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On the convergence of the Abel–Poisson means of multiple Fourier series

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Abstract: Let $A_\varepsilon(x,f)$ be the Abel–Poisson means of an integrable function $f(x)$ on n–dimensional torus $\mathbf{T}^n$, $-\pi < x_i \leq \pi$, $i = 1,\dots,n$ $(n \geq 2)$ in the Euclidean n–space. The famous Bochner's theorem asserts that for any function $f \in L^1(\mathbf{T}^n)$ the Abel–Poisson means $A_\varepsilon(x,f)$ are pointwise converge to $f(x)$ a.e., that is,

$$\lim_{\varepsilon\to 0^+} A_\varepsilon(x,f) = f(x), \quad a.e.\ x \in \mathbf{T}^n.$$

In this paper we investigate the rate of convergence of Abel–Poisson means at the so-called μ–smoothness point of f .

Key words: Abel–Poisson means, multiple Fourier series, Poisson summation formula

1. Introduction and formulation of main results

Let $\mathbb{R}^n$ denote the $n-$ dimensional Euclidean space,

$$\mathbf{T}^n = \{x \in \mathbb{R}^n \mid -\pi < x_i \leq \pi, \ \ i = 1,\dots n\}$$

be n–dimensional torus, and $\mathbb{Z}^n$ be the integral lattice of $\mathbb{R}^n$. For any $f(x) \in L^1(\mathbf{T}^n)$, we form the Fourier series of its periodical continuation by

$$f(x) \sim \sum a_m e^{im\cdot x}, \tag{1.1}$$

with Fourier coefficient $a_m = \int_{\mathbf{T}^n} f(x)\, e^{-im\cdot x} dx$, where $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$, and $m\cdot x = m_1x_1 + \cdots + m_nx_n$. For a function $f \in L^1(\mathbf{T}^n)$, define the Abel–Poisson means of $f(x)$ by

$$A_\varepsilon(x,f) = \sum_{m\in\mathbf{Z}^n} e^{-\varepsilon|x|} a_m e^{im\cdot x}. \tag{1.2}$$

It is well known that the Fourier transform of the function $e^{-\varepsilon|x|}$ has the following form

$$\mathrm{P}_\varepsilon(y) \equiv \mathrm{P}(y;\varepsilon) = (e^{-\varepsilon|\cdot|})^\wedge(y) = \frac{c_n\varepsilon}{(\varepsilon^2+|y|^2)^{(n+1)/2}}\ ,\ c_n = \pi^{-(n+1)/2}\Gamma(\frac{n+1}{2}). \tag{1.3}$$

The function $\mathrm{P}_\varepsilon(y)$ is called the Poisson kernel and has the following properties [12, p. 253].

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Proposition 1.1

$$\begin{array}{ll} (a) & \int_{\mathbb{R}^n} \mathrm{P}_\varepsilon(y)dy = 1, \ \textit{for all } \varepsilon > 0. \\ (b) & \mathrm{P}_1(y) \le A(1+|y|)^{-(n+\delta)}, \ \textit{for some } \delta > 0. \end{array} \tag{1.4}$$

For $A_\varepsilon(x,f)$, the Abel–Poisson means of $f(x)$, defined by (1.2), we have the following integral representation via Poisson kernel;

Lemma 1.2 *Let the function $f \in L^1(\mathbf{T}^n)$ be periodically continued to $\mathbb{R}^n$ and $\mathrm{P}_\varepsilon(y)$ be the Poisson kernel, defined by (1.3). Then*

$$A_\varepsilon(x,f) = \int_{\mathbb{R}^n} f(x-t)\mathrm{P}_\varepsilon(t)dt. \tag{1.5}$$

The formula (1.5) is well known and its proof is based on the following Poisson summation formula:

$$\sum_{m\in\mathbb{Z}^n} \Phi(\varepsilon m)e^{im\cdot t} = (2\pi)^n \sum_{m\in\mathbb{Z}^n} \mathrm{P}_\varepsilon(t+2\pi m), \quad \text{where } \Phi(x) = e^{-|x|}. \tag{1.6}$$

Many works with different perspectives on the summation of multiple Fourier series and integrals have been studied in great detail in the papers [6, 7, 11, 13] (see also Stein, E.M. and Weiess, G. [12] and Weiess, F. [14, 15]). The purpose of this paper is as follows: First we introduce the notion of μ–smoothness point of a function f, which is also a Lebesgue point of f. Then we estimate the error of approximation of $f(x)$ by its Abel–Poisson means $A_\varepsilon(x,f)$ as $\varepsilon \to 0^+$ at the μ–smoothness point of function f. Note that the analogous problem for the Gauss–Weierstrass means of the relevant higher dimensional Fourier series has been studied in [2]. Some aspects of the rate of convergence, in the case of truncated hypersingular integrals generated by the Poisson and metaharmonic semigroups have been studied in [4], and for truncated hypersingular integrals generated by the Gauss–Weierstrass semigroups have been studied in [3]. Also, the nice papers by Golubov, B.I. [9, 10], by Aliev I.A. [1] and by Bayrakci S., Shafiev M.F., Aliev I.A [5] should be mentioned.

Definition 1.3 *(cf.[1-3]) Let $\rho \in (0,1)$ be fixed parameter and the function $\mu(r)$, $(0 \le r \le \rho)$ be continuous on $[0,\rho]$, be positive on $(0,\rho]$, and $\mu(0) = 0$. We say that a function $\varphi \in L^1_{loc}(\mathbb{R}^n)$ has the μ–smoothness property at a point $x^0 \in \mathbb{R}^n$ if*

$$D_\mu(x^0) \equiv \sup_{0<r\le\rho} \frac{1}{r^n\mu(r)} \int_{|x|\le r} |\varphi(x^0-x) - \varphi(x^0)|dx < \infty. \tag{1.7}$$

In the sequel it will be assumed that $\mu(r) \ge \alpha r$ for an $\alpha > 0$, and $\mu(r) = \mu(\rho)$ for $\rho \le r < \infty$.

Remark 1.4 *It is clear that every μ–smoothness point x^0 is also a Lebesgue point of f:*

$$\frac{1}{r^n}\int_{|x|\le r} |f(x^0-x) - f(x^0)|dx \overset{(1.7)}{\le} \mu(r)D_\mu(x^0) \to 0, \ \textit{as } r \to 0^+.$$

Now, we state the main results of the paper.

Theorem 1.5 *Let a periodical function $f \in L^1(\mathbf{T}^n)$ has the μ–smoothness property at $x^0 \in \mathbf{T}^n$, and suppose that $A_\varepsilon(x^0, f)$ is the Abel–Poisson means of f defined by (1.5). Then*

$$|A_\varepsilon(x^0, f) - f(x^0)| \le cD_\mu(x^0)\left(\varepsilon + \int_0^{\rho/\varepsilon} \frac{\mu(\varepsilon r)}{1+r^2} dr\right), \ \varepsilon \to 0^+, \tag{1.8}$$

where c does not depend on ε.

Corollary 1.6 *Let the function μ be continuous on $[0, \rho]$, positive on $(0, \rho]$ and $\mu(0) = 0$. Also, suppose $\mu(t) \ge \alpha t$, $(0 \le t \le \rho,\ \alpha > 0)$ and be extended on $[\rho, \infty)$ as $\mu(t) = \mu(\rho)$. In addition, assume that there exists a locally bounded function $w(t) > 0$ such that*

$$\mu(\varepsilon t) \le \mu(\varepsilon) w(t),\ (0 < \varepsilon < \rho,\ 0 < t < \rho/\varepsilon),\ \text{and} \ \int_0^\infty \frac{w(t)}{1+t^2} dt < \infty. \tag{1.9}$$

If $x^0 \in \mathbf{T}^n$ is a μ–smoothness point of $f \in L^1(\mathbf{T}^n)$, then the following estimate holds

$$|A_\varepsilon(x^0, f) - f(x^0)| = O\left(\mu(\varepsilon)\right),\ \varepsilon \to 0^+. \tag{1.10}$$

Corollary 1.7 *(a) Let $0 < \gamma < 1$, $0 < \rho < 1$, and $\mu(\varepsilon) = \varepsilon^\gamma$, $0 < \varepsilon \le \rho$. If $x^0 \in \mathbf{T}^n$ is μ–smoothness point of $f \in L^1(\mathbf{T}^n)$, then following estimate holds*

$$|A_\varepsilon(x^0, f) - f(x^0)| = O\left(\varepsilon^\gamma\right),\ \varepsilon \to 0^+. \tag{1.11}$$

(b) Let $0 < \gamma < 1$, $0 < \rho < 1$, $0 < \beta < \infty$, and $\mu(\varepsilon) = \varepsilon^\gamma (\log \frac{1}{\varepsilon})^\beta$, $0 < \varepsilon \le \rho$. If $x^0 \in \mathbf{T}^n$ is μ–smoothness point of $f \in L^1(\mathbf{T}^n)$, then following estimate holds

$$|A_\varepsilon(x^0, f) - f(x^0)| = O\left(\varepsilon^\gamma (\log \frac{1}{\varepsilon})^\beta\right),\ \varepsilon \to 0^+. \tag{1.12}$$

2. Proofs of the main results

We will use of some techniques from [2] and [4].

Proof of Theorem 1.5: Let $x^0 \in \mathbf{T}^n$ be a $\mu-$smoothness point of f. By (1.4) and (1.5), we have

$$\begin{aligned}
\left|A_\varepsilon\left(x^0, f\right) - f(x^0)\right| &= \left|\int_{\mathbb{R}^n} P_\varepsilon(x)(f(x^0 - x) - f(x^0)) dx\right| \\
&\le \int_{|x| \le \rho} |P_\varepsilon(x)| \left|f(x^0 - x) - f(x^0)\right| dx + \int_{|x| > \rho} |P_\varepsilon(x)| \left|f(x^0 - x) - f(x^0)\right| dx \\
&= I_1(\varepsilon) + I_2(\varepsilon).
\end{aligned} \tag{2.1}$$

In order to estimate $I_1(\varepsilon)$ we introduce the function

$$\Psi\left(x, x^0\right) = \begin{cases} \left|f(x^0 - x) - f(x^0)\right|, & \text{if} \quad |x| \le \rho, \\ 0, & \text{if} \quad |x| > \rho. \end{cases}$$

Hence, using this function, we have

$$I_1(\varepsilon) = \int\limits_{\mathbb{R}^n} P_\varepsilon(x)\Psi\left(x, x^0\right) dx.$$

In the last integral, by change of variables $x = r\theta$, where $r = |x|$, and remembering that the function P_ε is radial, we obtain that

$$I_1(\varepsilon) = \int\limits_0^\infty r^{n-1} \left|P_\varepsilon(r)\right| \left(\int\limits_{|\theta|=1} \Psi\left(r\theta, x^0\right) d\theta \right) dr.$$

Here, $d\theta$ is the Euclidean area element of the unit sphere $S_{n-1} = \{\theta \in \mathbb{R}^n : \ |\theta| = 1\}$. If we let

$$g(r) = \int\limits_{|\theta|=1} \Psi\left(r\theta, x^0\right) d\theta,$$

and

$$h(r) = \int\limits_0^r g(t)t^{n-1}dt$$

then, we have

$$I_1(\varepsilon) = \int\limits_0^\infty r^{n-1} \left|P_\varepsilon(r)\right| g(r)dr = \int\limits_0^\infty \left|P_\varepsilon(r)\right| dh(r).$$

Since $P_\varepsilon(r) > 0$ and differentiable on $[0, \infty)$, we get, by integrating by parts,

$$I_1(\varepsilon) = \int\limits_0^\infty P_\varepsilon(r)dh(r) = P_\varepsilon(r)h(r)\,|_0^\infty - \int\limits_0^\infty P_\varepsilon'(r)h(r)dr. \tag{2.2}$$

Because that $h(0) = 0$, $h(r) = h(\rho)$ (for $r \geq \rho$), and $P_\varepsilon(r) \to 0$ as $r \to \infty$

$$P_\varepsilon(r)h(r)\,|_0^\infty = 0.$$

Now, observing that

$$h(r) = \int\limits_0^r g(t)t^{n-1}dt = \int\limits_{|x|\leq r} \Psi\left(x, x^0\right) dx = \int\limits_{|x|\leq r} \left|f(x^0 - x) - f(x^0)\right| dx$$

$$= r^n\mu(r)\frac{1}{r^n\mu(r)} \int\limits_{|x|\leq r} \left|f(x^0 - x) - f(x^0)\right| dx \leq r^n\mu(r)D_\mu(x^0) \tag{2.3}$$

where $D_\mu(x^0)$ is defined as in (1.7), and using the fact $h(r) = h(\rho) = constant$ for $r \geq \rho$, we have

$$I_1(\varepsilon) \leq \int_0^\infty h(r)|P'_\varepsilon(r)|dr = \int_0^\rho h(r)|P'_\varepsilon(r)|dr + \int_\rho^\infty h(r)|P'_\varepsilon(r)|dr$$

$$\leq D_\mu(x^0) \int_0^\rho r^n \mu(r)|P'_\varepsilon(r)|dr + h(\rho) \int_\rho^\infty |P'_\varepsilon(r)|dr. \tag{2.4}$$

By (1.3),

$$P'_\varepsilon(r) = c_1 \varepsilon r \left(\varepsilon^2 + r^2\right)^{-(n+3)/2}.$$

Thus, from (2.3) and (2.4), we obtain

$$I_1(\varepsilon) \leq c_1 D_\mu(x^0) \int_0^\rho r^{n+1} \mu(r) \varepsilon \frac{1}{(\varepsilon^2 + r^2)^{(n+3)/2}} dr$$

$$+ c_1 h(\rho) \varepsilon \int_\rho^\infty \frac{r}{(\varepsilon^2 + r^2)^{(n+3)/2}} dr.$$

Since $n \geq 1$, the latter integral in the above expression is finite, hence

$$I_1(\varepsilon) \leq c_1 D_\mu(x^0) \varepsilon \int_0^\rho r^{n+1} \mu(r) \left(\varepsilon^2 + r^2\right)^{-(n+3)/2} dr + c_2 \varepsilon.$$

Finally, denoting $c_3 = \max\left\{c_1; \frac{c_2}{D_\mu(x^0)}\right\}$, we obtain

$$I_1(\varepsilon) \leq c_3 D_\mu(x^0) \left(\varepsilon + \varepsilon \int_0^\rho r^{n+1} \mu(r) \left(\varepsilon^2 + r^2\right)^{-(n+3)/2} dr\right)$$

$$= c_3 D_\mu(x^0) \left(\varepsilon + \int_0^{\rho/\varepsilon} r^{n+1} \mu(\varepsilon r) \left(1 + r^2\right)^{-(n+3)/2} dr\right)$$

$$\leq c_4 D_\mu(x^0) \left(\varepsilon + \int_0^{\rho/\varepsilon} \frac{\mu(\varepsilon r)}{1 + r^2} dr\right). \tag{2.5}$$

Let us now estimate the term $I_2(\varepsilon)$:

$$I_2(\varepsilon) \leq |f(x^0)| \int_{|x|>\rho} |P_\varepsilon(x)|\, dx + \int_{|x|>\rho} |P_\varepsilon(x)|\, |f(x^0 - x)| dx. \tag{2.6}$$

For the first integral,

$$\left|f(x^0)\right| \int\limits_{|x|>\rho} |P_\varepsilon(x)|\,dx = c_5 \int\limits_{\rho}^{\infty} r^{n-1}\,|P_\varepsilon(r)|\,dr$$

$$= c_6 \int\limits_{\rho}^{\infty} r^{n-1} \frac{\varepsilon}{(\varepsilon^2+r^2)^{(n+1)/2}}dr \le c_7\varepsilon \int\limits_{\rho}^{\infty} \frac{1}{r^2}dr = c_8\varepsilon. \tag{2.7}$$

For the second integral, by applying the Hölder's inequality, we get

$$\int\limits_{|x|>\rho} |P_\varepsilon(x)|\,|f(x^0-x)|dx \le \left(\int\limits_{|x|>\rho} |f(x^0-x)|^p dx\right)^{1/p} \left(\int\limits_{|x|>\rho} |P_\varepsilon(x)|^q\,dx\right)^{1/q}$$

$$\le \|f\|_p\,\varepsilon \left(\int\limits_{|x|>\rho} \frac{dx}{|x|^{(n+1)q}}\right)^{1/q} = c_9\varepsilon. \tag{2.8}$$

Hence, by (2.7) and (2.8) we obtain

$$I_2(\varepsilon) \le c_{10}\varepsilon. \tag{2.9}$$

Finally, using (2.5) and (2.9) in (2.1), we have

$$\left|A_\varepsilon(x^0,f)-f(x^0)\right| \le I_1(\varepsilon)+I_2(\varepsilon) \le cD_\mu(x^0)\left(\varepsilon + \int\limits_{0}^{\rho/\varepsilon} \frac{\mu(\varepsilon r)}{1+r^2}dr\right),$$

where the coefficient c does not depend on $\varepsilon > 0$. This clearly proves the theorem.

Proof of Corollary 1.6:

By (1.8) and the condition $\mu(\varepsilon t) \le \mu(\varepsilon)\,w(t)$,

$$\left|A_\varepsilon(x^0,f)-f(x^0)\right| \le cD_\mu(x^0)\left(\varepsilon + \mu(\varepsilon)\int\limits_{0}^{\rho/\varepsilon} \frac{w(t)}{1+t^2}dt\right).$$

Since

$$\int\limits_{0}^{\infty} \frac{w(t)}{1+t^2}dt < \infty \qquad \text{and} \qquad \mu(\varepsilon) \ge \alpha\varepsilon, \quad (0<\varepsilon<\rho<1),$$

we obtain

$$\left|A_\varepsilon(x^0,f)-f(x^0)\right| \le c_1\mu(\varepsilon), \qquad (0<\varepsilon<\rho<1),$$

as desired.

Proof of Corollary 1.7:

a) It is easy to verify that the function

$$\mu(t) = \begin{cases} t^\gamma, & \text{if} \quad 0 \le t \le \rho < 1, \\ \rho^\gamma, & \text{if} \qquad\quad t \ge \rho, \end{cases}$$

satisfies all the conditions of Corollary 1.6 with the function $w(t) = t^{\gamma}, \quad (0 < t < \infty)$.

b) If we let

$$\mu(t) = \begin{cases} 0, & \text{if} \quad t = 0, \\ t^{\gamma}(\ln(1/t))^{\beta}, & \text{if} \quad 0 < t \leq \rho < 1, \\ \rho^{\gamma}(\ln(1/\rho))^{\beta}, & \text{if} \quad t \geq \rho, \end{cases}$$

and

$$w(t) = t^{\gamma}\left(1 + \frac{|\ln t|}{|\ln \rho|}\right)^{\beta}, \quad 0 < t < \infty,$$

then

$$\mu(\varepsilon t) = (\varepsilon t)^{\gamma}\left(\ln\frac{1}{\varepsilon t}\right)^{\beta} = \varepsilon^{\gamma}\left(\ln\frac{1}{\varepsilon}\right)^{\beta} t^{\gamma}\left(1 + \frac{\ln t^{-1}}{\ln \varepsilon^{-1}}\right)^{\beta}$$

$$\leq \varepsilon^{\gamma}\left(\ln\frac{1}{\varepsilon}\right)^{\beta} t^{\gamma}\left(1 + \frac{\ln|t|}{|\ln \rho|}\right)^{\beta}, \quad (0 < \varepsilon < \rho, \quad 0 < t < \rho < 1).$$

Consequently, for the function

$$w(t) = t^{\gamma}\left(1 + \frac{|\ln t|}{|\ln \rho|}\right), \quad 0 < t < \rho/\varepsilon,$$

we observe that

$$w(\varepsilon t) \leq \mu(\varepsilon)\, w(t), \quad (0 < \varepsilon < \rho, \quad 0 < t < \rho/\varepsilon),$$

and therefore

$$\int_0^{\infty} \frac{w(t)}{1+t^2} dt = c \int_0^{\infty} \frac{t^{\gamma}(\ln \rho + |\ln t|)^{\beta}}{1+t^2} dt < \infty.$$

This proves the Corollary 1.7-(b).

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