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Gradient estimates of a nonlinear elliptic equation for the V-Laplacian on noncompact Riemannian manifolds

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Abstract: In this paper, we consider gradient estimates for positive solutions to the following equation

$$\triangle_V u + au^p \log u = 0$$

on complete noncompact Riemannian manifold with k-dimensional Bakry-Émery Ricci curvature bounded from below. Using the Bochner formula and the Cauchy inequality, we obtain upper bounds of $|\nabla u|$ with respect to the lower bound of the Bakry-Émery Ricci curvature.

Key words: Gradient estimates, V-Laplacian, Riemannian manifolds, Bakry-Émery Ricci curvature

1. Introduction

It is an interesting problem to consider gradient estimates for equations on Riemannian manifolds. Li and Yau [8] derived some parabolic gradient estimates for positive solutions to the following equation

$$u_t = \triangle u \tag{1.1}$$

on Riemannian manifold with Ricci curvature bounded from below. Hamilton [5] obtained some elliptic type gradient estimates for positive solutions to (1.1). Nowadays, gradient estimate of Li-Yau type or of Hamilton type was extended to other equations on Riemannian manifolds with various curvature conditions. Ma [9] got some gradient estimates for positive solutions to the following equation

$$\triangle u + au \log u = 0 \tag{1.2}$$

on complete noncompact Riemannian manifold with Ricci curvature bounded below. Dung [3] got a sharp gradient estimates for the following equation

$$u_t = \triangle u + au \log u. \tag{1.3}$$

Yang [12], Guo-Ishida [4], and Chen [2] discussed gradient estimates for positive solutions to the following equation

$$u_t = \triangle u + au \log u + bu. \tag{1.4}$$

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Wang [10] studied gradient estimates for the following equation

$$u_t(x,t) = \triangle u(x,t) + au(x,t)\log u(x,t) + b(x,t)u(x,t). \tag{1.5}$$

Huang-Ma [7] and Wu [11] studied gradient estimates for the following equation

$$u_t = \triangle_f u + au\log u + bu \tag{1.6}$$

on complete noncompact Riemannian manifold with k-dimensional Bakry-Émery Ricci curvature bounded from below. Chen and Qiu [1] obtained some gradient estimates for positive solutions to the following equation

$$u_t = \triangle_V u + au\log u. \tag{1.7}$$

Zhao [13] discussed gradient estimates for positive solutions to the following equation

$$\triangle_V(u^p) + bu = 0. \tag{1.8}$$

Huang and Li [6] studied gradient estimates for positive solutions to the following equation

$$\triangle_V u + au\log u = 0. \tag{1.9}$$

Motivated by [6, 9, 11, 13], we study gradient estimates for positive solutions to the following equation

$$\triangle_V u + au^p\log u = 0 \tag{1.10}$$

on complete noncompact Riemannian manifold with k-dimensional Bakry-Émery Ricci curvature bounded from below.

2. Preliminaries and notations

Let (M^n, g) be a complete n-dimensional Riemannian manifold. The V-Laplacian $\triangle_V$ is defined by

$$\triangle_V = \triangle + \langle V, \nabla\cdot\rangle,$$

where V is a smooth vector field on (M^n, g), $\triangle$ is the Laplacian operator, ∇ is the gradient operator. If $V = \nabla f$, then $\triangle_V$ is called the f-Laplacian. Therefore the V-Laplacian $\triangle_V$ could be viewed as a generalization of the f-Laplacian $\triangle_f$. For $k \geq n$, we can define the Bakry-Émery Ricci curvature as follows:

$$\mathrm{Ric}_V = \mathrm{Ric} - \frac{1}{2}\mathfrak{L}_V g,$$

$$\mathrm{Ric}_V^k = \mathrm{Ric}_V - \frac{1}{k-n}V^* \otimes V^*, \tag{2.1}$$

where Ric is the Ricci curvature of (M^n, g), $\mathfrak{L}$ denotes the Lie derivative and V^* is the dual 1-form of V. Throughout this paper, we denote by $B_q(R)$ the geodesic ball centered at q with radius R and use the convention that $k = n$ if and only if $V \equiv 0$. Usually, Ric_V^k is called the k-dimensional Bakry-Émery Ricci curvature and Ric_V can seen the ∞-dimensional Bakry- mery Ricci curvature. For Ric_V^k, there is an important formula which can be read as

$$\frac{1}{2}\triangle_V|\nabla u|^2 = |\mathrm{Hess}u|^2 + \langle\nabla u, \nabla\triangle_V u\rangle + \mathrm{Ric}_V(\nabla u, \nabla u), \tag{2.2}$$

where $\mathrm{Hess}u$ is the Hessian of u. Formula (2.1) is usually called the Bochner formula for Ric_V^k.

3. Main results and their proof

Theorem 3.1 *Let* (M^n, g) *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}_V^k(B_q(2R)) \geq -A$*, where* $A \geq 0$ *is a constant. Suppose that* u *is a positive solution to (1.10) on* $B_q(2R)$*,* $p \neq \frac{k+2}{k}$*,* $\epsilon = 1 - \frac{kp}{k+2}$ *and* $h = u^\varepsilon$*. Then on* $B_q(2R)$*, the following inequality holds:*

$$\frac{1}{2}\triangle_V|\nabla h|^2 \geq \frac{kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right)\frac{|\nabla h|^4}{h^2} + \frac{kp}{kp-k-2}\frac{\langle\nabla h, \nabla|\nabla h|^2\rangle}{h}$$

$$-ah^{\frac{(p-1)(2+k)}{2+k-kp}}|\nabla h|^2 - A|\nabla h|^2. \tag{3.1}$$

Proof Direct calculation shows that

$$\triangle_V h = \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} + \varepsilon u^{\varepsilon-1}\triangle_V u = \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} - ah^{\frac{\varepsilon+p-1}{\varepsilon}}\log h. \tag{3.2}$$

Therefore

$$\langle\nabla h, \nabla\triangle_V h\rangle = \frac{\varepsilon-1}{\varepsilon}\langle\nabla h, \nabla\frac{|\nabla h|^2}{h}\rangle - a\langle\nabla h, \nabla(h^{\frac{\varepsilon+p-1}{\varepsilon}}\log h)\rangle$$

$$= \frac{\varepsilon-1}{\varepsilon h}\langle\nabla h, \nabla|\nabla h|^2\rangle - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2} - \frac{a(\varepsilon+p-1)}{\varepsilon}h^{\frac{p-1}{\varepsilon}}|\nabla h|^2\log h - ah^{\frac{p-1}{\varepsilon}}|\nabla h|^2. \tag{3.3}$$

By (2.1), (2.2), (3.2) and (3.3), we have

$$\frac{1}{2}\triangle_V|\nabla h|^2 \geq \frac{1}{k}(\triangle_V h)^2 + \langle\nabla h, \nabla\triangle_V h\rangle + \mathrm{Ric}_V^k(\nabla h, \nabla h)$$

$$= \frac{1}{k}\left[\frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^2}{h} - ah^{\frac{\varepsilon+p-1}{\varepsilon}}\log h\right]^2 + \frac{\varepsilon-1}{\varepsilon h}\langle\nabla h, \nabla|\nabla h|^2\rangle - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h^2}$$

$$-\frac{a(\varepsilon+p-1)}{\varepsilon}h^{\frac{p-1}{\varepsilon}}|\nabla h|^2\log h - ah^{\frac{p-1}{\varepsilon}}|\nabla h|^2 + \mathrm{Ric}_V^k(\nabla h, \nabla h)$$

$$= \left[\frac{(\varepsilon-1)^2}{k\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon}\right]\frac{|\nabla h|^4}{h^2} - \left[\frac{2(\varepsilon-1)}{k\varepsilon} + \frac{\varepsilon+p-1}{\varepsilon}\right]ah^{\frac{p-1}{\varepsilon}}|\nabla h|^2\log h$$

$$+\frac{a^2}{k}(h^{\frac{\varepsilon+p-1}{\varepsilon}}\log h)^2 + \frac{\varepsilon-1}{\varepsilon h}\langle\nabla h, \nabla|\nabla h|^2\rangle - ah^{\frac{p-1}{\varepsilon}}|\nabla h|^2 + \mathrm{Ric}_V^k(\nabla h, \nabla h). \tag{3.4}$$

Since $\epsilon = 1 - \frac{kp}{k+2}$, we have $\frac{\varepsilon-1}{\varepsilon} = \frac{kp}{kp-k-2}$ and

$$\frac{2(\varepsilon-1)}{k\varepsilon} + \frac{\varepsilon+p-1}{\varepsilon} = 0, \quad \frac{(\varepsilon-1)^2}{k\varepsilon^2} - \frac{\varepsilon-1}{\varepsilon} = \frac{kp}{kp-k-2}\left(\frac{p}{kp-k-2}-1\right).$$

Therefore, by (3.4) and the condition $\mathrm{Ric}_V^k(B_q(2R)) \geq -A$, we conclude that (3.1) is true. The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_V^k(B_q(2R)) \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10) on $B_q(2R)$, $1 \leq p < \frac{k+2}{k-1}$ and $p \neq \frac{k+2}{k}$, then the following inequalities hold on $B_q(R)$:*
(1) If $a > 0$, then

$$|\nabla u| \leq M\sqrt{\beta(M^{p-1}a + A) + \frac{C}{R^2}(1 + \sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R))}; \tag{3.5}$$

(2) If $a < 0$, then

$$|\nabla u| \leq M\sqrt{\beta\max\{0, m^{p-1}a + A\} + \frac{C}{R^2}(1 + \sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R))}, \tag{3.6}$$

where $M = \sup_{x\in B_q(2R)} u(x)$, $m = \inf_{x\in B_q(2R)} u(x)$, $\beta = \frac{2(k+2)^2}{kp(p+k+2-kp)}$ and C is a constant depending only on k and p.

Proof As in [1] and [6], we define a cut-off function $\psi \in C^2[0, +\infty)$ by

$$\psi(t) = \begin{cases} 1, & t \in [0, 1] \\ 0, & t \in [2, +\infty) \end{cases}$$

satisfying $\psi(t) \in [0, 1]$, and

$$\psi'(t) \leq 0, \quad \psi''(t) \geq -C_1, \quad \frac{|\psi'(t)|^2}{\psi(t)} \leq C_1.$$

Let

$$\varphi = \psi(\frac{r(x)}{R}), \quad G = \varphi|\nabla h|^2.$$

According to [1] and [6], we have

$$\frac{|\nabla\varphi|^2}{\varphi} \leq \frac{C_1}{R^2} \tag{3.7}$$

and

$$-\triangle_V\varphi \leq \frac{\sqrt{C_1(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R)}{R^2} + \frac{C_1}{R^2}. \tag{3.8}$$

Assume that G achieves its maximum at the point $x_0 \in B_q(2R)$ and assume that $G(x_0) > 0$ (otherwise the proof is trivial). Then at the point x_0, it holds that

$$\nabla G = 0, \quad \triangle_V G \leq 0, \quad \nabla|\nabla h|^2 = -\frac{|\nabla h|^2}{\varphi}\nabla\varphi.$$

Direct calculation shows that

$$\triangle_V G = \varphi\triangle_V|\nabla h|^2 + \frac{\triangle_V\varphi}{\varphi}G - \frac{2|\nabla\varphi|^2}{\varphi^2}G$$

$$\geq 2\varphi[\frac{kp}{kp-k-2}(\frac{p}{kp-k-2}-1)\frac{|\nabla h|^4}{h^2}+\frac{kp}{kp-k-2}\frac{\langle\nabla h,\nabla|\nabla h|^2\rangle}{h}$$

$$-ah^{\frac{(p-1)(2+k)}{2+k-kp}}|\nabla h|^2-A|\nabla h|^2]+\frac{\triangle_V\varphi}{\varphi}G-\frac{2|\nabla\varphi|^2}{\varphi^2}G$$

$$=\frac{2kp}{kp-k-2}(\frac{p}{kp-k-2}-1)\frac{G^2}{h^2\varphi}-\frac{2kp}{kp-k-2}\frac{G}{\varphi}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}$$

$$-2ah^{\frac{(p-1)(2+k)}{2+k-kp}}G-2AG+\frac{\triangle_V\varphi}{\varphi}G-\frac{2|\nabla\varphi|^2}{\varphi^2}G.$$

Therefore at the point x_0, it holds that

$$\frac{2kp}{kp-k-2}(\frac{p}{kp-k-2}-1)\frac{G}{h^2}\leq\frac{2kp}{kp-k-2}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}+2ah^{\frac{(p-1)(2+k)}{2+k-kp}}\varphi+2A\varphi-\triangle_V\varphi+\frac{2|\nabla\varphi|^2}{\varphi}. \tag{3.9}$$

If $1\leq p<\frac{k+2}{k}$, then $\frac{2kp}{kp-k-2}<0$. Therefore, by the Cauchy inequality we have

$$\frac{2kp}{kp-k-2}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}\leq-\frac{2kp}{kp-k-2}[\frac{kp-k-2}{2(kp-k-2-p)}\frac{|\nabla\varphi|^2}{\varphi}+\frac{kp-k-2-p}{2(kp-k-2)}\frac{G}{h^2}]. \tag{3.10}$$

According to (3.9) and (3.10), we get

$$\frac{kp}{kp-k-2}(\frac{p}{kp-k-2}-1)\frac{G}{h^2}\leq\frac{-kp}{kp-k-2-p}\frac{|\nabla\varphi|^2}{\varphi}+2au^{p-1}\varphi+2A\varphi-\triangle_V\varphi+\frac{2|\nabla\varphi|^2}{\varphi}. \tag{3.11}$$

If $\frac{k+2}{k}<p<\frac{k+2}{k-1}$, then $\frac{p}{kp-k-2}>1$, $p+k+2-kp>0$ and

$$\frac{2kp}{kp-k-2}\frac{\langle\nabla h,\nabla\varphi\rangle}{h}\leq\frac{2kp}{kp-k-2}[\frac{kp-k-2}{2(p+k+2-kp)}\frac{|\nabla\varphi|^2}{\varphi}+\frac{p+k+2-kp}{2(kp-k-2)}\frac{G}{h^2}]. \tag{3.12}$$

According to (3.9) and (3.12), we get

$$\frac{kp}{kp-k-2}(\frac{p}{kp-k-2}-1)\frac{G}{h^2}\leq\frac{kp}{p+k+2-kp}\frac{|\nabla\varphi|^2}{\varphi}+2au^{p-1}\varphi+2A\varphi-\triangle_V\varphi+\frac{2|\nabla\varphi|^2}{\varphi}. \tag{3.13}$$

If $a>0$, by (3.7), (3.8), (3.11) and (3.13) we conclude that there exists a constant C depending only on k and p such that

$$\frac{kp}{kp-k-2}(\frac{p}{kp-k-2}-1)G(x)\leq h^2(x_0)[2M^{p-1}a+2A+\frac{C}{R^2}(1+\sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R))] \tag{3.14}$$

holds for $x\in B_p(2R)$.

If $x\in B_p(R)$, then $\varphi=1$ and $G(x)=|\nabla h|^2(x)=\varepsilon^2u^{2\varepsilon-2}(x)|\nabla u|^2(x)$. Therefore, by (3.14) we get

$$|\nabla u|^2(x)\leq\frac{(k+2)^2}{kp(p+k+2-kp)}u^{2-2\varepsilon}(x)u^{2\varepsilon}(x_0)H\leq\frac{(k+2)^2}{kp(p+k+2-kp)}M^2H, \tag{3.15}$$

where

$$H = 2M^{p-1}a + 2A + \frac{C}{R^2}(1 + \sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R)). \tag{3.16}$$

By (3.15) and (3.16), we conclude that (3.5) is true. On the other hand, if $a < 0$ then $2am^{p-1}\varphi + 2A\varphi \leq 2\max\{0, am^{p-1} + A\}$. Similar to the case of $a > 0$, we can get (3.11) and (3.13). Therefore, we conclude that (3.6) is true. The proof of Theorem 3.2 is complete. □

Theorem 3.3 *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_V^k(B_q(2R)) \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10) on $B_q(2R)$, then the following inequalities hold on $B_q(R)$:*
(1) If $a > 0$, $0 < p \leq 1$ and $\inf_{x \in B_q(2R)} u(x) \neq 0$, then

$$|\nabla u| \leq M\sqrt{\beta(am^{p-1} + A) + \frac{C}{R^2}(1 + \sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R))}; \tag{3.17}$$

(2)If $a < 0$ and $0 < p \leq 1$, then

$$|\nabla u| \leq M\sqrt{\beta\max\{0, aM^{p-1} + A\} + \frac{C}{R^2}(1 + \sqrt{(k-1)A}\coth(\sqrt{\frac{A}{k-1}}R))}, \tag{3.18}$$

where $M = \sup_{x \in B_q(2R)} u(x)$, $m = \inf_{x \in B_q(2R)} u(x)$, $\beta = \frac{2(k+2)^2}{kp(p+k+2-kp)}$ and C is a constant depending only on k and p.

Proof As in the proof of Theorem 3.2, we can arrive at (3.9). If $0 < p \leq 1$, then $\frac{2kp}{kp-k-2} < 0$. Therefore, by the Cauchy inequality we can get (3.11). If $a > 0$, $0 < p \leq 1$ and $\inf_{x \in B_q(2R)} u(x) \neq 0$, then

$$ah^{\frac{(p-1)(2+k)}{2+k-kp}}\varphi = au^{p-1}\varphi \leq am^{p-1}. \tag{3.19}$$

According to (3.11), (3.19) and the methods in the proof of Theorem 3.2, we conclude that (3.17) is true. If $a < 0$ and $0 < p \leq 1$, then

$$ah^{\frac{(p-1)(2+k)}{2+k-kp}}\varphi + A\varphi = au^{p-1}\varphi + A\varphi \leq \max\{0, aM^{p-1} + A\}. \tag{3.20}$$

By (3.11), (3.20) and the methods in the proof of Theorem 3.2, we conclude that (3.18) is true. The proof of Theorem 3.3 is complete. □

Letting $R \to +\infty$, we obtain the following global estimates on complete noncompact Riemannian manifolds:

Theorem 3.4 *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_V^k \geq -A$, where $A \geq 0$ is a constant. Suppose that u is a positive solution to (1.10), $1 \leq p < \frac{k+2}{k-1}$ and $p \neq \frac{k+2}{k}$, then the following inequalities hold*
(1) If $a > 0$, then

$$|\nabla u| \leq M\sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)}(M^{p-1}a + A)};$$

(2) If $a<0$ *, then*

$$|\nabla u| \leq M\sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)}\max\{0, m^{p-1}a+A\}},$$

where $M=\sup_{x\in M^n} u(x)$ *,* $m=\inf_{x\in M^n} u(x)$ *.*

Theorem 3.5 *Let* (M^n,g) *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}_V^k \geq -A$ *, where* $A\geq 0$ *is a constant. Suppose that* u *is a positive solution to (1.10), then the following inequalities hold*
(1) If $a>0$ *,* $0<p\leq 1$ *and* $\inf_{x\in M^n} u(x)\neq 0$ *, then*

$$|\nabla u| \leq M\sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)}(am^{p-1}+A)};$$

(2) If $a<0$ *and* $0<p\leq 1$ *, then*

$$|\nabla u| \leq M\sqrt{\frac{2(k+2)^2}{kp(p+k+2-kp)}\max\{0, aM^{p-1}+A\}},$$

where $M=\sup_{x\in M^n} u(x)$ *,* $m=\inf_{x\in M^n} u(x)$ *.*

In particular, we have

Theorem 3.6 *Let* (M^n,g) *be an* n*-dimensional complete Riemannian manifold with* $\mathrm{Ric}_V^k \geq -A$ *, where* $A\geq 0$ *is a constant. Suppose that* u *is a bounded nonconstant positive solution to (1.10),* $M=\sup_{x\in M^n} u(x)$ *and* $m=\inf_{x\in M^n} u(x)$ *. If* $a<0$ *and* $0<p\leq 1$ *, then* $M^{1-p} > -\frac{a}{A}$ *. If* $a<0$ *,* $1\leq p<\frac{k+2}{k-1}$ *and* $p\neq \frac{k+2}{k}$ *, then* $m^{p-1} < -\frac{A}{a}$ *.*

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