Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1342 – 1359
© TÜBİTAK
doi:10.3906/mat-2106-113

Existence and multiplicity of solutions for p(.)-Kirchhoff-type equations

Rabil AYAZOĞLU[1,2,*], **Sezgin AKBULUT**[3], **Ebubekir AKKOYUNLU**[1]
[1]Faculty of Education, Bayburt University, Bayburt, Turkey
[2]Institute of Mathematics and Mechanics of ANAS, Baku, Azerbaijan
[3]Faculty of Science, Ataturk University, Erzurum, Turkey

Received: 28.06.2021 • **Accepted/Published Online:** 18.03.2022 • **Final Version:** 05.05.2022

Abstract: This paper is concerned with the existence and multiplicity of solutions of a Dirichlet problem for $p(.)$-Kirchhoff-type equation

$$\begin{cases} M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\left(-\Delta_{p(x)}u\right) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Using the mountain pass theorem, fountain theorem, dual fountain theorem and the theory of the variable exponent Sobolev spaces, under appropriate assumptions on f and M, we obtain results on existence and multiplicity of solutions.

Key words: Lebesgue and Sobolev spaces with variable exponent, $p(.)$-Laplacian, Kirchhoff-type equation, mountain pass theorem, fountain theorem, dual fountain theorem

1. Introduction

In this paper, we deal with the existence of multiple solutions for the nonhomogeneous, nonlocal elliptic $p(.)$-Kirchhoff-type problem given by

$$\begin{cases} M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\left(-\Delta_{p(x)}u\right) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega\ , \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $\Delta_{p(.)}u = \operatorname{div}(|\nabla u|^{p(.)-2}\nabla u)$ is the $p(.)$-Laplacian operator, $p : \Omega \to \mathbb{R}^+$ is Lipschitz continuous and bounded, the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a Kirchhoff function. The main theorems used here are mountain pass theorem, fountain theorem, and dual fountain theorem.

The operator $\Delta_{p(.)}$ is said to be $p(.)$-Laplacian which is a natural generalization of the p-Laplacian operator. The $p(.)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to the problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically

*Correspondence: rabilmashiyev@gmail.com
2010 *AMS Mathematics Subject Classification:* 35A15, 35D30, 35J35, 35J60

change the mechanical properties under the influence of an exterior electromagnetic field [38, 42]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [2]. Another field of application of equations with variable exponent growth conditions is image processing [8]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [5, 19, 21, 23, 26, 34, 35] for the study of the $p(.)$-Laplacian equations and the corresponding variational problems.

Problem (1.1) is a generalization of a model for the stationary case introduced by Kirchhoff (see [27]). More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

was firstly presented by Kirchhoff, where ρ is the mass density, ρ_0 is the initial tension, L is the length of the string, h is the area of the crosssection, while E is the Young modulus of the material. A distinguish feature of the Kirchhoff equation (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, which depends on the average $\frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ of the kinetic energy $\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2$ on $[0, L]$, and hence the equation is no longer a pointwise identity and therefore it is often called a nonlocal problem. This equation extends the classical D'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

On the other hand, the equation given as

$$\begin{cases} -\left(a + b \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

is related to the stationary analog of Equation (1.2). The early classical studies dedicated to Kirchhoff equations were given by Pohozaev [37]. However, Equation (1.3) received much attention only after the paper by Lions [29], where an abstract framework to the problem was proposed. For some interesting results, we refer to [1, 9, 17, 24, 25, 32, 41]. Also, nonlocal boundary value problems like (1.3) can be used for modeling several physical and biological systems where u describes a process that depends on the average of itself, such as the population density. We refer the reader to [3, 6, 10] for some related works.

Recently, the studies of the Kirchhoff equations and the Kirchhoff systems have been considered by variational method in the case involving the p-Laplacian operator [13, 30, 31]. Moreover, due to the increasing amount of attention towards partial differential equations with nonstandard growth conditions, it was further extended to the $p(.)$-Laplacian operator $\Delta_{p(.)}$,defined by $\Delta_{p(.)} u := \mathrm{div}(|\nabla u|^{p(.)-2} \nabla u)$ [4, 11, 12, 40].

In addition to these, in [15], the authors investigated the nonlocal $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -\left(a + b \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \mathrm{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

and showed the existence of a sequence of positive, homoclinic weak solutions of (1.4) under some suitable conditions on f by using the variational methods.

In [14], by a direct variational approach, Dai and Hao established conditions ensuring the existence and multiplicity of solutions for the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{|\nabla u|^{p(x)}dx}{p(x)}\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u), \text{ in } \Omega, \\ u = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases}$$

In [16], Dai and Ma dealt with the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff type problem with Neumann boundary data of the following form

$$\begin{cases} -M\left(\int_\Omega \frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)\left(\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) - |u|^{p(x)-2}u\right) = f(x,u), \text{ in } \Omega, \\ \frac{\partial u}{\partial v} = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

By using a direct variational approach and the theory of the variable exponent Sobolev spaces, under appropriate assumptions on f and M, they obtained several results on the existence and multiplicity of solutions for the problem. In particular, they also obtained some results which can be considered as extensions of the classical result named “combined effects of concave and convex nonlinearities”. Moreover, the positive solutions and the regularity of weak solutions of the problem were considered by authors.

In [23], the authors studied the nonlocal $p(x)$-Laplacian problem of the following form

$$\left\{ M\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}dx\right)\left(\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u\right) = f(x,u), \text{ in } \mathbb{R}^N, \ u \in W^{1,p(.)}\left(\mathbb{R}^N\right). \right.$$

By using the method of weight function and the theory of the variable exponent Sobolev space, under appropriate assumptions on f and M, they obtained some results on the existence and multiplicity of solutions of this problem.

In the present paper, motivated by the above, we show the existence and multiplicity of nontrivial weak solutions of Problem (1.1) by using mountain pass theorem, fountain theorem and dual fountain theorem. Problem (1.1) is called nonlocal because of the presence of the term M, which implies that the equation in (1.1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particularly interesting.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, using the variational methods, we give the existence and multiplicity of nontrivial weak solutions of problem (1.1).

2. Preliminaries

In this section, we recall some results on Lebesgue and Sobolev spaces with variable exponent. The reader is referred to (see [18, 20, 28, 33]), and references therein for more details.

Set

$$C_+(\overline{\Omega}) = \left\{p : p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\right\},$$

and denote for any $p \in C_+(\overline{\Omega})$

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ := \max_{x\in\overline{\Omega}} p(x) < +\infty. \tag{2.1}$$

Let us denote by $\Re(\Omega)$ the set of all measurable real functions defined on Ω, elements in $\Re(\Omega)$ which are equal to each other almost everywhere are considered as one element. For any $p \in C_+(\overline{\Omega})$, we define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ by

$$L^{p(.)}(\Omega) = \left\{ u : u \in \Re(\Omega), \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\},$$

with the norm, so-called Luxemburg norm on $L^{p(.)}(\Omega)$

$$\|u\|_{L^{p(.)}(\Omega)} := |u|_{p(.)} = \inf \left\{ \mu > 0 : \sigma\left(\frac{u}{\mu}\right) \leq 1 \right\},$$

where $\sigma : L^{p(.)}(\Omega) \to \mathbb{R}$ with

$$\sigma(u) = \int_\Omega |u|^{p(x)}\, dx.$$

Sobolev space with variable exponent $W^{1,p(.)}(\Omega)$ is defined by

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)} := \|u\|_{1,p(.)} = |u|_{p(.)} + |\nabla u|_{p(.)}.$$

We define $W_0^{1,p(.)}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p(.)}(\Omega)} := E$. We can define an equivalent norm $\|u\|_E = |\nabla u|_{p(.)}$. The modular of E that is the mapping $\Lambda : E \to \mathbb{R}$ is defined by

$$\Lambda(u) = \int_\Omega |\nabla u|^{p(x)}\, dx,$$

for all $u \in E$. It is well known that if $1 < p^- \leq p^+ < +\infty$ then $L^{p(.)}(\Omega)$, $W^{1,p(.)}(\Omega)$ and E are separable and reflexive Banach spaces. We refer to [18, 28, 38] for the elementary properties of the spaces $L^{p(.)}(\Omega)$, $W^{1,p(.)}(\Omega)$ and E.

Proposition 2.1 (see [18, 28]). *For $u,\ u_k \in L^{p(.)}(\Omega)$ $(k = 1, 2, ...)$ we have:*

i) $\min\left\{ |u|_{p(.)}^{p^-}, |u|_{p(.)}^{p^+} \right\} \leq \sigma(u) \leq \max\left\{ |u|_{p(.)}^{p^-}, |u|_{p(.)}^{p^+} \right\}$;

ii) $\min\left\{ \sigma^{1/p^-}(u), \sigma^{1/p^+}(u) \right\} \leq |u|_{p(.)} \leq \max\left\{ \sigma^{1/p^-}(u), \sigma^{1/p^+}(u) \right\}$;

iii $\sigma\left(u/|u|_{p(.)} \right) = 1$ *if* $|u|_{p(.)} \neq 0$;

iv) $|u_k|_{p(.)} \to 0 \Leftrightarrow \sigma(u_k) \to 0$*;*

v) $|u_k|_{p(.)} \to \infty \Leftrightarrow \sigma(u_k) \to \infty$.

Proposition 2.2 (see [18, 28]). *For $u,\ u_k \in E$ $(k = 1, 2, ...)$ we have:*

i) $\min\left\{ \|u\|_E^{p^-}, \|u\|_E^{p^+} \right\} \leq \Lambda(u) \leq \max\left\{ \|u\|_E^{p^-}, \|u\|_E^{p^+} \right\}$*;*

ii) $\min\left\{\Lambda^{1/p^-}(u),\Lambda^{1/p^+}(u)\right\}\leq\|u\|_E\leq\max\left\{\Lambda^{1/p^-}(u),\Lambda^{1/p^+}(u)\right\}$*;*

iii) $\|u_k\|_E\to 0\Leftrightarrow\Lambda(u_k)\to 0$*;*

iv) $\|u_k\|_E\to\infty\Leftrightarrow\Lambda(u_k)\to\infty$*;*

v) $\Lambda(u/\|u\|_E)=1$ *if* $\|u\|_E\neq 0$*.*

Denote

$$L_+^\infty(\Omega):=\left\{h\in L^\infty(\Omega):h^->1\right\}.$$

Proposition 2.3 (Hölder-type inequality, see [18, 28]). *(i) Let* $p\in L_+^\infty(\Omega)$*. The conjugate space to* $L^{p(.)}(\Omega)$ *is* $L^{p'(.)}(\Omega)$*, where* $1/p(x)+1/p'(x)=1$ *for almost every (a.e.)* $x\in\Omega$*. Moreover, the following inequality hold*

$$\left|\int_\Omega u(x)\upsilon(x)dx\right|\leq 2\,|u|_{p(.)}\,|\upsilon|_{p'(.)}\,,$$

for all $u\in L^{p(.)}(\Omega)$ *and* $\upsilon\in L^{p'(.)}(\Omega)$*.*

(ii) If $p_1,\ p_2\in C_+(\overline{\Omega})$*,* $p_1(x)\leq p_2(x)$ *for any* $x\in\overline{\Omega}$*, then* $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$*, and the embedding is continuous.*

Proposition 2.4 (see [7]). *Assume that* $r\in L_+^\infty(\Omega)$ *and* $p\in C_+(\overline{\Omega})$*. If* $|u|^{r(.)}\in L^{p(.)}(\Omega)$*, then we have*

$$\min\left\{|u|_{r(.)p(.)}^{r^-},|u|_{r(.)p(.)}^{r^+}\right\}\leq\left||u|^{r(.)}\right|_{p(.)}\leq\max\left\{|u|_{r(.)p(.)}^{r^-},|u|_{r(.)p(.)}^{r^+}\right\}.$$

In particular, if $r(.)\equiv r$ *is a constant, then*

$$||u|^r|_{p(.)}=|u|_{rp(.)}^r\,.$$

Proposition 2.5 (see [20, 21, 28]). *Let* Ω *be a bounded domain in* $\mathbb{R}^N$*,* $p\in C(\overline{\Omega})$*,* $p^+<N$*. Then for any* $q\in L_+^\infty(\Omega)$ *with* $q\ll p^*=\begin{cases}\frac{Np(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\geq N\end{cases}$ *, there is a compact embedding* $E\hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$*, where the symbol* $q\ll p^*$ *to denote* $ess\inf_{x\in\overline{\Omega}}\{p^*(x)-q(x)\}>0$*.*

Lemma 2.6(see [22]). *Let* E *be a separable and reflexive Banach space. Let define the functional*

$$\psi(u)=\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}dx.$$

i) $\psi:E\to\mathbb{R}$ *is convex. The mapping* $\psi':E\to E^*$ *is a strictly monotone, bounded homeomorphism, and of* (S_+) *type, namely* $u_n\rightharpoonup u$ *(weakly) and* $\overline{\lim_{n\to\infty}}\langle\psi'(u_n),u_n-u\rangle\leq 0$ *implies* $u_n\to u$ *(strongly) on* E*;*

ii) The functional ψ *is well defined on* E;

iii) The functional ψ *is of* $C^1(E,\mathbb{R})$ *and*

$$\langle\psi'(u),v\rangle=\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla v dx,\ \text{for all } u,v\in E;$$

iv) The functional ψ *is weakly lower semi-continuous on* E*.*

Since the M function is continuous and bounded, Lemma 2.6 is implemented for $M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\psi(u)$. Since the proof of Lemma 2.6 is very similar to the proof of Theorem 3.1 given in [22], we omit it.

Let $p \in C_+\left(\overline{\Omega}\right)$ with condition (2.1). We assume that M and f satisfy the following conditions:

(M_1): $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continous function and satisfies the (polynomial growth) condition

$$m_1 t^{\beta-1} \leq M(t) \leq m_2 t^{\beta-1},$$

for all $t > 0$ and m_1, m_2 real numbers such that $m_2 \geq m_1 \geq 0$ and $\beta > 1$;

(f_1): $q \in C_+\left(\overline{\Omega}\right)$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists a constant $C_1 > 0$ such that

$$|f(x,t)| \leq C_1\left(1 + |t|^{q(x)-1}\right), \text{ for all } (x,t) \in \Omega \times \mathbb{R};$$

(f_2): $f(x,t) = o\left(|t|^{\beta p^+-1}\right)$ as $t \to 0$, uniformly in $x \in \Omega$;

(f_3): $\inf_{\{x\in\Omega, |t|=1\}} F(x,t) > 0$;

(f_4): $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$;

(AR): $\theta > \frac{m_2\beta\left(p^+\right)^\beta}{m_1\left(p^-\right)^{\beta-1}}$ such that

$$0 < \theta F(x,t) := \theta\int_0^t f(x,s)\,ds \leq t f(x,t), \ \forall t \in \mathbb{R}, \ x \in \Omega.$$

We are now in a position to state our main results.

3. Main results and proofs

Since our approach is variational, we define the Euler–Lagrange energy functional associated with the problem (P), $I : (E, \|u\|_E) \to \mathbb{R}$ by

$$I(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u)dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$. In a standard way, we find that $I \in C^1(E,\mathbb{R})$ and its Gateaux derivative is given by

$$\langle I'(u), v\rangle = M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx - \int_\Omega f(x,u)v dx,$$

for all $u, v \in E$.

Now we give the definition of weak solutions of Problem (1.1).

Definition 3.1. We call that $u \in E$ is a weak solution of (1.1), if

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega f(x,u)v dx,$$

for all $v \in E$.

It is clear that critical points of I are weak solutions of (1.1).

Theorem 3.2. *Assume that* (M_1) *and* (f_1) *conditions hold. If* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and* $q^+ < \beta p^- < (p^-)^*$, *then there exists a weak solution of Problem* (1.1).

Theorem 3.3. *If* M *satisfies* (M_1), f *satisfies* (f_1), (f_2), (f_3) *and* (AR). *If* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and* $\beta p^+ < q^-$. *Then Problem* (1.1) *has a nontrivial weak solution.*

Theorem 3.4. *Assume that* (M_1), (f_1), (f_2), (f_3), (f_4) *and* (AR) *hold. If* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *and* $\beta p^+ < q^-$, *then* I *has a sequence of critical points* (u_k) *such that* $I(u_k) \to \infty$ *as* $k \to \infty$, *and Problem* (1.1) *has infinite many pairs of solutions.*

Theorem 3.5. *Assume that the conditions* (M_1), (f_1), (f_2), (f_4) *hold, and* f *satisfies*

(f_5): $f(x,t) \geq a\,|t|^{\varrho(x)-1}$, $t \to 0$, *where* $\varrho \in C_+(\overline{\Omega})$, $\varrho^+ < \frac{m_2}{m_1}\beta p^-$ *for all* $x \in \Omega$.

Then problem (1.1) *has a sequence of solutions* $(\pm v_k)_{k=1}^{\infty}$ *such that* $I(\pm vk) < 0$, *and* $I(\pm vk) \to 0$ *as* $k \to \infty$.

Lemma 3.6. *Let* $p \in C_+\left(\overline{\Omega}\right)$.

i) The functional I *is well defined on* E;

ii) The functional I *is weakly lower semicontinuous on* E.

Since the proof of Lemma 3.6 is very similar to the proof of Lemma 2.7 given in [36], we omit it.

Proof of Theorem 3.2. By Proposition 2.5 $E \hookrightarrow L^{\beta p^+}(\Omega)$, $E \hookrightarrow L^{q(\cdot)}(\Omega)$ are continuous embeddings, there exist positive constants C_2, C_3, such that

$$|u|_{\beta p^+} \leq C_2 \|u\|_E, \ \forall u \in E, \tag{3.1}$$

and

$$|u|_{q(.)} \leq C_3 \|u\|_E, \ \forall u \in E. \tag{3.2}$$

From (M_1), we have

$$\frac{m_1}{\beta} t^\beta \leq \widehat{M}(t), \tag{3.3}$$

and from (f_1), we have

$$|F(x,t)| \leq C_4 + C_5 |t|^{q(x)}, \ \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.4}$$

Then, by using (3.2), (3.3), (3.4), Proposition 2.1, 2.2, with $\|u\|_E > 1$, we get

$$\begin{aligned} I(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) - \int_\Omega F(x,u)dx \\ &\geq \frac{m_1}{\beta}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right)^\beta - \int_\Omega \left(C_4 + C_5 |u|^{q(x)}\right) dx \\ &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p^-} - C_5 \max\left\{|u|_{q(.)}^{q^-}, |u|_{q(.)}^{q^+}\right\} - C_4 |\Omega| \\ &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p^-} - C_5 C_3 \max\left\{\|u\|_E^{q^-}, \|u\|_E^{q^+}\right\} - C_4 |\Omega| \\ &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p^-} - C_6 \|u\|_E^{q^+} - C_7. \end{aligned}$$

Since $\beta p^- > q^+$, then $I(u) \to \infty$ as $\|u\|_E \to \infty$. Therefore the functional I is bounded from below and coercive. Since I is weakly semicontinuous on E, I has minimum point u in E, and u is a weak solution of problem (1.1). The proof of Theorem 3.2 is completed. ■

Definition 3.7. We say that the functional I satisfies the Palais–Smale condition ($(PS)_c$, $c \in \mathbb{R}$ for short) if every sequence $(u_n) \subset E$ such that

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ as } n \to \infty \text{ in } E^*,$$

where E^* is the dual space of E, contains a convergent subsequence in the norm of E.

We have the following lemma.

Lemma 3.8. *Under the hypotheses of Theorem 3.3, I satisfies the $(PS)_c$ condition.*

Proof. Let assume that there is a Palais–Smale sequence (u_n) in E. Then, by (3.3), (AR), (M_1) and Proposition 2.2, we get

$$\begin{aligned}
& c+1+\|u_n\|_E \\
\geq\ & I(u_n) - \frac{1}{\theta}\left\langle I'(u_n), u_n \right\rangle \\
=\ & \widehat{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u_n)dx \\
& -\frac{1}{\theta}M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx + \frac{1}{\theta}\int_\Omega f(x,u_n)u_n dx \\
\geq\ & \frac{m_1}{\beta}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)^\beta \\
& -\frac{1}{\theta}M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
& -\int_\Omega \left(F(x,u_n) - \frac{1}{\theta}f(x,u_n)u_n\right)dx \\
\geq\ & \frac{m_1}{\beta\,(p^+)^\beta}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^\beta \\
& -\frac{m_2}{\theta\,(p^-)^{\beta-1}}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^{\beta-1}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right) \\
& -\int_\Omega \left(F(x,u_n) - \frac{1}{\theta}f(x,u_n)u_n\right)dx \\
\geq\ & \left(\frac{m_1}{\beta\,(p^+)^\beta} - \frac{m_2}{\theta\,(p^-)^{\beta-1}}\right)\|u_n\|_E^{\beta p^-} - C_8.
\end{aligned}$$

Hence, (u_n) is bounded in E. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $\langle I'(u_n), (u_n - u)\rangle \to 0$.

Thus we have

$$\begin{aligned} & \left\langle I'\left(u_n\right),\left(u_n-u\right)\right\rangle \\ = \; & M\left(\int_\Omega \frac{\left|\nabla u_n\right|^{p(x)}}{p(x)} dx\right)\left(\int_\Omega\left|\nabla u_n\right|^{p(x)-2} \nabla u_n\left(\nabla u_n-\nabla u\right) dx\right)-\int_\Omega f\left(x, u_n\right)\left(u_n-u\right) dx \rightarrow 0, \end{aligned} \tag{3.5}$$

since $\left(u_n\right)$ is bounded $\left|u_n\right|_{q(.)} \leq \widetilde{C}_1$. Therefore, from $\left(f_1\right)$, Propositions 2.3, 2.4, we get

$$\begin{aligned} & \left|\int_\Omega f\left(x, u_n\right)\left(u_n-u\right) dx\right| \\ \leq \; & C_1\left|\int_\Omega\left(1+\left|u_n\right|^{q(x)-1}\right)\left(u_n-u\right) dx\right| \\ = \; & C_1\left(\int_\Omega\left|u_n-u\right| dx+\int_\Omega\left|\left(\left|u_n\right|^{q(x)-1}\right)\left(u_n-u\right)\right| dx\right) \\ \leq \; & 2 C_1\left|u_n-u\right|_{q(.)}|1|_{q'(.)}+2 C_1\left|u_n^{q(x)-1}\right|_{q'(.)}\left|u_n-u\right|_{q(.)} \\ \leq \; & \widetilde{C_2}\left|u_n-u\right|_{q(.)}+2 C_1\left\{\left|u_n\right|_{q(.)}^{q^{+}-1}+\left|u_n\right|_{q(.)}^{q^{-}-1}\right\}\left|u_n-u\right|_{q(.)} \rightarrow 0. \end{aligned} \tag{3.6}$$

From Proposition 2.5, $\left(u_n\right)$ converges strongly to u in $L^{q(.)}(\Omega)$, that is $\left|u_n-u\right|_{q(.)} \longrightarrow 0$ as $n \rightarrow \infty$. From, (3.5) and (3.6) we get

$$M\left(\int_\Omega \frac{\left|\nabla u_n\right|^{p(x)}}{p(x)} dx\right) \int_\Omega\left(\left|\nabla u_n\right|^{p(x)-2} \nabla u_n\left(\nabla u_n-\nabla u\right)\right) dx \rightarrow 0.$$

From condition $\left(M_1\right)$, it follows

$$\int_\Omega\left|\nabla u_n\right|^{p(x)-2} \nabla u_n\left(\nabla u_n-\nabla u\right) dx \rightarrow 0.$$

Eventually, by Lemma 2.6 (i), we get $u_n \rightarrow u$ in E. In conclusion, I satisfies the $(PS)_c$ condition. The proof of Lemma 3.8 is completed. ■

We will use the following "mountain pass theorem" to prove Theorem 3.3.

Lemma 3.9 (Mountain pass theorem see [39]**).** *Let* E *be a Banach space,* $I \in C^1(E, \mathbb{R})$ *with* $I(0)=0$. *Suppose that:*

$\left(H_1\right)$ *There exist* $\alpha, r>0$ *such that* $I(u) \geq \alpha>0$, $u \in E$ *with* $\|u\|_E=r$*;*

$\left(H_2\right)$ *There exists* $e \in E$ *such that* $\|e\|_E>r$, $I(e)<0$.

Then there is a sequence $\left(u_n\right) \subset E$ *such that*

$$I\left(u_n\right) \rightarrow c \text{ and } I'\left(u_n\right) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ in } E^*,$$

where

$$0<c=\inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \quad ,$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Lemma 3.10. *Suppose M satisfies (M_1), f satisfies (f_1), (f_2). Then the following statements hold:*

i) There exist two positive real numbers ρ and α such that $I(u) \geq \alpha > 0$, $u \in E$ with $\|u\|_E = \rho \in (0,1)$;

ii) There exists $u \in E$ such that $\|u\|_E > \rho$, $I(u) < 0$.

Proof. *i)* Let $\|u\|_E < 1$. Then by (M_1), we have

$$\begin{aligned} I(u) &= \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) - \int_\Omega F(x,u)dx \\ &\geq \frac{m_1}{\beta}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right)^\beta - \int_\Omega F(x,u)dx \\ &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p+} - \int_\Omega F(x,u)dx. \end{aligned}$$

Let ε, $\varepsilon_1 > 0$ be small enough such that $0 < \varepsilon_1 < \frac{m_1}{2\beta (p^+)^\beta C_2^{\beta p^+}}$. It follows from (f_1) and (f_2) for any ε, $\varepsilon_1 > 0$, choose C_ε, $C_{\varepsilon_1} > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{\beta p^+ - 1} + C_\varepsilon |t|^{q(x)-1}, \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

and

$$|F(x,t)| \leq \varepsilon_1 |t|^{\beta p^+} + C_{\varepsilon_1} |t|^{q(x)}, \text{ for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.7}$$

Therefore, from (3.1), (3.2), (3.7) and Proposition 2.2, we get

$$\begin{aligned} I(u) &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p^+} - \int_\Omega F(x,u)dx \\ &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p^+} - \varepsilon_1 \int_\Omega |u|^{\beta p^+} dx - C_{\varepsilon_1} \int_\Omega |u|^{q(x)} dx \\ &\geq \frac{m_1}{\beta (p^+)^\beta} \|u\|_E^{\beta p^+} - \varepsilon_1 C_2^{\beta p^+} \|u\|_E^{\beta p^+} - C_{\varepsilon_1} C_3^{q^-} \|u\|_E^{q^-} \\ &\geq \left(\frac{m_1}{\beta (p^+)^\beta} - \varepsilon_1 C_2^{\beta p^+}\right) \|u\|_E^{\beta p^+} - C_9 \|u\|_E^{q^-} \\ &\geq \frac{m_1}{2\beta (p^+)^\beta} \rho^{\beta p^+} - C_9 \rho^{q^-} = \rho^{\beta p^+}\left(\frac{m_1}{2\beta (p^+)^\beta} - C_9 \rho^{q^- - \beta p^+}\right), \end{aligned}$$

Since $\|u\|_E < 1$ and $\beta p^+ < q^-$, the function $t \to \frac{m_1}{2\beta (p^+)^\beta} - C_9 \rho^{q^- - \beta p^+}$ is strictly positive in a neighborhood of zero. It follows that there exist two positive real numbers ρ and α such that $I(u) \geq \alpha > 0$, $u \in E$ with $\|u\|_E = \rho \in (0,1)$.

$ii)$ On the other hand, when $s \geq 1$, from condition (M_1) we can easily obtain

$$\widehat{M}(s) \leq m_2 \frac{s^\beta}{\beta} \leq \frac{m_2}{\beta} s^{\frac{m_2}{m_1}\beta}. \tag{3.8}$$

For each $x \in \Omega$ and $t \in \mathbb{R}$, let us define the function

$$\eta_1(s) = s^{-\theta} F(x, st) - F(x,t),$$

for all $s \geq 1$. Then we deduce from (AR) that

$$\eta_1'(s) = s^{-\theta} \left[f(x, st)t - \frac{\theta}{s} F(x,t) \right] \geq 0,$$

for all $s \geq 1$. So η_1 is increasing function on $[1, +\infty)$ and $\eta_1(\tau) \geq \eta_1(1) = 0$ for all $\tau \in [1, +\infty)$. Hence,

$$F(x, st) \geq s^\theta F(x,t), \ \forall x \in \Omega, \ t \in \mathbb{R}, \ s \geq 1. \tag{3.9}$$

Thus, for any fixed $\omega \in E$, $\omega \geq 0$, $\omega \neq 0$, $s \geq 1$ and from (3.8) and (3.9) conditions, we get

$$\begin{aligned} I(s\omega) &= \widehat{M}\left(\int_\Omega \frac{|\nabla (s\omega)|^{p(x)}}{p(x)} dx \right) - \int_\Omega F(x, s\omega) dx \\ &\leq \frac{m_2}{\beta} \left(\int_\Omega \frac{1}{p(x)} |\nabla (s\omega)|^{p(x)} dx \right)^{\frac{m_2}{m_1}\beta} - \int_\Omega F(x, s\omega) dx \\ &\leq \frac{m_2}{\beta (p^-)^{\frac{m_2}{m_1}\beta}} s^{\frac{m_2}{m_1}\beta p^+} \left(\int_\Omega |\nabla \omega|^{p(x)} dx \right)^{\frac{m_2}{m_1}\beta} - s^\theta \int_\Omega F(x, \omega) dx. \end{aligned}$$

By (AR), we have $\theta > \frac{m_2 \beta (p^+)^\beta}{m_1 (p^-)^{\beta - 1}} > \frac{m_2}{m_1} \beta p^+$ then we get $I(s\omega) \to -\infty$ as $s \to +\infty$. The proof of Lemma 3.10 is completed. ■

Proof of Theorem 3.3. From Lemma 3.8, Lemma 3.10, Lemma 2.6 (vi), and the fact that $I(0) = 0$, I satisfies the mountain pass theorem. Therefore, I has at least one nontrivial critical point, i.e. Problem (1.1) has a nontrivial weak solution. The proof of Theorem 3.3 is completed. ■

We will use the following "fountain theorem" to prove Theorem 3.4.

Because E is reflexive and separable Banach space, then there are $e_j \in E$ and $e_j^* \in E^*$ such that

$$E = \overline{span\{e_j : j = 1, 2, ...\}}, \ E^* = \overline{span\{e_j^* : j = 1, 2, ...\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

For convenience, we write $E_j = span\{e_j\}$, $Y_k = \oplus_{j=1}^k E_j$, $Z_k = \overline{\oplus_{j=k}^\infty E_j}$.

Lemma 3.11 (see [22]). *If* $\alpha \in C_+\left(\overline{\Omega}\right)$, $\alpha(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *denote*

$$\gamma_k = \sup\{|u|_{\alpha(.)} : \|u\|_{W_0^{1,p(.)}(\Omega)} = 1, \ u \in Z_k\},$$

then $\lim_{k\to\infty}\gamma_k = 0$.

Lemma 3.12 (Fountain theorem see [39]**).** *Assume that*

(A_1) *E is a Banach space, $I \in C^1(E,\mathbb{R})$ is an even functional;*

If for each $k = 1,\ 2,\ \ldots$ there exist $\rho_k > r_k > 0$ such that

(A_2) $a_k = \max_{u\in Y_k, \|u\|_E=\rho_k} I(u) \le 0$;

(A_3) $b_k = \inf_{u\in Z_k, \|u\|_E=r_k} I(u) \to \infty$ *as* $k\to\infty$;

(A_4) *I satisfies $(PS)_c$ condition for every $c > 0$,*

then I has a sequence of critical values tending to $+\infty$.

Proof of Theorem 3.4. According to (f_4) and Lemma 3.8, I is an even functional and satisfies $(PS)_c$ condition. We only need to prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_2) and (A_3) hold. Thus, the conclusion of Theorem 3.4 can be obtained from fountain theorem.

Verification of (A_2). For any $(x,t)\in\Omega\times\mathbb{R}$, set $\eta_2(\xi) = F(x,\xi^{-1}t)\xi^{\theta},\ \forall\xi\ge 1$. By (AR), we have

$$\begin{aligned}\eta_2'(\xi) &= -f(x,\xi^{-1}t)t\xi^{\theta-2} + \theta F(x,\xi^{-1}t)\xi^{\theta-1}\\ &= \xi^{\theta-1}\left[\theta F(x,\xi^{-1}t) - \xi^{-1}f(x,\xi^{-1}t)t\right] \le 0,\end{aligned}$$

so, η_2 is a nonincreasing function on $[1,+\infty)$. Thus, for any $|t|\ge 1$, we have $\eta_2(1)\ge\eta_2(|t|)$, that is

$$F(x,t) \ge F(x,t^{-1}t)\,|t|^{\theta} \ge \widehat{C}\,|t|^{\theta} \text{ for all } x\in\Omega, \tag{3.10}$$

where $\widehat{C} = \inf_{\{x\in\Omega,|t|=1\}} F(x,t) > 0$ given in (f_3) condition. Also, by (f_2), there exist constants $C_{10} > 0$, $t^* > 0$ such that

$$|F(x,t)| \le C_{10}\,|t|^{\beta p^+},\ \forall x\in\Omega,\ 0<|t|\le t^*, \tag{3.11}$$

and by (f_1), there exists $C_{11} > 0$ such that

$$\begin{aligned}|F(x,t)| &\le C_{11}\left(C_4 + C_5\,|t|^{q(x)}\right) \le C_{11}\left(C_4 + C_5\,|t|^{q^-}\right)\\ &\le C_{11}\,|t|^{\beta p^+},\ \forall x\in\Omega,\ t^*\le|t|\le 1.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$F(x,t) \ge -\left(C_{10}+C_{11}\right)|t|^{\beta p^+}, \forall x\in\Omega,\ \ |t|\in[0,1], \tag{3.13}$$

then we get from (3.10) and (3.13) that

$$F(x,t) \ge \widehat{C}\,|t|^{\theta} - C_{12}\,|t|^{\beta p^+}, \forall x\in\Omega,\ t\in\mathbb{R}, \tag{3.14}$$

Since $p^+ < \beta p^+ < (p^-)^*$, by using (3.8), (3.14) when $\rho_k \geq 1$, we get

$$\begin{aligned}
& I(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u)dx \\
\leq\ & \frac{m_2}{\beta}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)^{\frac{m_2}{m_1}\beta} - \int_\Omega \left(\widehat{C}\,|u|^\theta - C_{12}\,|u|^{\beta p^+}\right)dx \\
\leq\ & \frac{m_2}{\beta\,(p^-)^{\frac{m_2}{m_1}\beta}}\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{\frac{m_2}{m_1}\beta} - \widehat{C}\int_\Omega |u|^\theta\,dx + C_{12}\int_\Omega |u|^{\beta p^+}\,dx \\
\leq\ & \frac{m_2}{\beta\,(p^-)^{\frac{m_2}{m_1}\beta}}\,\|u\|_E^{\frac{m_2}{m_1}\beta p^+} - \widehat{C}\,\|u\|_E^\theta + C_{12}\,\|u\|_E^{\beta p^+} \\
\leq\ & \frac{C^0 m_2}{\beta\,(p^-)^{\frac{m_2}{m_1}\beta}}\,\|u\|_E^{\frac{m_2}{m_1}\beta p^+} - \widehat{C}\,\|u\|_E^\theta.
\end{aligned}$$

Because all norms are equivalent on the finite dimensional space $Y_k = k$, and $\theta > \frac{m_2}{m_1}\beta p^+ > 1$, there exists $\rho_k \geq 1$ large enough such that

$$a_k = \max_{u\in Y_k,\|u\|_E=\rho_k} I(u) \leq 0.$$

Hence, (A_2) holds.

Verification of (A_3). For any $u \in Z_k$, $\|u\|_E = r_k = \left(C_{13}C_3^{q^+}q^+{\gamma_k}^{q^+}m_1^{-1}(p^+)^\beta\right)^{\frac{1}{\beta p^- - q^+}} \geq 1$, by using (3.2), (3.4) and Proposition 2.1, we have

$$\begin{aligned}
& I(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,u)dx \\
\geq\ & \frac{m_1}{\beta}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)^{\beta} - C_{13}\int_\Omega \left(C + |u|^{q(x)}\right)dx \\
\geq\ & \frac{m_1}{\beta\,(p^+)^\beta}\,\|u\|_E^{\beta p^-} - C_{13}\max\left\{|u|_{q(.)}^{q^-}, |u|_{q(.)}^{q^+}\right\} - C_{13}C\,|\Omega| \\
\geq\ & \frac{m_1}{\beta\,(p^+)^\beta}\,\|u\|_E^{\beta p^-} - C_{13}\max\left\{C_3^{q^-}\,\|u\|_E^{q^-}, C_3^{q^+}\,\|u\|_E^{q^+}\right\} - C_{14} \\
\geq\ & \frac{m_1}{\beta\,(p^+)^\beta}\,\|u\|_E^{\beta p^-} - C_{13}C_3^{q^+}{\gamma_k}^{q^+}\,\|u\|_E^{q^+} - C_{14} \\
=\ & \frac{m_1}{\beta\,(p^+)^\beta}\left(C_{13}C_3^{q^+}q^+{\gamma_k}^{q^+}m_1^{-1}(p^+)^\beta\right)^{\frac{\beta p^-}{\beta p^- - q^+}} \\
& -C_{13}C_3^{q^+}{\gamma_k}^{q^+}\left(C_{13}C_3^{q^+}q^+{\gamma_k}^{q^+}m_1^{-1}(p^+)^\beta\right)^{\frac{q^+}{\beta p^- - q^+}} - C_{14} \\
=\ & \frac{m_1}{(p^+)^\beta}\left(\frac{1}{\beta} - \frac{1}{q^+}\right)r_k^{\frac{\beta p^-}{\beta p^- - q^+}} - C_{14}.
\end{aligned}$$

Because of $\gamma_k \to 0$ as $k \to \infty$ and $\beta < \beta p^- < q^+$, we have $r_k \to \infty$ as $k \to \infty$. Then for $u \in Z_k$ with $\|u\|_E = r_k$ such that $\rho_k > r_k > 0$, we get

$$b_k = \inf_{u\in Z_k, \|u\|_E = r_k} I(u) \geq \frac{m_1}{(p^+)^\beta}\left(\frac{1}{\beta} - \frac{1}{q^+}\right) r_k^{\frac{\beta p^-}{\beta p^- - q^+}} - C_{14} \to \infty$$

as $k \to \infty$.

So, (A_3) holds. The proof of Theorem 3.4 is completed. ∎

Lemma 3.13 (Dual fountain theorem, see [39]). *Suppose that* $I \in C^1(X, \mathbb{R})$ *satisfies* $I(-u) = I(u)$, *and for every* $k \geq k_0$, *there exist* $\rho_k > r_k > 0$ *such that*

(B_1) $c_k = \inf_{u\in Z_k, \|u\|_X = \rho_k} I(u) \geq 0$;

(B_2) $d_k = \max_{u\in Y_k, \|u\|_X = r_k} I(u) < 0$;

(B_3) $e_k = \inf_{u\in Z_k, \|u\|_X \leq \rho_k} I(u) \to 0$ *as* $k \to \infty$;

(B_4) I *satisfies* $(PS)_c^*$ *condition for every* $c \in [e_{k_0}, 0)$.

Then I *has a sequence of negative critical values converging to* 0.

Definition 3.14. We say that I satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $(u_{n_j}) \subset X$ such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $\left(I\,|_{Y_{n_j}}\right)'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of I.

Lemma 3.15. *Assume that the conditions* (M_1), (f_1) *and* (AR) *hold, then* I *satisfies the* $(PS)_c^*$ *condition.*

Proof. Suppose $(u_{n_j}) \subset E$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to \widetilde{c}$ and $\left(I\,|_{Y_{n_j}}\right)'(u_{n_j}) \to 0$. Similar to the process of verifying the $(PS)_c$ condition in the proof of Lemma 3.8, we can get the boundedness of $\|u_{n_j}\|_E$. Going, if necessary, to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in E. As $E = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}
\lim_{n_j\to\infty} \left\langle I'(u_{n_j}), (u_{n_j} - u)\right\rangle &= \lim_{n_j\to\infty} \left\langle I'(u_{n_j}), (u_{n_j} - v_{n_j})\right\rangle + \lim_{n_j\to\infty} \left\langle I'(u_{n_j}), (v_{n_j} - u)\right\rangle \\
&= \lim_{n_j\to\infty} \left\langle \left(I\,|_{Y_{n_j}}\right)'(u_{n_j}), (u_{n_j} - v_{n_j})\right\rangle = 0.
\end{aligned}$$

As I' is of (S_+) type, we can conclude that $u_{n_j} \to u$, furthermore we have $I'(u_{n_j}) \to I'(u)$.

Now, let us prove $I'(u) = 0$ below. Taking arbitrarily $w_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned}
\langle I'(u), w_k\rangle &= \left\langle I'(u) - I'(u_{n_j}), w_k\right\rangle + \left\langle I'(u_{n_j}), w_k\right\rangle \\
&= \left\langle I'(u) - I'(u_{n_j}), w_k\right\rangle + \left\langle \left(I\,|_{Y_{n_j}}\right)'(u_{n_j}), w_k\right\rangle.
\end{aligned}$$

Going to limit in the right side of above equation, we reach that

$$\langle I'(u), w_k\rangle = 0, \ \forall w_k \in Y_k.$$

So, $I'(u) = 0$. This shows that I satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. ∎

Proof of Theorem 3.5. We will prove the Theorem 3.5 with the help of Lemma 3.13.

From (f_4) and Lemma 3.15, we know that $I \in C^1(E,\mathbb{R})$ is an even functional and I satisfies (B_4), respectively.

Verification of (B_1). For any $v \in Z_k$, $\|v\|_E = 1$ and $0 < t < 1$, by using (3.7), (M_1) and Proposition 2.1, we have

$$
\begin{aligned}
& I(tv) = \widehat{M}\left(\int_\Omega \frac{|\nabla (tv)|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,tv)dx \\
\geq\ & \frac{m_1}{\beta}\left(\int_\Omega \frac{|\nabla (tv)|^{p(x)}}{p(x)}dx\right)^{\beta} - \varepsilon t^{\beta p^+}\int_\Omega |v|^{\beta p^+}dx - C_\varepsilon t^{q^-}\int_\Omega |v|^{q(x)}dx \\
\geq\ & \frac{m_1}{\beta (p^+)^{\beta}} t^{\beta p^+}\left(\int_\Omega |\nabla v|^{p(x)}dx\right)^{\beta} - \varepsilon t^{\beta p^+}\int_\Omega |v|^{\beta p^+}dx - C_\varepsilon t^{q^-}\int_\Omega |v|^{q(x)}dx \\
\geq\ & \frac{m_1}{\beta (p^+)^{\beta}} t^{\beta p^+}\|v\|_E^{\beta p^+} - \varepsilon C_2^{\beta p^+} t^{\beta p^+}\|v\|_E^{\beta p^+} - \begin{cases} C_\varepsilon t^{q^-}\gamma_k^{q^-}\|v\|_E^{q^-} & \text{if } |u|_{q(.)} \leq 1, \\ C_\varepsilon t^{q^-}\gamma_k^{q^+}\|v\|_E^{q^-} & \text{if } |u|_{q(.)} > 1, \end{cases} \\
\geq\ & \left(\frac{m_1}{\beta (p^+)^{\beta}} - \varepsilon C_2^{\beta p^+}\right) t^{\beta p^+} - \begin{cases} C_\varepsilon t^{q^-}\gamma_k^{q^-} & \text{if } |u|_{q(.)} \leq 1, \\ C_\varepsilon t^{q^-}\gamma_k^{q^+} & \text{if } |u|_{q(.)} > 1, \end{cases} \\
\geq\ & \left(\frac{m_1}{\beta (p^+)^{\beta}} - \varepsilon C_2^{\beta p^+}\right) t^{\beta p^+} - C_\varepsilon t^{q^-}\gamma_k^{q^-}.
\end{aligned} \tag{3.15}
$$

Let $\varepsilon > 0$ be small enough such that $0 < \varepsilon < \frac{m_1}{2\beta (p^+)^{\beta} C_2^{\beta p^+}}$ similarly proof of Lemma 3.10 (i). Thus, from (3.15), we get

$$
I(tv) \geq \frac{m_1}{2\beta (p^+)^{\beta}} t^{\beta p^+} - C_\varepsilon t^{q^-}\gamma_k^{q^-}. \tag{3.16}
$$

Since $q^- > \beta p^+$, taking $t = \rho_k$ small enough and sufficiently large k, for $v \in Z_k$ with $\|v\|_E = 1$ we have $I(tv) \geq 0$. So for sufficiently large k,

$$
c_k = \inf_{u \in Z_k, \|u\|_E = \rho_k} I(u) \geq 0,
$$

thus, (B_1) is satisfied.

Verification of (B_2). For $v \in Y_k$ with $\|v\|_E = 1$ and $0 < t < \rho_k < 1$, by using (f_5) and (M_1), we get

$$\begin{aligned} I(tv) &= \widehat{M}\left(\int_\Omega \frac{|\nabla(tv)|^{p(x)}}{p(x)}dx\right) - \int_\Omega F(x,tv)dx \\ &\le \frac{m_2}{\beta}\left(\int_\Omega \frac{|\nabla(tv)|^{p(x)}}{p(x)}dx\right)^{\frac{m_2}{m_1}\beta} - a\int_\Omega \frac{|tv|^{\varrho(x)}}{\varrho(x)}dx \\ &\le \frac{m_2}{\beta(p^-)^{\frac{m_2}{m_1}\beta}}t^{\frac{m_2}{m_1}\beta p^-}\left(\int_\Omega |\nabla v|^{p(x)}dx\right)^{\frac{m_2}{m_1}\beta} - \frac{at^{\varrho^+}}{\varrho^+}\int_\Omega |v|^{\varrho(x)}dx \\ &\le \frac{m_2}{\beta(p^-)^{\frac{m_2}{m_1}\beta}}t^{\frac{m_2}{m_1}\beta p^-} - \frac{at^{\varrho^+}}{\varrho^+}. \end{aligned}$$

Since $\varrho^+ < \frac{m_2}{m_1}\beta p^-$ implies that there exists a $r_k \in (0,\rho_k)$ such that $I(tv) < 0$ when $t = r_k$. Hence we get

$$d_k = \max_{u\in Y_k, \|u\|_E = r_k} I(u) < 0,$$

so (B_2) is satisfied.

Verification of (B_3). Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$e_k := \max_{u\in Z_k, \|u\|_E \le \rho_k} \inf I(u) \le b_k := \max_{u\in Y_k, \|u\|_E = r_k} I(u) < 0.$$

From (3.16), for $v \in Z_k$, $\|v\|_E = 1$, $0 \le t \le \rho_k$ and $u = tv$, we have

$$I(u) = I(tv) \ge \frac{m_1}{2\beta(p^+)^\beta}t^{\beta p^+} - C_\varepsilon t^{q^-}\gamma_k^{q^-} \ge -C_\varepsilon t^{q^-}\gamma_k^{q^-}.$$

Hence $e_k \to 0$, i.e., (B_3) is satisfied.

The conclusion of Theorem 3.5 is reached by the Dual fountain theorem. ■

Acknowledgement

Thanks are extended to the anonymous referees for their helpful suggestions on the quality improvement of our paper.

References

[1] Alves CO, Corrêa FJSA, Ma TF. Positive solutions for a quasilinear elliptic equation of Kirchhoff type. Computers & Mathematics with Applications 2005; 49 (1): 85-93.

[2] Antontsev SN, Rodrigues Jos F. On stationary thermo-rheological viscous ows. Annali dell'Universita di Ferrara 2006; 52 (1): 19-36.

[3] Arosio A, Pannizi S. On the well-posedness of the Kirchhoff string. Transactions of the American Mathematical Society 1996; 348: 305–330.

[4] Avci M, Cekic B, Mashiyev RA. Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory. Mathematical Methods in the Applied Sciences 2011; 34: 1751-1759.

[5] Boureanu MM, Rădulescu V, Repovš D. On a $p(\cdot)$-biharmonic problem with no-flux boundary condition. Computers and Mathematics with Applications 2016; 72: 2505-2515.

[6] Cavalcante MM, Cavalcante VN, Soriano JA. Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation. Advances in Differential Equations 2001; 6: 701-730.

[7] Cekic B, Mashiyev RA. Existence and Localization Results for $p(x)$-Laplacian via Topological Methods. Fixed Point Theory and Applications 2010; 2010 (1): 1-7.

[8] Chen Y, Levine S, Rao M. Variable exponent, linear growth functionals in image restoration. SIAM Journal on Applied Mathematics 2006; 66 (4): 1383-1406.

[9] Cheng B, Wu X. Existence results of positive solutions of Kirchhoff type problems. Nonlinear Analysis: Theory, Methods & Applications 2009; 71: 4883-4892.

[10] Chipot M, Rodrigues JF. On a class of nonlocal nonlinear elliptic problems. Mathematical Modelling and Numerical Analysis 1992; 26: 447-467.

[11] Chung NT. Multiplicity results for a class of $p(x)$-Kirchhoff-type equations with combined nonlinearities. Electronic Journal of Qualitative Theory of Differential Equations 2012; 42: 1-13.

[12] Chung NT. Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities. Complex Variables and Elliptic Equations 2013; 58 (12): 1637-1646.

[13] Corrêa FJSA, Figueiredo GM. On an elliptic equation of p-Kirchhoff type via variational methods. Bulletin of the Australian Mathematical Society 2006; 74 (2): 263-277.

[14] Dai G, Hao R. Existence of solutions for a $p(x)$-Kirchhoff-type equation. Journal of Mathematical Analysis and Applications 2009; 359: 275-284.

[15] Dai G, Liu D. Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation. Journal of Mathematical Analysis and Applications 2009; 359: 704-710.

[16] Dai G, Ma R. Solutions for a $p(x)$-Kirchhoff type equation with Neumann boundary data. Nonlinear Analysis: Reel World Applications 2011; 12 (5): 2666-2680.

[17] D'Ancona P, Spagnolo S. Global solvability for the degenerate Kirchhoff equation with real analytic data. Inventiones Mathematicae 1992; 108 (1): 247-262.

[18] Diening L, Harjulehto P, Hästö P, Růžička M. Lebesgue and Sobolev spaces with variable exponents. Berlin, Germany: Springer-Verlag, 2011.

[19] Ekincioglu I, Ayazoglu (Mashiyev) R. On Steklov boundary value problems for $p(x)$-Laplacian equations. Electronic Journal of Mathematical Analysis and Applications 2017; 5 (2): 289-297.

[20] Fan XL, Shen JS, Zhao D. Sobolev embedding theorems for spaces $W^{1,p(x)}(\Omega)$. Journal of Mathematical Analysis and Applications 2001; 262: 749-760.

[21] Fan XL, Zhao D. On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. Journal of Mathematical Analysis and Applications 2001; 263: 424-446.

[22] Fan XL, Zhang QH. Existence of solutions for $p(x)$-Laplacian Dirichlet problems. Nonlinear Analysis: Theory, Methods & Applications 2003; 52: 1843-1852.

[23] Guo E, Zhao P. Existence and multiplicity of solutions for nonlocal $p(x)$-Laplacian problems in $\mathbb{R}^N$. Boundary Value Problems 2012; 2012 (1): 1-15.

[24] He X, Zou W. Infinitely many positive solutions for Kirchhoff-type problems. Nonlinear Analysis: Theory, Methods & Applications 2009; 70: 1407-1414.

[25] Jin J, Wu X. Infinitely many radial solutions for Kirchhoff-type problems in $\mathbb{R}^N$. Journal of Mathematical Analysis and Applications 2010; 369 (2): 564-574.

[26] Kefi K. $p(x)$-Laplacian with indefinite weight. Proceedings of the American Mathematical Society 2011; 139 (12): 4351-4360.

[27] Kirchhoff G. Vorlesungen Uber Mechanik. Leipzig, Germany: Teubner, 1883.

[28] Kováčik O, Rákosnik J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Mathematical Journal 1991; 41 (116): 592-618.

[29] Lions JL. On some equations in boundary value problems of mathematical physics. In: North-Holland Mathematics Studies 1978; 30: 284-346.

[30] Liu D. On a p-Kirchhoff equation via fountain theorem and dual fountain theorem. Nonlinear Analysis: Theory, Methods & Applications 2010; 72 (1): 302-308.

[31] Liu D, Zhao P. Multiple nontrivial solutions to a p-Kirchhoff equation. Nonlinear Analysis: Theory, Methods & Applications, 2012; 75 (13): 5032-5038.

[32] Mao A, Zhang Z. Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition. Nonlinear Analysis: Theory, Methods & Applications 2009; 70: 1275-1287.

[33] Mashiyev RA. Some properties of variable Sobolev capacity. Taiwanese Journal of Mathematics 2008; 12 (3): 671-678.

[34] Mashiyev RA, Alisoy G, Ogras S. Some properties of the first eigenvalue of the $p(x)$-Laplacian on Riemannian manifolds. Turkish Journal of Mathematics 2009; 33: 351-358.

[35] Mashiyev RA, Çekiç B, Buhrii OM. Existence of solutions for $p(x)$-Laplacian equations. Electronic Journal of Qualitative Theory of Differential Equations 2010; 65: 1-13.

[36] Mashiyev RA, Çekiç B, Avci M, Yücedağ Z. Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition. Complex Variables and Elliptic Equations 2012; 57 (5): 579-595.

[37] Pohozaev S. On a class of quasilinear hyperbolic equations. Matematicheskii Sbornik 1975; 96: 152-166.

[38] Ružička M. Electrorheological Fluids: Modeling and Mathematical Theory. Berlin, Germany: Springer-Verlag, 2002.

[39] Willem M. Minimax Theorems. Basel: Bierkhauser,1996.

[40] Yucedag Z, Avci M, Mashiyev RA.On an Elliptic System of $p(x)$-Kirchhoff-type under Neumann Boundary Condition. Mathematical Modelling and Analysis 2012; 17 (2): 161-170.

[41] Zhang Z, Perera K. Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow. Journal of Mathematical Analysis and Applications 2006; 317: 456-463.

[42] Zhikov VV. Averaging of functionals of the calculus of variations and elasticity theory. Mathematics of the USSR-Izvestiya 1987; 29 (1): 33-66.