykur in [5], respectively. By the fact that X_1 is simply-connected, it can be concluded that

$$\begin{aligned} e(X_1) &= 2-2b_1(X_1)+b_2^+(X_1)+b_2^-(X_1) \\ &= 2+b_2^+(X_1)+b_2^-(X_1)=11 \text{ and} \\ \sigma(X_1) &= -7=b_2^+(X_1)-b_2^-(X_1), \end{aligned}$$

which give that $(b_2^+(X_1),b_2^-(X_1))=(1,8)$. By Freedman's classification, the 4-manifold X_1 is homeomorphic to the rational surface $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$. However, the 4-manifold $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$ does not admit a genus-4 Lefschetz fibration by Baykur's result [5, Lemma 2]. Hence, it cannot be diffeomorphic to X_1, which implies that X_1 is an exotic copy of $\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$. □

3.2. Constructing a small hyperelliptic genus-4 Lefschetz fibration on a simply-connected 4-manifold

Consider the generalized Matsumoto's Lefschetz fibration for $g=4$ with the monodromy factorization (2.4)

$$V_4=t^2_{C'}(t_{\alpha_0}t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_{\alpha_4})(t_{\beta_0}t_{\beta_1}t_{\beta_1}t_{\beta_3}t_{\beta_4})=t_{\delta_1}t_{\delta_2}, \tag{3.21}$$

where we denote the simple closed curves A_i, B_i, and C by α_i, β_i, and C', respectively, to distuguish them from some of which appearing before.

We then embed the relation (3.3), $T=t_{\delta_1}t_{\delta_2}$, in Mod_2^2 into Mod_4^2 in such a way that the boundary parallel curve δ_2 shown in Figure 1 is mapped to the curve C' shown in Figure 6. In this case, we get the relation $T=t_{\delta_1}t_{C'}$ in Mod_4^2. The conjugation of this relation by $\varphi=t_{b_2}$ gives the following relation:

$$\begin{aligned} T^{\varphi} &= t_{\varphi(e)}t_{\varphi(x_1)}t_{\varphi(x_2)}t_{\varphi(x_3)}t_{\varphi(d)}t_{\varphi(B_2)}t_{\varphi(C)}=t_{\varphi(\delta_1)}t_{\varphi(C')} \\ &= t_ft_{y_1}t_{y_2}t_{x_3}t_dt_{D_2}t_C=t_{\delta_1}t_{C'}, \end{aligned} \tag{3.22}$$

where all curves containing the relation (3.22) are depicted in Figures 6 and 7 (here we denote the curves $\varphi(e)$, $\varphi(x_1)$, $\varphi(x_2)$, and $\varphi(B_2)$ by f, y_1, y_2, and D_2, respectively).

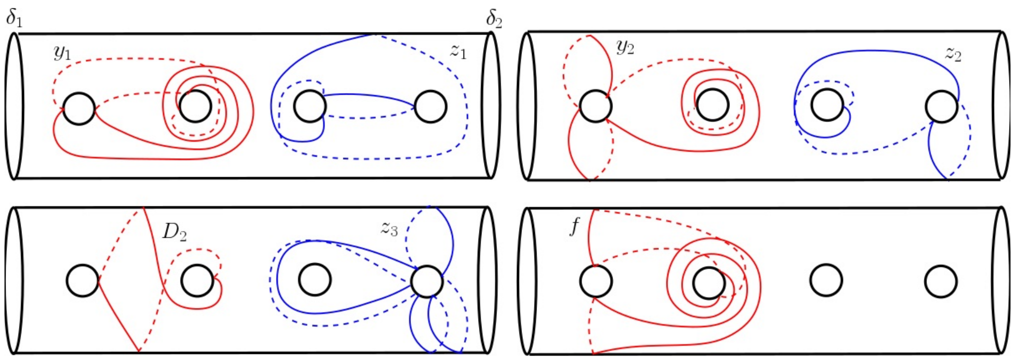

Figure 7. The curves on the surface Σ_4^2.

In a similar way, we consider the relation (3.2), $T' = t_{\delta_1} t_{\delta_2}$, in Mod_2^2, where the curves that appear in the factorization T' are shown in Figures 1 and 3. By conjugating this relation with the diffeomorphism $\psi = t_{a_2}^{-1}$, we get the relation

$$\begin{aligned} (T')^{\psi} &= t_{\psi(e')} t_{\psi(x_1')} t_{\psi(x_2')} t_{\psi(x_3')} t_{\varphi(d')} t_{\psi(B_2)} t_{\psi(C)} = t_{\psi(\delta_1)} t_{\psi(\delta_2)} \\ &= t_{e'} t_{\psi(x_1')} t_{\psi(x_2')} t_{\psi(x_3')} t_{d'} t_{B_2} t_C = t_{\delta_1} t_{\delta_2} \end{aligned}$$

in Mod_2^2. Now let us embed this relation into Mod_4^2 so that the curves δ_1, $e', \psi(x_i')$, d', B_2, and C are mapped to the curves C', e'', z_i, d'', B_2'', and C'', respectively, given in Figures 6 and 7. Thus, we get the following relation in Mod_4^2:

$$t_{e''} t_{z_1} t_{z_2} t_{z_3} t_{d''} t_{B_2''} t_{C''} = t_{C'} t_{\delta_2}. \tag{3.23}$$

The relations (3.21), (3.22), and (3.23) give rise to the following relation:

$$t_{C'}^2 (t_{\alpha_0} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4})(t_{\beta_0} t_{\beta_1} t_{\beta_1} t_{\beta_3} t_{\beta_4})(t_f t_{y_1} t_{y_2} t_{x_3} t_d t_{D_2} t_C)(t_{e''} t_{z_1} t_{z_2} t_{z_3} t_{d''} t_{B_2''} t_{C''}) = (t_{\delta_1} t_{\delta_2})(t_{\delta_1} t_{C'})(t_{C'} t_{\delta_2}),$$

which can be written as

$$t_{C'}^{-2} t_{C'}^2 (t_{\alpha_0} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4})(t_{\beta_0} t_{\beta_1} t_{\beta_1} t_{\beta_3} t_{\beta_4})(t_f t_{y_1} t_{y_2} t_{x_3} t_d t_{D_2} t_C)(t_{e''} t_{z_1} t_{z_2} t_{z_3} t_{d''} t_{B_2''} t_{C''}) = t_{\delta_1}^2 t_{\delta_2}^2$$

since $t_{C'}$, t_{δ_1}, and t_{δ_2} all commute with each other, we have the following relation in Mod_4^2:

$$(t_{\alpha_0} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4})(t_{\beta_0} t_{\beta_1} t_{\beta_1} t_{\beta_3} t_{\beta_4})(t_f t_{y_1} t_{y_2} t_{x_3} t_d t_{D_2} t_C)(t_{e''} t_{z_1} t_{z_2} t_{z_3} t_{d''} t_{B_2''} t_{C''}) = t_{\delta_1}^2 t_{\delta_2}^2.$$

By capping off both boundary components δ_1 and δ_2, we get the following factorization of identity in Mod_4:

$$(t_{\alpha_0} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4})(t_{\beta_0} t_{\beta_1} t_{\beta_1} t_{\beta_3} t_{\beta_4})(t_f t_{y_1} t_{y_2} t_{x_3} t_d t_{D_2} t_C)(t_{e''} t_{z_1} t_{z_2} t_{z_3} t_{d''} t_{B_2''} t_{C''}) = 1. \tag{3.24}$$

Let W_2 denote the positive factorization (3.24) and let (X_2, f_2) be the genus-4 Lefschetz fibration with the monodromy W_2. It admits 24 singular fibers with $(n, s) = (18, 6)$. Since its vanishing cycles are invariant under the hyperelliptic involution ι, the Lefschetz fibration (X_2, f_2) is hyperelliptic. Moreover, it admits two sections of (-2) self-intersection.

We now compute the fundamental group $\pi_1(X_2)$ of the 4-manifold X_2. Since the Lefschetz fibration (X_2, f_2) admits a section, its fundamental group $\pi_1(X_2)$ is isomorphic to the quotient of $\pi_1(\Sigma_4)$ by the normal subgroup generated by its vanishing cycles.

Consider the generators a_i, b_i of $\pi(\Sigma_4)$ shown in Figure 4. Thus, $\pi_1(X_2)$ has a presentation with generators $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ and with defining relation (3.10) and the relations

$$\alpha_i = \beta_i = z_j = y_1 = y_2 = x_3 = f = d = D_2 = C = e'' = d'' = B_2'' = C'' = 1, \quad i = 0, \dots, 4 \text{ and } j = 1, 2, 3,$$

where the curves shown in Figures 6 and 7. One can get that $\pi_1(X_2)$ has the following relations (among many others):

$$\begin{aligned}
\beta_0 &= b_1b_2b_3b_4 = 1, && (3.25)\\
\beta_1 &= a_1b_1b_2b_3b_4a_4 = 1, && (3.26)\\
\beta_2 &= a_1b_2b_3b_4a_4b_4^{-1} = 1, && (3.27)\\
\beta_3 &= a_2b_2b_3[b_4, a_4]a_3 = 1, && (3.28)\\
\beta_4 &= a_3^{-1}a_2b_2^{-1}a_2^{-1}[a_1, b_1^{-1}]b_2a_2^{-1} = 1, && (3.29)\\
y_1 &= b_1b_2^2a_2^{-1}a_1b_2^2a_2^{-1}a_1, && (3.30)\\
D_2 &= b_2a_2^{-1}[a_1, b_1^{-1}]a_1^{-1} = 1, && (3.31)\\
C &= [a_1, b_1] = 1, && (3.32)\\
z_1 &= b_3a_3^{-1}a_4b_3a_3^{-1}a_4b_4a_4^{-1} = 1, && (3.33)\\
C'' &= [a_4, b_4] = 1, && (3.34)
\end{aligned}$$

The relations (3.25) and (3.26) imply that $a_1a_4 = 1$. Thus, we get $b_2b_3 = 1$ using the relation (3.34) and (3.27). This gives the relations $b_1b_4 = 1$ and $a_2a_3 = 1$ by the relations (3.25) and (3.28), respectively. We then have $[a_2, b_2] = 1$ from (3.29) and (3.32). Since $a_2 = a_3^{-1}$ and $b_2 = b_3^{-1}$, we conclude that $[a_3, b_3] = 1$. By the relations (3.20) and (3.34), we get $a_3a_4 = 1$. Hence, we have $a_1 = a_2^{-1} = a_3 = a_4^{-1}$. This implies that $b_2 = b_3 = 1$ using the relations (3.31) and (3.32). From this, we obtain the relations $b_1 = a_1^{-4}$ and $b_1 = a_1^{-3}$ using the relations (3.30) and (3.33), respectively. This implies that $a_1 = 1$ and so $b_1 = b_4 = 1$. We, therefore, get $\pi_1(X_2) = 1$.

Theorem 3.3 *The 4-manifold X_2 is an exotic copy of $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$.*

Proof The Euler characteristic $e(X_2)$ of X_2 is given by

$$\begin{aligned}
e(X_2) &= 4 - 4g + n + s_1 + s_2\\
&= 4 - 4(4) + 18 + 6 + 0 = 12
\end{aligned}$$

for $(n, s_1, s_2) = (18, 6, 0)$, where $s = s_1 + s_2$. Since the Lefschetz fibration (X_2, f_2) is hyperelliptic, we compute the signature $\sigma(X_2)$ of X_2 using the signature formula given in Lemma 2.2. Thus, the signature $\sigma(X_2)$ is given by

$$\sigma(X_2) = \frac{1}{9}(-5n + 3s_1 + 7s_2) = -8.$$

It follows from $\pi_1(X_2) = 1$ that one can conclude that

$$\begin{aligned} e(X_2) &= 2 - 2b_1(X_2) + b_2^+(X_2) + b_2^-(X_2) \\ &= 2 + b_2^+(X_2) + b_2^-(X_2) = 12 \text{ and} \\ \sigma(X_2) &= b_2^+(X_2) - b_2^-(X_2) = -8, \end{aligned}$$

which imply that $(b_2^+(X_2), b_2^-(X_2)) = (1, 9)$. By Freedman's classification, the 4-manifold X_2 is homeomorphic to the 4-manifold $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. However, the rational surface $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ does not admit a genus-4 Lefschetz fibration by Baykur's result [5, Lemma 2]. Thus, it cannot be diffeomorphic to X_2. We, therefore, conclude that X_2 is an exotic $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. □

4. The minimal number of singular fibers in Lefschetz fibrations on simply-connected 4-manifolds

In this section, we examine the minimal number of singular fibers in Lefschetz fibrations on simply connected 4-manifolds. We remind that N_g (respectively M_g) denotes the minimal number of singular fibers in all genus-g (respectively hyperelliptic) Lefschetz fibratons on a simply connected 4-manifold. Also, let us recall that n and s denote the number of nonseparating and separating singular fibers in a genus-g Lefschetz fibration, respectively. In the following lemma, based on the Cadavid's signature inequality (2.2), we reach a lower bound for n.

Lemma 4.1 *Let (X, f) be a genus-g Lefschetz fibration over $\mathbb{S}^2$. Then $n \geq 4g - 2b_1(X) + b_2^+(X) - 1$. In particular, for a simply connected X, $n \geq 4g$.*

Proof For a 4-manifold X that admits a genus-g Lefschetz fibration on $\mathbb{S}^2$, we have the following inequality:

$$\begin{aligned} e(X) + \sigma(X) &\leq \big(4 - 4g + n + s\big) + \big(n - s - 2(2g - b_1(X))\big) \\ &= 4 - 8g + 2n + 2b_1(X). \end{aligned}$$

On the other hand, $e(X) + \sigma(X)$ can be written as

$$\begin{aligned} e(X) + \sigma(X) &= \big(2 - 2b_1(X) + b_2^+(X) + b_2^-(X)\big) + \big(b_2^+(X) - b_2^-(X)\big) \\ &= 2 - 2b_1(X) + 2b_2^+(X). \end{aligned}$$

Hence, we obtain the required inequality

$$n \geq 4g - 2b_1(X) + b_2^+(X) - 1.$$

Moreover, since X is a symplectic 4-manifold, it satisfies $b_2^+(X) > 0$. Then we get

$$n \geq 4g - 2b_1(X).$$

In particular, for a simply connected 4-manifold, one can conclude that $n \geq 4g$. □

Remark 4.2 *One can observe that any genus-* $g \geq 2$ *hyperelliptic Lefschetz fibration on a simply connected* 4*-manifold* X *must satisfy* $n+s > 4g$. *Otherwise, such a fibration admits* $n+s = 4g$ *singular fibers (in this case* $s=0$ *by Lemma 4.1), then* $(n,s) = (4g, 0)$ *does not satisfy the equation in Lemma 2.4, which leads to a contradiction. Hence, we conclude that* $M_g \geq 4g+1$.

Remark 4.3 *If there exists a genus-*$g \geq 2$ *Lefschetz fibration on a simply connected* 4*-manifold* X *with* $n = 4g$, *then it follows from the proof of Lemma 4.1 that* $\chi_h(X) = 1$. *Therefore,* X *satisfies* $(b_2^+(X), b_2^-(X)) = (1, 1+s)$ *and* $(e(X), \sigma(X)) = (4+s, -s)$. *Now if* $\sigma(X) = -s \not\equiv 0 \pmod{16}$, *then* X *is spin by Rokhlin's theorem, and in turn, homeomorphic to* $\mathbb{C}P^2 \# (1+s)\overline{\mathbb{C}P^2}$ *by Freedman. Also, if* $0 < s \leq 8$, *then the* 4*-manifold* X *is an exotic copy of* $\mathbb{C}P^2 \# (1+s)\overline{\mathbb{C}P^2}$ *by [5, Lemma 2].*

It has been known that the minimal number of singular fibers in all torus Lefschetz fibrations is 12. One can conclude that $N_1 = M_1 = 12$ by the existence of torus Lefschetz fibrations with 12 singular fibers on the elliptic surface $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$. Baykur and Korkmaz [6] constructed a genus-2 Lefschetz fibration of type of $(8,6)$, so that its total space is an exotic copy of $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$. By [6, Theorem 2] one can conclude that it is the smallest simply connected 4-manifold which admits a genus-2 Lefschetz fibration. Thus, it follows immediately that $M_2 = N_2 = 14$. In Proposition 4.4 (a), we obtain the same result by using slightly different arguments than those used in the proof of [6, Theorem 2].

In the case $g=3$, Baykur constructed a genus-3 hyperelliptic Lefschetz fibration on a 4-manifold which is an exotic $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$ with $(n,s) = (12,6)$ [5]. Therefore, using also Lemma 4.1 and Remark 4.2, we get $12 \leq N_3 \leq 18$ and $13 \leq M_3 \leq 18$. In Proposition 4.4 (b), we find the exact value of M_3.

For $g=4$, we have constructed the nonhyperelliptic Lefschetz fibration (X_1, f_1) of genus-4 with 23 singular fibers whose total space is an exotic copy of $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$ (see Theorem 3.2). We thus get $16 \leq N_4 \leq 23$. Similarly, the genus-4 hyperelliptic Lefschetz fibration (X_2, f_2) with 24 singular fibers whose the total space is an exotic $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$ (see Theorem 3.3) gives an upper bound for the number M_4. Therefore, using the lower bound mentioned in Remark 4.2, we get $17 \leq M_4 \leq 24$. The following proposition gives a better lower bound for the number M_4.

Proposition 4.4 *For the numbers* M_2, M_3, *and* M_4 *the following holds.*

(a) $N_2 = M_2 = 14$,

(b) $M_3 = 18$,

(c) $21 \leq M_4 \leq 24$.

Proof We first prove part (a). Suppose that there exists a genus-2 Lefschetz fibration on a simply connected 4-manifold X with $n+s < 14$. Note that the Lefschetz fibration may have only type-1 separating vanishing cycles, $s = s_1$. By lemmata 2.2 and 2.4 together with the inequality (2.2), we get:

$$e(X) = n+s-4, \quad \sigma(X) = -\frac{1}{5}(3n+s) \leq n-s-8 \text{ and } n+2s \equiv 0 \pmod{10}.$$

We have also $n \geq 8$ by Lemma 4.1. Thus, the possible values of (n,s) are $(8,1)$ and $(10,0)$. On the other hand, by the proof of Lemma 4.1, the 4-manifold X must satisfy $\chi_h(X) \geq 1$. However, both values $(8,1)$ and $(10,0)$ of (n,s) have $\chi_h(X) = 0$, which leads to a contradiction. Hence, we conclude that $N_2 = M_2 = 14$.

For the proof of (b), suppose that there exists a genus-3 hyperelliptic Lefschetz fibration on a simply-connected 4-manifold X with $n+s < 18$. The 4-manifold X may admit only separating vanishing cycles of type-1, so $s = s_1$. It follows from Lemmata 2.4 and 2.2 and the inequality (2.2) that we have:

$$e(X) = n + s - 8, \quad \sigma(X) = \frac{1}{7}(4n - s) \leq n - s - 12 \text{ and } n + 12s \equiv 0 \pmod{28}.$$

Also, Lemma 2.4 implies that $n \geq 12$. Thus, we have only one possible value $(16,1)$ of (n,s). On the other hand, by the proof of Lemma 4.1, the 4-manifold X must have $\chi_h(X) \geq 1$. However, the pair $(16,1)$ satisfies $\chi_h(X) = 0$, which gives a contradiction. Hence, $M_3 = 18$.

The proof of (c) follows slightly along the same lines. Suppose that there exists a genus-4 hyperelliptic Lefschetz fibration on a simply connected 4-manifold Y with $n+s < 24$, where $s = s_1 + s_2$. By Lemmata 2.4 and 2.2, we have the following:

$$e(Y) = n + s_1 + s_2 - 12, \quad \sigma(Y) = \frac{1}{9}(-5n + 3s_1 + 7s_2) \text{ and } n + 12s_1 + 4s_2 \equiv 0 \pmod{18}.$$

It follows from Lemma 2.4 that $n \geq 16$. From these, we obtain the possible six values $(16,1,2)$, $(16,0,5)$, $(16,4,2)$, $(18,3,0)$, $(18,2,3)$ and $(20,1,1)$ of (n,s_1,s_2). The decompositions $(16,1,2)$, $(18,3,0)$, and $(20,1,1)$ satisfy $\chi_h(Y) = 0$, a contradiction. By considering the possible values $(16,0,5)$, $(16,4,2)$, and $(18,2,3)$ of (n,s_1,s_2), it can be concluded that $21 \leq M_4 \leq 24$. □

As we have already proved above, $M_2 = 14$ and $M_3 = 18$. Moreover, the total spaces of corresponding hyperelliptic Lefschetz fibrations are exotic copies of the rational surface $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$. Since we know that $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$ does not admit a genus-g Lefschetz fibration for $g \geq 2$ [5, Lemma 2], the following question appears naturally:

question 4.5 *Does there exist a genus-g Lefschetz fibration whose total space is an exotic copy of $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$ for $g \geq 4$?*

If there exists such a fibration, then it admits $4g+6$ singular fibers with $(n,s) = (4g,6)$, which implies that $4g \leq N_g \leq 4g+6$. Moreover, if it is also hyperelliptic, one can conclude that $4g+1 \leq M_g \leq 4g+6$. Since $M_g = 4g+6$ holds for $g = 2$ and 3, it is also natural to ask the following question:

question 4.6 *Is it true that $M_g = 4g+6$ for $g \geq 4$?*

Appendix

In this appendix, we give a proof of Lemma 3.1 using Alexander's method.

Proof of Lemma 3.1.

Since the collection of simple closed curves and simple proper arcs $\{\gamma_i\}$ fills the genus-2 surface Σ_2^2 as shown in Figure 3, we will prove the relation (3.1) by showing that the oriented three curves γ_2, γ_3, γ_4 and the two arcs γ_1 and γ_5 are fixed (up to isotopy) under the map

$$t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{e'}t_{x'_1}t_{x'_2}t_{x'_3}t_{d'}t_{B_2}t_C.$$

Indeed, Figures 8 and 9 show that the collection $\{\gamma_i\}$ are fixed and also their given orientations are preserved under this map. This finishes the proof.

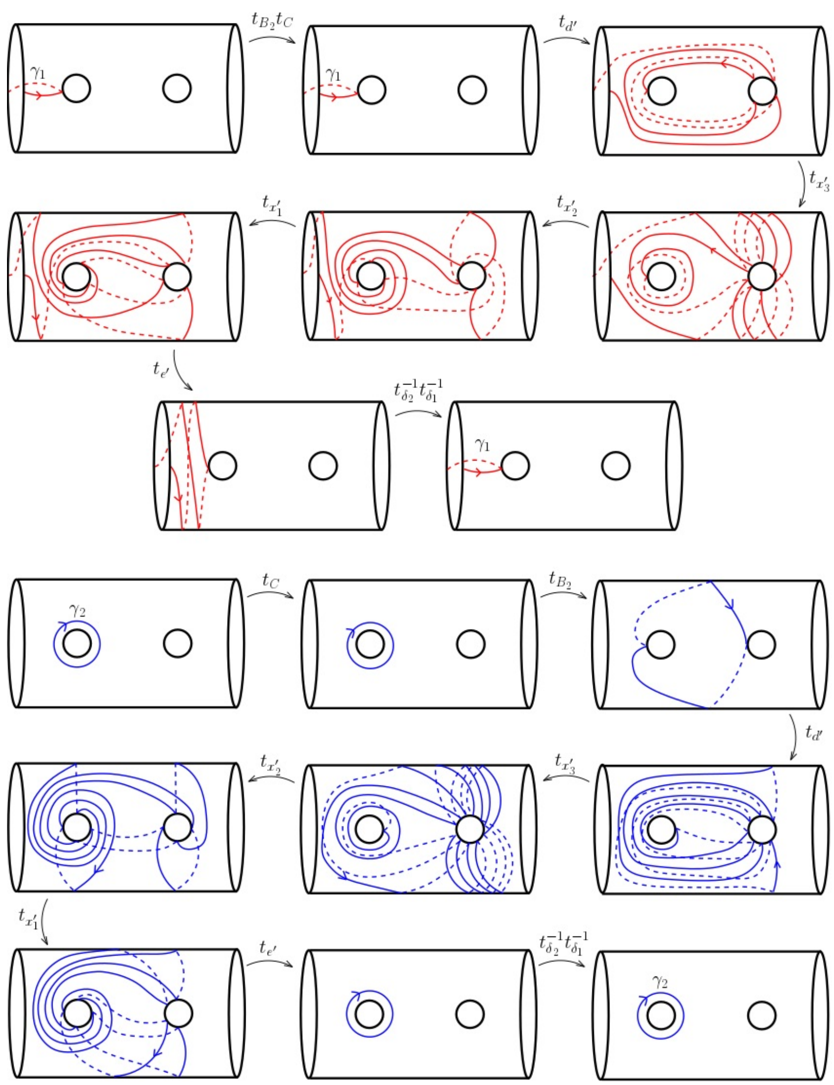

Figure 8. $t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{e'}t_{x'_1}t_{x'_2}t_{x'_3}t_{d'}t_{B_2}t_C(\gamma_i) = \gamma_i$, for $i = 1, 2$.

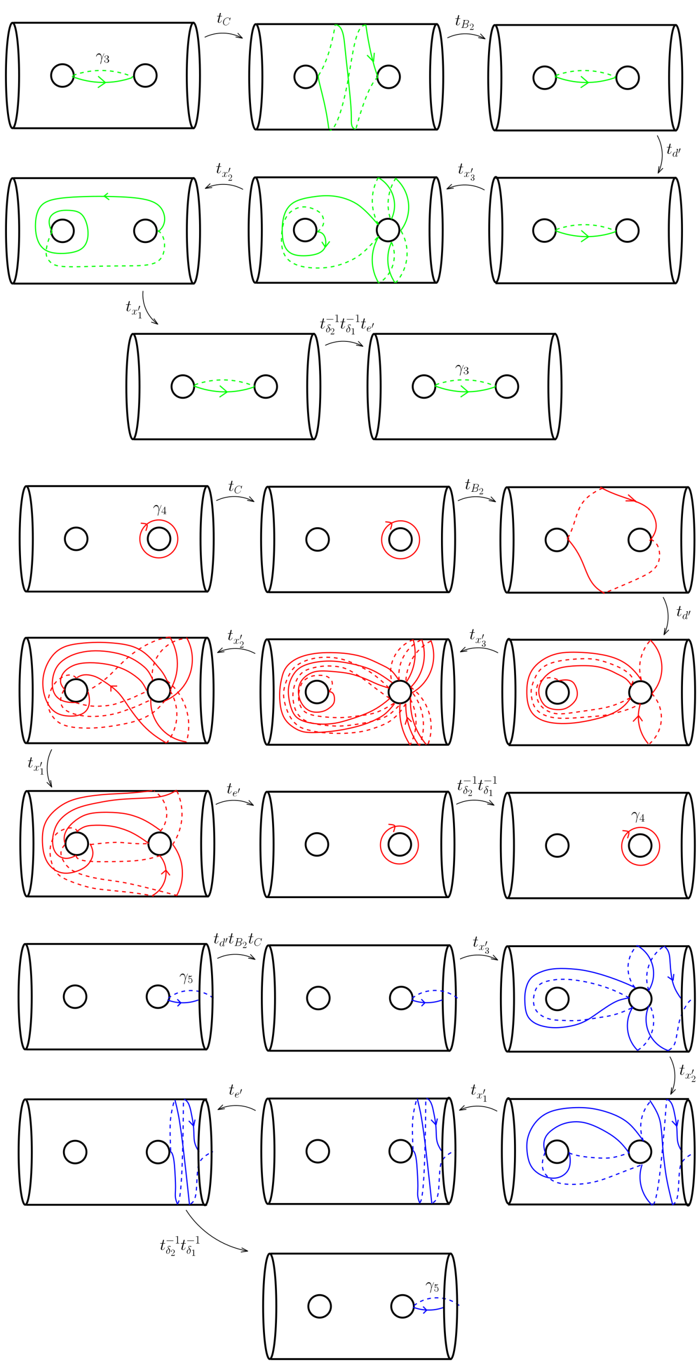

Figure 9. $t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{e'}t_{x'_1}t_{x'_2}t_{x'_3}t_{d'}t_{B_2}t_C(\gamma_i) = \gamma_i$, for $i = 3, 4, 5$.

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