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An application of modified sigmoid function to a class of $q-$ starlike and $q-$ convex analytic error functions

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Abstract: In this study, in the open unit disc Λ, by applying the $q-$ derivative operator and the fractional $q-$ derivative operator and by using the principle of subordination between analytic functions, we introduce some new interesting subclasses of $q-$ starlike and $q-$ convex analytic functions associated with error functions and modified sigmoid functions.

Key words: univalent function, analytic function, $q-$ starlike error function, $q-$ convex error function, modified sigmoid function, subordination, convolution.

1. Introduction

Let $\mathcal{A}$ be the family of functions of the form

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are normalized analytic functions by the conditions $g(0) = g\prime(0) - 1 = 0$ in the open unit disc $\Lambda = \{z : |z| < 1\}$ and let S be the subclass of $\mathcal{A}$ consisting of the form (1) which are also univalent in Λ. To remember the concept of subordination between analytic functions, let h and g be analytic in Λ. Then for $h, g \in \Lambda$, g is subordinate to h if there exists a Schwarz function $w \in S$ given by the form

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 \cdots, (z \in \Lambda), \tag{2}$$

such that $g(z) = h(w(z)), (z \in \Lambda)$, where $w(0) = 0, |w(z)| < 1, z \in \Lambda$. This subordination is denoted by

$$g(z) \prec h(z). \tag{3}$$

Specially, if h is univalent in Λ, mentioned subordination is equivalent to $g(0) = h(0)$ and $g(\Lambda) \subset h(\Lambda)$.

In the field of Geometric Function Theory, different subclasses of $\mathcal{A}$ have been considered from various aspects. The $q-$ calculus and the fractional q-calculus play imperative role in the theory of hypergeometric series, quantum physics and the operator theory. Srivastava first used the $q-$calculus in the situation of Geometric Function Theory and the basic (or $q-$) hypergeometric functions in a book chapter [28]. With

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the help of $q-$ calculus theory, we can describe the theory of univalent functions with different ways (see, for example, [12–14, 24, 25]). Jackson [12, 13] initiated the application of $q-$calculus and systematically developed $q-$derivative and $q-$ integral. Also these operators play virtual role in the theory of relativity, astronomy, atomic physics, nuclear physics and quantum mechanics.

Now, we recall some basic concepts to present with the fundamental content:

For the function $g \in \mathcal{A}$, the Jackson's q-derivative is defined by [12]

$$D_q g(z) = \begin{cases} \frac{g(z)-g(qz)}{(1-q)z} & z \neq 0 \\ g'(0), & z = 0 \end{cases} \tag{4}$$

where $0 < q < 1$ and $\lim_{q \to 1^-} D_q g(z) = g'(z)$ if g is differentiable at z.

Among the special functions, the error function is an important one. This function take places in different areas of science. The function

$$erg(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n)!} z^{2n+1} \tag{5}$$

was defined by Abramowitz [1] and then it became the topic of compherensive studies and applications. One may find different properties and inequalities of error function in [4, 5, 8]. Error function also its inverse, introduced by Carlitz [6] and denoted by $inverg$, were widely studied in applied mathematics and mathematical physics, for example, data analysis [11], probability, and statistics [7], concentration-dependent diffusion problems [22], in heat conduction problem [7] and solutions to Einstein's scalar-field equations. Very recently, normalized analytic error function by the form of

$$E_r g(z) = \frac{\sqrt{\pi z}}{2} erg(\sqrt{z}) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n \tag{6}$$

was studied by Ramachandran et al. [27] and by using convolution, the family of analytic functions defined as

$$\varepsilon = \mathcal{A} * E_r g = \left\{ \mathcal{G} \colon \mathcal{G}(z) = (g * E_r g)(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} a_n z^n, \; g \in \mathcal{A} \right\} \tag{7}$$

where $E_r g$ denotes the class consisting of a single function and the symbol $*$ denotes the well known Hadamard product of two analytic functions. By using equation (3), one can conclude that

$$D_q \mathcal{G}(z) = 1 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} [n]_q a_n}{(2n-1)(n-1)!} z^{n-1}, \tag{8}$$

where $[n]_q = \frac{1-q^n}{1-q}, (n \in N)$. As well as error functions, activation function can be given an important example

of special functions. Activation function plays role as a crashing function, such that in a neural network, the output of a neuron is between certain values (generally 0 and 1, or -1 and 1). The sigmoid function is the most popular function among the activation functions and it is usually used in gradient descendent type learning algorithms. There are several possibilities for using sigmoid function, for instance, truncated series expansion, look-up tables, or piecewise approximation. The sigmoid function is given by the form

$$k(z) = \frac{1}{1+\exp(-z)}. \tag{9}$$

This function is called the sigmoidal curve or logistic function. In light of the properties mentioned in [9] and [21], we can say that sigmoid function is very useful in geometric function theory. Recently, sigmoid function for various classes of analytic and univalent functions was studied by Oladipo [18], Murugusundaramoorthy, and Janani [17], Olantunji et al. [19], Olatunji [20], Ramachandran and Dhanalakshmi [26], and Kamali et al. [15]. We need the following lemmas to derive our main results.

Lemma 1 *[23] If a function $p \in \mathcal{P}$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, (z \in \Lambda),$$

then $|p_k| \leq 2, k \in \mathbb{N}$ where $\mathcal{P}$ is the family of all analytic functions in Λ for which $p(0) = 1$ and $\Re\{p(z)\} > 0$.

We recall the series form of a modified sigmoid function defined in equation (9) is given by (see [21])

$$\Upsilon(z) = 2k(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m. \tag{10}$$

Lemma 2 *[21] Let $k(z)$ be a Sigmoid function defined in equation (9) and $\Upsilon(z) = 2k(z)$. Then $\Upsilon(z) \in \mathcal{P}, z \in \Lambda$, where $\Upsilon(z)$ is a modified sigmoid function.*

Lemma 3 *[21] Let*

$$\Upsilon_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m.$$

Then $|\Upsilon_{n,m}(z)| < 2, z \in \Lambda$.

Letting $m = 1$, Fadipe et al. [21] remarked that

$$\Upsilon_{n,1}(z) = \Phi(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, (z \in \Lambda), \tag{11}$$

where $d_n = \frac{(-1)^{n+1}}{2n!}$.

In the literature, celebrated Fekete-Szegö functional for normalized univalent functions of the form given by (1) is well known for its rich history. The Fekete-Szegö problem is the problem of maximizing the value of the nonlinear functional $\left|a_2 a_4 - \mu a_3^2\right|$ [10]. The equality is valid for the Koebe function. The sharp upper bound for Fekete-Szegö functional was found by Keogh and Merkes [16] for some subclasses of univalent function classes.

In recent decades, coefficient estimates and famous celebrated Fekete-Szegö inequality were studied for the class of univalent functional defined by some special functions like error functions, sigmoid functions, q-derivative operators as well as with their convolution. All of these special functions occur widely in multiple branches of mathematics and sciences. More examples endowed with the special functions which are mentioned above can be found in [2, 3, 9, 15, 17, 19, 20, 26, 27]

The aim of this work is to benefit from the error function, modified sigmoid function, and the principle of the subordination, to introduce new interesting subclasses of univalent functions and derive Taylor-Maclaurin coefficient inequalities for functions belonging these new subclasses. Also we will consider the famous Fekete-Szegö problem.

2. The classes $\mathcal{S}_q(k,\Upsilon)$ and $\mathcal{C}_q(b,\Upsilon)$

In the rest of the paper, unless otherwise stated, the function $g\in\mathcal{A}$ is explained by (1) and modified logistic sigmoid activation function is shown by $\Upsilon(z)\in\mathcal{A}$.

Definition 4 *A function $g\in\mathcal{A}$ is said to be in the class $\mathcal{S}_q(k,\Upsilon)$ if it satisfies the following subordination condition*

$$1+\frac{1}{k}\left(\frac{zD_q\mathcal{G}(z)}{\mathcal{G}(z)}-1\right)\prec\Upsilon(z), \tag{12}$$

where $\Upsilon(z)$ as given in (10), $k\in\mathbb{C}\backslash\{0\}$ and the real numbers $0<q<1$.

Definition 5 *A function $g\in\mathcal{A}$ is said to be in the class $\mathcal{C}_q(b,\Upsilon)$ if it satisfies the following subordination condition*

$$1+\frac{1}{k}\left(\frac{D_q\left(zD_q\mathcal{G}(z)\right)}{D_q\mathcal{G}(z)}-1\right)\prec\Upsilon(z), \tag{13}$$

where $\Upsilon(z)$ as given in (10), $k\in\mathbb{C}\backslash\{0\}$ and the real numbers $0<q<1$.

As $q\to 1^-$, $[n]_q\to n$ and $\lim_{q\to 1^-} D_q g(z)=g'(z)$, we can give some new classes related to above defined classes:

Remark 6 *1. For $k\in\mathbb{C}\backslash\{0\}$, the real numbers $0<q<1$ and $\Upsilon(z)$ as given in (10),*

i) $\lim_{q\to 1^-}\mathcal{S}_q(k,\Upsilon)=\mathcal{S}(k,\Upsilon)$ and this new class consists of the functions $g\in\mathcal{A}$ of the form

$$1+\frac{1}{k}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}-1\right)\prec\Upsilon(z).$$

ii) $\lim_{q\to 1^-}\mathcal{C}_q(k,\Upsilon)=\mathcal{C}(k,\Upsilon)$ and this new class consists of the functions $g\in\mathcal{A}$ of the form

$$1+\frac{1}{k}\left(\frac{\left(z\mathcal{G}'(z)\right)'}{\mathcal{G}'(z)}-1\right)\prec\Upsilon(z)$$

2. For $k=1$, the real numbers $0<q<1$ and Υ as given in (10), *i)* $\lim_{q\to 1^-} \mathcal{S}_q(k,\Upsilon) = \mathcal{S}(\Upsilon)$ *and this new class consists of the functions* $g \in \mathcal{A}$ *of the form*

$$\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)} \prec \Upsilon(z).$$

ii) $\lim_{q\to 1^-} \mathcal{C}_q(k,\Upsilon) = \mathcal{C}(\Upsilon)$ *and this new class consists of the functions* $g \in \mathcal{A}$ *of the form*

$$\frac{(z\mathcal{G}'(z))'}{\mathcal{G}'(z)} \prec \Upsilon(z).$$

Theorem 7 *Let* $\Upsilon(z)$ *be given as equation (10). For* $g \in \mathcal{A}$ *if* $g \in \mathcal{S}_q(k,\Upsilon)$*, then*

$$|a_2| \leq \frac{3}{2\left([2]_q - 1\right)} |k|, \tag{14}$$

$$|a_3| \leq \left(1 + \frac{1}{2\left([2]_q - 1\right)} |k|\right) \frac{5}{\left([3]_q - 1\right)} |k|. \tag{15}$$

Particularly, taking $m = 1$ *for* $\Upsilon(z)$ *given by (10), if Schwarz function is chosen* $w(z) = \Phi(z) - 1$*, then*

$$\begin{aligned} a_2 &= \frac{3}{4\left(1 - [2]_q\right)} k, \\ a_3 &= \left(1 + \frac{1}{2\left(1 - [2]_q\right)} k\right) \frac{5k}{4\left(1 - [3]_q\right)}, \end{aligned}$$

where $\Phi(z)$ *is given by (11).*

Proof Assume that $g \in \mathcal{S}_q(k,\Upsilon)$. Then, from (10) and principle of subordination, there exists a function $w(z)$ satisfying the conditions of the Schwarz lemma such that

$$1 + \frac{1}{k}\left(\frac{zD_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right) = \Upsilon(w(z)), \tag{16}$$

where $\Upsilon(z)$ is a modified sigmoid function given by

$$\Upsilon(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + \cdots. \tag{17}$$

By taking

$$\begin{aligned} w(z) &= c_1 z + c_2 z^2 + c_3 z^3 \cdots, (z \in \Lambda), \\ w^3(z) &= c_1^3 z^3 + 3c_1^2 c_2 z^4 + (3c_1^2 c_3 + 3c_1 c_2^2) z^5 + \cdots, \\ w^5(z) &= c_1^5 z^5 + \cdots, \end{aligned} \tag{18}$$

and putting $w(z)$, $w^3(z)$ and $w^5(z)$ in the equality

$$\Upsilon(w(z)) = 1 + \frac{1}{2}w(z) - \frac{1}{24}w(z)^3 + \frac{1}{240}w(z)^5 - \frac{17}{40320}w(z)^7 + \cdots,$$

we obtain

$$\Upsilon(w(z)) = 1 + \frac{c_1}{2}z + \frac{c_2}{2}z^2 + \left(\frac{c_3}{2} - \frac{c_1^3}{24}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1^2 c_2}{8}\right)z^4 + \cdots . \tag{19}$$

It is well known that if $|w(z)| < 1$, then

$$|c_j| \leq 1, \tag{20}$$

for all $j \in \mathbb{N}$ and

$$\left|c_2 - \rho c_1^2\right| \leq \max\{1, |\rho|\}. \tag{21}$$

From (7) and (8), we have

$$zD_q\mathcal{G}(z) - \mathcal{G}(z) = \sum_{n=2}^{\infty} \left([n]_q - 1\right) A_n a_n z^n, \tag{22}$$

where

$$A_n = \frac{(-1)^{n-1}}{(2n-1)(n-1)!}, \tag{23}$$

$$zD_q\mathcal{G}(z) - \mathcal{G}(z) = k\mathcal{G}(z)\left[\Upsilon\left(w\left(z\right)\right) - 1\right]. \tag{24}$$

Using (19) and (22) in equation (24) and taking expanding in series forms, we get

$$\sum_{n=2}^{\infty} \left([n]_q - 1\right) A_n a_n z^n = k\left[z + \sum_{n=2}^{\infty} A_n a_n z^n\right]\left[\frac{c_1}{2}z + \frac{c_2}{2}z^2 + \left(\frac{c_3}{2} - \frac{c_1^3}{24}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1^2 c_2}{8}\right)z^4 + \cdots\right], \tag{25}$$

$$\begin{aligned} &\Rightarrow \quad \left([2]_q - 1\right) A_2 a_2 z^2 + \left([3]_q - 1\right) A_3 a_3 z^3 + \left([4]_q - 1\right) A_4 a_4 z^4 + \cdots \\ &= \quad k\left[z + A_2 a_2 z^2 + A_3 a_3 z^3 + A_4 a_4 z^4 + \cdots\right]\left[\frac{c_1}{2}z + \frac{c_2}{2}z^2 + \left(\frac{c_3}{2} - \frac{c_1^3}{24}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1^2 c_2}{8}\right)z^4 + \cdots\right] \end{aligned} \tag{26}$$

Comparing the coefficients of z^2, z^3, and z^4 in (26), using (23) for $n = 2, 3, 4$, writing $A_2 = -\frac{1}{3}, A_3 = \frac{1}{10}$, after simplifying the above, we have

$$\begin{aligned} a_2 &= \frac{c_1}{2A_2\left([2]_q - 1\right)}k \\ &= \frac{-3c_1 k}{2\left([2]_q - 1\right)}, \end{aligned} \tag{27}$$

$$\begin{aligned} a_3 &= \left(\frac{c_2}{2} + \frac{A_2 a_2 c_1}{2}\right)k \\ &= \left(\frac{c_2}{2} + \frac{c_1^2}{4\left([2]_q - 1\right)}k\right)\frac{k}{\left([3]_q - 1\right)A_3} \\ &= \left(c_2 + \frac{c_1^2}{2\left([2]_q - 1\right)}k\right)\frac{5k}{\left([3]_q - 1\right)}. \end{aligned} \tag{28}$$

Applying absolute value to equalities (27), (28) and using the inequality (20), we obtain

$$\begin{aligned} |a_2| &= \left| \frac{-3c_1 k}{2\left([2]_q - 1\right)} \right| \\ &\leq \frac{3}{2\left([2]_q - 1\right)} |k|, \\ |a_3| &= \left| \left(c_2 + \frac{c_1^2}{2\left([2]_q - 1\right)} k \right) \frac{5k}{\left([3]_q - 1\right)} \right| \\ &\leq \left(1 + \frac{1}{2\left([2]_q - 1\right)} |k| \right) \frac{5}{\left([3]_q - 1\right)} |k|. \end{aligned}$$

Also, if we take $m = 1$ in (10), we have

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, d_n = \frac{(-1)^{n+1}}{2n!},$$

$$w(z) = \Phi(z) - 1 = \frac{1}{2}z - \frac{1}{4}z^2 + \frac{1}{12}z^3 - \cdots. \tag{29}$$

Comparing with (18) and (29), we get

$$c_1 = \frac{1}{2}; c_2 = -\frac{1}{4}; c_3 = \frac{1}{12}; \cdots. \tag{30}$$

Putting (30) in (27) and (28), we obtain

$$\begin{aligned} a_2 &= \frac{3}{4\left(1 - [2]_q\right)} k, \\ a_3 &= \left(1 + \frac{1}{2\left(1 - [2]_q\right)} k \right) \frac{5k}{4\left(1 - [3]_q\right)}. \end{aligned}$$

□

Corollary 8 *Letting $q \to 1^-$ in Theorem 7, we have*

$$|a_2| \leq \frac{3}{2} |k|, \tag{31}$$

$$|a_3| \leq \left(1 + \frac{1}{2} |k| \right) \frac{5}{2} |k|. \tag{32}$$

Particularly, taking $m = 1$ for $\Upsilon(z)$ given by (10), we have

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, d_n = \frac{(-1)^{n+1}}{2n!}.$$

If Schwarz function is chosen $w(z) = \Phi(z) - 1$, *then*

$$\begin{aligned} a_2 &= -\frac{3}{4}k \ , \\ a_3 &= \left(\frac{1}{2}k - 1\right)\frac{5k}{8}. \end{aligned}$$

Theorem 9 *Let* μ *be a nonzero complex number and let* $g \in \mathcal{S}_q(k, \Upsilon)$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{5\left|k\right|}{\left([3]_q - 1\right)} \max\left\{1, \left|\frac{10\left([2]_q - 1\right)}{\left([3]_q - 1\right)} - \mu\right| \frac{9\left([3]_q - 1\right)}{20\left([2]_q - 1\right)^2}\left|k\right|\right\}. \tag{33}$$

Proof From (27) and (28), we get

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &= \left|\left(c_2 + \frac{c_1^2}{2\left([2]_q - 1\right)}k\right)\frac{5k}{\left([3]_q - 1\right)} - \mu\left(\frac{-3c_1 k}{2\left([2]_q - 1\right)}\right)^2\right| \\ &\leq \frac{5\left|k\right|}{\left([3]_q - 1\right)} \max\left\{c_2 - c_1^2\left(\frac{9\mu\left([3]_q - 1\right)}{20\left([2]_q - 1\right)^2}k - \frac{1}{2\left([2]_q - 1\right)}k\right)\right\} \end{aligned}$$

From inequalities (20) and (21), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{5\left|k\right|}{\left([3]_q - 1\right)} \max\left\{1, \left|\frac{9\mu\left([3]_q - 1\right)}{20\left([2]_q - 1\right)^2}k - \frac{1}{2\left([2]_q - 1\right)}k\right|\right\}.$$

By a simple calculation, we obtain the desired result. □

Corollary 10 *Letting* $\mu = 1$ *in Theorem 9, we have*

$$\left|a_3 - a_2^2\right| \leq \frac{5\left|k\right|}{\left([3]_q - 1\right)} \max\left\{1, \left|\frac{10\left([2]_q - 1\right)}{\left([3]_q - 1\right)} - 1\right| \frac{9\left([3]_q - 1\right)}{20\left([2]_q - 1\right)^2}\left|k\right|\right\}..$$

Corollary 11 *Letting* $q \to 1^-$ *in Theorem 9, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{5\left|k\right|}{2} \max\left\{1, \left|\frac{5}{9} - \mu\right| \frac{10}{9}\left|k\right|\right\}.$$

Now, we give an example for Theorems 7 and 9.

Example 12 *Let* $\Upsilon(z)$ *be given as equation (10). For* $g \in \mathcal{S}_q(k, \Upsilon)$, *if the Schwarz function is* $w(z) = z$, *then we can show that the following estimates can be obtained by the same process with Theorems 7 and 9.*

$$\left|a_2\right| \leq \frac{3}{2\left([2]_q - 1\right)}\left|k\right|, \tag{34}$$

$$\left|a_3\right| \leq \frac{5}{2\left([3]_q - 1\right)}\left|k\right|^2. \tag{35}$$

and the Fekete-Szegö inequality is

$$\left|a_3-\mu a_2^2\right| \leq \left\{ \begin{array}{ccc} \frac{5}{2([3]_q-1)}\left|k\right|^2\left(1+\frac{9([3]_q-1)}{10([2]_q-1)^2}\mu\right) & , & \mu\geq 0 \\ \frac{5}{2([3]_q-1)}\left|k\right|^2\left(1-\frac{9([3]_q-1)}{10([2]_q-1)^2}\mu\right) & , & \mu<0 \end{array} \right\}. \tag{36}$$

Also, letting $q\to 1^-$ inequalities in example 12, we have

$$\left|a_2\right| \leq \frac{3}{2}\left|k\right|, \tag{37}$$

$$\left|a_3\right| \leq \frac{5}{4}\left|k\right|^2. \tag{38}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \left\{ \begin{array}{ccc} \frac{5}{4}\left|k\right|^2\left(1+\frac{9}{5}\mu\right) & , & \mu\geq 0 \\ \frac{5}{4}\left|k\right|^2\left(1-\frac{9}{5}\mu\right) & , & \mu<0 \end{array} \right. \tag{39}$$

Theorem 13 *Let $\Upsilon(z)$ be given as equation (10). For $g\in\mathcal{A}$ if $g\in\mathcal{C}_q(k,\Upsilon)$, then*

$$\left|a_2\right| \leq \frac{3}{2[2]_q\left([2]_q-1\right)}\left|k\right|, \tag{40}$$

$$\left|a_3\right| \leq \left(1+\frac{1}{2\left([2]_q-1\right)}\left|k\right|\right)\frac{5}{[3]_q\left([3]_q-1\right)}\left|k\right|. \tag{41}$$

Particularly, taking $m=1$ for $\Upsilon(z)$ given by (10), if Schwarz function is chosen $w(z)=\Phi(z)-1$, then

$$a_2 = \frac{3}{4[2]_q\left(1-[2]_q\right)}k,$$

$$a_3 = \left(1+\frac{1}{2\left(1-[2]_q\right)}k\right)\frac{5}{4[3]_q\left(1-[3]_q\right)}k,$$

where $\Phi(z)$ is given by (11).

Proof The proof is obtained by following the same process with Theorem 7. □

Corollary 14 *Letting $q\to 1^-$ in Theorem 13, we have*

$$\left|a_2\right| \leq \frac{3}{4}\left|k\right|,$$

$$\left|a_3\right| \leq \left(1+\frac{1}{2}\left|k\right|\right)\frac{5}{6}\left|k\right|.$$

Particularly, taking $m=1$ for $\Upsilon(z)$ given by (10), we have

$$\Phi(z)=1+\sum_{n=1}^{\infty}d_nz^n, d_n=\frac{(-1)^{n+1}}{2n!}.$$

If Schwarz function is chosen $w(z) = \Phi(z) - 1$, *then*

$$\begin{aligned} a_2 &= -\frac{3}{8}k \ , \\ a_3 &= \left(\frac{1}{2}k - 1\right)\frac{5}{24}k, \end{aligned}$$

Theorem 15 *Let* μ *be a nonzero complex number and let* $g \in \mathcal{C}_q(k, \Upsilon)$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{5\left|k\right|}{[3]_q\left([3]_q - 1\right)} \max\left\{1, \left|\frac{10[2]_q^2\left([2]_q - 1\right)}{9[3]_q\left([3]_q - 1\right)} - \mu\right| \frac{9[3]_q\left([3]_q - 1\right)}{20[2]_q^2\left([2]_q - 1\right)^2}\left|k\right|\right\}. \tag{42}$$

Proof The proof is obtained by following the same process with Theorem 9. □

Corollary 16 *Letting* $\mu = 1$ *in Theorem 15, we get*

$$\left|a_3 - a_2^2\right| \leq \frac{5\left|k\right|}{[3]_q\left([3]_q - 1\right)} \max\left\{1, \left|\frac{10[2]_q^2\left([2]_q - 1\right)}{9[3]_q\left([3]_q - 1\right)} - 1\right| \frac{9[3]_q\left([3]_q - 1\right)}{20[2]_q^2\left([2]_q - 1\right)^2}\left|k\right|\right\}.$$

Corollary 17 *Letting* $q \to 1^-$ *in Theorem 15, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{5\left|k\right|}{6} \max\left\{1, \left|\frac{20}{27} - \mu\right| \frac{27}{40}\left|k\right|\right\}.$$

Now, we give an example for Theorems 13 and 15.

Example 18 *Let* $\Upsilon(z)$ *be given as equation (10). For* $g \in \mathcal{A}$ *if* $g \in \mathcal{C}_q(k, \Upsilon)$ *and the Schwarz function* $w(z) = z$, *then we can show that the following estimates can be obtained by the same process with Theorems 13 and 15.*

$$\left|a_2\right| \leq \frac{3}{2[2]_q\left([2]_q - 1\right)}\left|k\right|, \tag{43}$$

$$\left|a_3\right| \leq \frac{5}{2[3]_q\left([3]_q - 1\right)\left([2]_q - 1\right)}\left|k\right|^2. \tag{44}$$

and the Fekete-Szegö inequality is

$$\left|a_3 - \mu a_2^2\right| \leq \left\{\begin{array}{lcl} \frac{5}{2([3]_q-1)([2]_q-1)[2]_q}\left|k\right|^2\left(1 + \frac{9([3]_q-1)}{10([2]_q-1)[2]_q}\mu\right) & , & \mu \geq 0 \\ \frac{5}{2([3]_q-1)([2]_q-1)[2]_q}\left|k\right|^2\left(1 - \frac{9([3]_q-1)}{10([2]_q-1)[2]_q}\mu\right) & , & \mu < 0 \end{array}\right\}. \tag{45}$$

Also, letting $q \to 1^-$ *in inequalities above, we have*

$$\begin{aligned} \left|a_2\right| &\leq \frac{3}{4}\left|k\right|, \\ \left|a_3\right| &\leq \frac{5}{12}\left|k\right|^2. \end{aligned}$$

and the Fekete-Szegö inequality is

$$\left|a_3 - \mu a_2^2\right| \leq \left\{\begin{array}{lcl} \frac{5}{8}\left|k\right|^2\left(1 + \frac{9}{10}\mu\right) & , & \mu \geq 0 \\ \frac{5}{8}\left|k\right|^2\left(1 - \frac{9}{10}\mu\right) & , & \mu < 0 \end{array}\right\}. \tag{46}$$

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