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Li-Yorke chaos and topological distributional chaos in a sequence

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Abstract: We study here the topological notion of Li-Yorke chaos defined for uniformly continuous self-maps defined on uniform Hausdorff spaces, which are not necessarily compact metrizable. We prove that a weakly mixing uniformly continuous self-map defined on a second countable Baire uniform Hausdorff space without isolated points is Li-Yorke chaotic. Further, we define and study the notion of topological distributional chaos in a sequence for uniformly continuous self-maps defined on uniform Hausdorff spaces. We prove that Li-Yorke chaos is equivalent to topological distributional chaos in a sequence for uniformly continuous self-maps defined on second countable Baire uniform Hausdorff space without isolated points. As a consequence, we obtain that Devaney chaos implies topological distributional chaos in a sequence.

Key words: Devaney chaos, distributional chaos in a sequence, Li-Yorke chaos, uniform space, weakly mixing

1. Introduction

The idea that many simple nonlinear deterministic systems can behave in an apparently unpredictable and chaotic manner was first noticed by the great French mathematician Henri Poincaré. However, the term chaos in connection with a map was first used by Li and Yorke [9]. Since then various definitions of chaos have been introduced and studied. A common idea of them is to describe the complexity and unpredictability of the behavior of the orbits. Devaney gave another widely accepted definition of chaos, popularly known as Devaney's chaos [4]. Huang and Ye proved that the notion of Devaney chaos is stronger than Li-Yorke chaos [6].

Various extensions of the definition of Li-Yorke chaos have been studied, for example, dense chaos, generic chaos. The above-cited extensions were mainly based on the size of a scrambled set. In 1994, Schweizer and Smítal extended Li-Yorke's approach by introducing the notion of distributional chaos, which involves a probabilistic measure of the distance between trajectories of points [11]. Since then it has evolved into three variants of the so-called distributional chaos namely $DC1$, $DC2$, and $DC3$ (ordered from strongest to weakest). Note that $DC1$, $DC2$, and $DC3$ are all equivalent for continuous self-maps defined on intervals. There are examples justifying that Li-Yorke chaos and Devaney chaos need not imply any version of distributional chaos [10, 11]. Wang et al. introduced a generalized version of distributional chaos, popularly known as distributional chaos in a sequence in [13]. For continuous self-maps defined on the intervals, Li-Yorke chaos and distributional chaos in a sequence are equivalent.

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Recently, various notions of chaos are extended to general topological spaces and their interrelations are being explored. Devaney chaos and Li-Yorke chaos are studied for group actions on uniform spaces in [2]. The notions of shadowing and specification for uniformly continuous self-maps on uniform spaces are studied in [3]. The concepts of topological shadowing and topological chain transitivity are further explored in [1]. In [12], the authors have defined the topological notion of distributional chaos for uniformly continuous self-maps on uniform spaces and explored the relation between topological notions of weak specification and distributional chaos. Wu et al. have extended the classical Auslander-Yorke dichotomy theorem for uniformly continuous maps defined on uniform spaces [15].

The main objective of this paper is to define and study the topological notion of distributional chaos in a sequence for uniformly continuous maps defined on uniform spaces. The paper comprises four sections. In section 2, we introduce the basic definitions and terminologies required for the development of the paper. In section 3, we revisit the notion of Li-Yorke chaos defined for uniformly continuous self-maps on uniform Hausdorff spaces. We prove that a weakly mixing uniformly continuous self-map defined on a second countable Baire uniform Hausdorff space without isolated points is Li-Yorke chaotic. In section 4, we define the notion of topological distributional chaos in a sequence for uniformly continuous self-maps defined on uniform spaces and study the equivalence of Li-Yorke chaos and topological distributional chaos in a sequence for uniformly continuous self-maps defined on second countable Baire uniform Hausdorff spaces without isolated points. As a consequence, we obtain that Devaney chaos implies topological distributional chaos in a sequence for uniform homeomorphism defined on second countable Baire uniform Hausdorff spaces without isolated points.

2. Preliminaries

For completion, we give here the metric definitions of Li-Yorke chaos and distributional chaos in a sequence.

2.1. Li-Yorke chaos

Let f be a continuous self-map defined on a metric space (X, d). A set $S \subset X$ containing at least two points is called a Li-Yorke scrambled set if for any two distinct points x, y in S,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \tag{2.1}$$

The function f is said to be chaotic in the sense of Li-Yorke if there exists an uncountable Li-Yorke scrambled set [9]. A pair (x, y) satisfying (2.1) is called a Li-Yorke scrambled pair. The definition of a scrambled pair can be rephrased in the following manner: A pair (x, y) is a Li-Yorke scrambled pair if there exist an increasing sequence $\{n_i\}$ such that $d(f^{n_i}(x), f^{n_i}(y)) \to 0$ as $n_i \to \infty$, another increasing sequence $\{m_i\}$ and a positive number $\epsilon > 0$ such that $d(f^{m_i}(x), f^{m_i}(y)) \geqslant \epsilon$, for all $i \in \mathbb{N}$.

2.2. Distributional chaos in a sequence

Let f be a continuous self-map defined on a metric space (X, d) and let $\{p_i\}$ be an increasing sequence of positive integers. For $x, y \in X$ and $t > 0$, let

$$F_{xy}(t, \{p_i\}) = \liminf_{n\to\infty} \frac{1}{n} |\{0 \leqslant i < n \mid d(f^{p_i}(x), f^{p_i}(y)) < t\}|,$$

$$F^*_{xy}(t, \{p_i\}) = \limsup_{n\to\infty} \frac{1}{n} |\{0 \leqslant i < n \mid d(f^{p_i}(x), f^{p_i}(y)) < t\}|,$$

where $|A|$ denotes the cardinality of the set A. A set $D \subset X$ is said to be distributionally scrambled set in the sequence $\{p_i\}$ for f if for every pair of distinct points x, y in D,

i. $F_{xy}(\delta, \{p_i\}) = 0$, for some $\delta > 0$ and

ii. $F^*_{xy}(\delta, \{p_i\}) = 1$, for every $\delta > 0$.

A map f is said to be distributionally chaotic in a sequence $\{p_i\}$ if f has an uncountable distributionally scrambled set with respect to the sequence $\{p_i\}$ [13]. These definitions generalize the well-known notion of distributional chaos introduced by Schweizer and Smítal in [11]. Note that distributionally scrambled sets in a natural sequence are distributionally scrambled sets.

2.3. Uniform spaces

Let X be a nonempty set. The set $\triangle_X = \{(x, x) \mid x \in X\}$ denotes the diagonal of $X \times X$. For a subset M of $X \times X$, define $M^T = \{(y, x) \mid (x, y) \in M\}$. A set $M \subseteq X \times X$ is said to be symmetric if $M = M^T$. The composite $U \circ V$ of two subsets U and V of $X \times X$ is defined to be the set $\{(x, y) \in X \times X \mid$ there exists $z \in X$ satisfying $(x, z) \in U$ and $(z, y) \in V\}$.

Definition 2.1 *[7] Let X be a nonempty set. A uniform structure on X is a nonempty set $\mathcal{U}$ of subsets of $X \times X$ satisfying the following conditions:*

i. if $U \in \mathcal{U}$ then $\triangle_X \subset U$,

ii. if $U \in \mathcal{U}$ then $U^T \in \mathcal{U}$,

iii. if $U \in \mathcal{U}$ then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$,

iv. if $U \in \mathcal{U}$ and $V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$,

v. if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$ then $V \in \mathcal{U}$.

The elements of $\mathcal{U}$ are then called the entourages of the uniform structure and the pair $(X,\mathcal{U})$ is called a uniform space.

If U is a neighborhood of $\triangle_X$, then $U \cap U^T$ is a symmetric neighborhood of $\triangle_X$, thus we can often work with symmetric neighborhoods without loss of generality. Note that the fact that points x and y are close (in terms of distance) in a metric space X is equivalent to the fact that point (x, y) is close to the diagonal $\triangle_X$ of $X \times X$ in a uniform space $(X,\mathcal{U})$. If $(X,\mathcal{U})$ is a uniform space, then there is an induced topology on X characterized by the fact that the neighborhoods of an arbitrary point $x \in X$ consists of the sets $U[x]$, where U varies over all entourages of X. The set $U[x] = \{y \in X \mid (x, y) \in U\}$ is called the cross-section of U at $x \in X$. The uniform space $(X,\mathcal{U})$ is said to be Hausdorff if $\bigcap\{U \mid U \in \mathcal{U}\} = \triangle_X$.

3. Li-Yorke chaos on uniform spaces

Henceforth by a dynamical system, we mean a pair (X, f), where X is a uniform Hausdorff space without isolated points and $f : X \to X$ is a uniformly continuous map. We denote the map $f \times f$ by F and by $F^i(x, y)$

we mean $(f^i(x), f^i(y))$. The orbit of an element $x \in X$ is given by $O(x) = \{f^n(x) \mid n \in \mathbb{N} \cup \{0\}\}$. A map f is said to be topologically transitive if for any nonempty open subsets U and V of X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. If $f \times f$ is transitive then the map f is said to be weakly mixing.

In [2], authors have studied Li-Yorke chaos for a uniformly continuous self-maps f defined on a uniform space X. The set of proximal pairs, the set of asymptotic pairs and the set of distal pairs with respect to f are denoted by PR, AR, and DR, respectively and are defined as follows:

$$PR = \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, i \in \mathbb{N} \text{ such that } F^i(x, y) \in E\},$$

$$AR = \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, k \in \mathbb{N} \text{ such that } F^i(x, y) \in E, \forall i \geqslant k\},$$

$$DR = X \times X \setminus PR = \{(x, y) \in X \times X \mid \exists E \in \mathcal{U} \text{ such that } F^i(x, y) \notin E, \forall i \in \mathbb{N}\}.$$

A subset S of a uniform space X is a Li-Yorke scrambled set if for any pair of distinct elements $x, y \in S$, $(x, y) \in PR \setminus AR$. A map $f : X \to X$ is said to be Li-Yorke chaotic if there exists an uncountable scrambled set for f.

For an increasing sequence $\{p_i\}$ of positive integers we define proximal relation $PR(f, \{p_i\})$ and asymptotic relation $AR(f, \{p_i\})$ with respect to sequence $\{p_i\}$, respectively as follows:

$$PR(f, \{p_i\}) = \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, i \in \mathbb{N} \text{ such that } F^{p_i}(x, y) \in E\},$$

$$AR(f, \{p_i\}) = \{(x, y) \in X \times X \mid \forall E \in \mathcal{U}, \exists\, k \in \mathbb{N} \text{ such that } F^{p_i}(x, y) \in E, \forall i \geqslant k\}.$$

The set $X \times X \setminus PR(f, \{p_i\})$ is denoted by $DR(f, \{p_i\})$ and is called the distal relation with respect to sequence $\{p_i\}$.

In the above defined terminologies, we can say that a pair $(x, y) \in X \times X$ is Li-Yorke scrambled if there exists increasing sequences $\{m_i\}$, $\{n_i\}$ such that

$$(x, y) \in DR(f, \{m_i\}) \cap AR(f, \{n_i\}).$$

Proof for the following lemma is similar to the result proved in [14] for metric spaces.

Lemma 3.1 *Let X be a second countable Baire space and let f be a continuous self-map on X. If $(Y, f_{|_Y})$ is transitive subsystem of (X, f) and $x \in X$ is such that $O(x)$ is dense in Y then for each open set U in Y, the set $\{t \mid f^t(x) \in U\}$ is not bounded above.*

Theorem 3.2 *Let $(X,\mathcal{U})$ be a second countable Baire uniform Hausdorff space without isolated points and let f be a uniformly continuous self-map defined on X. If f is weakly mixing then f is Li-Yorke chaotic.*

Proof Since X is a second countable space, $X \times X$ has a countable open base say $\{G_1, G_2, \ldots\}$. Consider the set $D = \bigcap_{n=1}^{\infty} \bigcup_{t \in \mathbb{N}} F^{-t}(G_n)$. For each $n \in \mathbb{N}$, the set $\bigcup_{t \in \mathbb{N}} F^{-t}(G_n)$ is open in $X \times X$. Since f is weakly mixing, $f \times f$ is transitive and hence $\bigcup_{t \in \mathbb{N}} F^{-t}(G_n)$ is dense in $X \times X$. Thus, D is a countable intersection of dense sets in $X \times X$. By choice of D, orbit of any pair in D is dense in $X \times X$. Select any x_0, $y_0 \in X$, with $x_0 \neq y_0$. For any $(x, y) \in D$ with $x \neq y$, it follows by Lemma 3.1 that any open set containing (x_0, x_0) contains infinite number of points of type $F^t(x, y)$. Therefore there exists an increasing sequence $\{n_i\}$ such that $F^{n_i}(x, y) \to (x_0, x_0)$

as $n_i \to \infty$. Thus for any $E \in \mathcal{U}$, $F^{n_i}(x,y) \in E$ for all but finitely many i's. Therefore $(x,y) \in AR(f,\{n_i\})$. Using similar arguments, there exists an increasing sequence $\{m_i\}$ such that $F^{m_i}(x,y) \to (x_0,y_0)$ as $m_i \to \infty$. Since $x_0 \neq y_0$, there exists $E \in \mathcal{U}$ such that $(x_0,y_0) \notin E$. Since $F^{m_i}(x,y) \to (x_0,y_0)$, there exists $k \in \mathbb{N}$ such that $F^{m_i}(x,y) \notin E$, for all $i > k$. Then consider the sequence $m_i^{'} = m_{i+k}$, $i \in \mathbb{N}$. Then $F^{m_i^{'}}(x,y) \notin E$, for each $i \in \mathbb{N}$. Thus $(x,y) \in DR(f,\{m_i^{'}\})$. This implies that $(x,y) \in DR(f,\{m_i^{'}\}) \cap AR(f,\{n_i\})$. Hence (x,y) is a Li-Yorke scrambled pair. Since $(x,y) \in D$ with $x \neq y$ is arbitrary, it follows that f is Li-Yorke chaotic. □

The following example justifies that Theorem 3.2 need not be true if the underlying space is not Hausdorff.

Example 3.3 *Let $f : S^1 \to S^1$ be defined by $f(e^{i\theta}) = e^{2i\theta}$, where S^1 is equipped with the cofinite topology. Then S^1 with cofinite topology is not Hausdorff. Clearly, f is mixing and hence weakly mixing. Note that, for any two distinct points $x, y \in S^1$, the pair (x,y) is not Li-Yorke scrambled. Thus, f is not Li-Yorke chaotic.*

4. Topological distributional chaos in a sequence

Let $(X,\mathcal{U})$ be a uniform Hausdorff space and let $f : X \to X$ be a uniformly continuous map. For an increasing sequence $\{p_i\}$ of positive integers, $U \in \mathcal{U}$ and $x, y \in X$, define the lower and upper distribution functions $F_{xy}(U, \{p_i\})$ and $F^*_{xy}(U, \{p_i\})$, respectively as follows:

$$F_{xy}(U, \{p_i\}) = \liminf_{n\to\infty} \frac{1}{n}|\{0 \leqslant i < n \mid F^{p_i}(x,y) \in U\}|,$$

$$F^*_{xy}(U, \{p_i\}) = \limsup_{n\to\infty} \frac{1}{n}|\{0 \leqslant i < n \mid F^{p_i}(x,y) \in U\}|,$$

where $|A|$ denotes the cardinality of the set A.

Definition 4.1 *A subset D of X is said to be topologically distributionally scrambled set in an increasing sequence $\{p_i\}$ if for any $x, y \in D$ with $x \neq y$ we have*

i. $F_{xy}(U, \{p_i\}) = 0$, *for some $U \in \mathcal{U}$ and*

ii. $F^*_{xy}(U, \{p_i\}) = 1$, *for all $U \in \mathcal{U}$.*

Such a pair (x,y) is called topologically distributionally scrambled pair for f in a sequence $\{p_i\}$. We denote by $DCR(f,\{p_i\})$ the collection of all points $(x,y) \in X \times X$ such that (x,y) is topologically distributionally scrambled pair for f in a sequence $\{p_i\}$ and call it the topologically distributionally chaotic relation with respect to a sequence $\{p_i\}$. If f has an uncountable topologically distributionally scrambled set in an increasing sequence $\{p_i\}$ then f is said to be topologically distributionally chaotic in sequence $\{p_i\}$.

Remark 4.2 *(i) If we consider the uniform space $(X,\mathcal{U})$, where (X,d) is a metric space and $\mathcal{U}$ is the natural uniformity generated by the family $\{d^{-1}[0,\epsilon] \mid \epsilon > 0\}$, then every entourage E contains $E_\epsilon = d^{-1}[0,\epsilon]$, for some $\epsilon > 0$ and any E_ϵ is an entourage. Therefore, in this case, topological distributional chaos in a sequence coincides with the metric notion of distributional chaos in a sequence.*
(ii) If $\{p_i\}$ is the sequence of positive integers, then f is topologically distributionally chaotic of type 1 as defined in [12].

The mappings $f : X \to X$ and $g : Y \to Y$ are said to be topologically conjugate if there exists a uniform homeomorphism $h : X \to Y$ such that $gh = hf$. The following proposition can be proved along the lines of [12, Proposition 5].

Proposition 4.3 *Let* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *be uniform Hausdorff spaces. Suppose* $f : X \to X$ *and* $g : Y \to Y$ *are topologically conjugate. Then* f *is topologically distributionally chaotic in a sequence* $\{p_i\}$ *implies* g *is topologically distributionally chaotic in a sequence* $\{p_i\}$.

Recently, many authors have used the families of subsets of positive integers to study the properties of dynamical systems. Recall that, a Furstenberg family $\mathcal{F}$ is a family consisting of some subsets of the set of positive integers which are hereditary upwards, that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A class of Furstenberg families can also be defined by considering the upper density with respect to a sequence. For strictly increasing sequence $Q = \{n_i\}$ of positive integers and $P \subseteq \mathbb{N}$, the upper density of P with respect to Q is given by

$$\bar{d}(P \mid Q) = \limsup_{k\to\infty} \frac{|\{P \cap \{n_1, n_2, \cdots, n_k\}\}|}{k},$$

where $|A|$ denotes cardinality of the set A. For every $a \in [0,1]$, define $\overline{\mathcal{M}}_Q(a) = \{P \subseteq \mathbb{N} \mid P \cap Q$ is infinite and $\bar{d}(P \mid Q) \geq a\}$. Note that $\overline{\mathcal{M}}_Q(a)$ is a Furstenberg family. For $x \in X$ and $A \subseteq X$, define $N(x, A) = \{n \in \mathbb{N} \mid f^n(x) \in A\}$ and $\overline{\mathcal{M}}_Q(a, A) = \{x \in X \mid N(x, A) \in \overline{\mathcal{M}}_Q(a)\}$.

We can now rephrase the definition of a topologically distributionally scrambled pair in a sequence Q as follows: Let $(X,\mathcal{U})$ be a uniform Hausdorff space consisting of symmetric open entourages, f a uniformly continuous self-map on X and Q an increasing sequence of positive integers. Then $(x, y) \in X \times X$ is a topologically distributionally scrambled pair in sequence Q if

i. for some $U \in \mathcal{U}$, $(x, y) \in \overline{\mathcal{M}}_Q(1, X \times X \setminus \overline{U})$, and

ii. for any $U \in \mathcal{U}$, $(x, y) \in \overline{\mathcal{M}}_Q(1, U)$

Following lemma can be proved along the lines of [8, Lemma 3.2].

Lemma 4.4 *Let* X *be a topological space and* f *a continuous self-map on* X, Q *strictly increasing sequences of positive integers and* $a \in [0,1]$. *Then for any nonempty open subset* W *of* X, $\overline{\mathcal{M}}_Q(a, W)$ *is a* G_δ *set.*

The following result follows from Lemma 4.4.

Lemma 4.5 *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space,* f *a uniformly continuous self-map on* X *and* Q *an increasing sequence of positive integers. Then the set of all topologically distributionally scrambled pairs in the sequence* Q *is a* G_δ *subset of* $X \times X$.

Lemma 4.6 *Let* $(X,\mathcal{U})$ *be a uniform Hausdorff space and let* f *be an uniformly continuous self-map on* X. *If* $\{m_i\}$ *and* $\{n_i\}$ *are increasing sequences of positive integers then there exists an increasing sequence* $\{p_i\}$ *of positive integers such that* $DR(f, \{m_i\}) \cap AR(f, \{n_i\}) \subset DCR(f, \{p_i\})$.

Proof Let $b_1 = 2$, $b_i = 2^{b_1+b_2+\ldots+b_{i-1}}$ for $i > 1$. Then $\{b_i\}$ is an increasing sequence of positive integers. Let

$$p_i = \begin{cases} m'_i, & \text{if } i \leqslant b_1 \text{ or } \sum\limits_{j=1}^{2k} b_j < i \leqslant \sum\limits_{j=1}^{2k+1} b_j,\ k \in \mathbb{N} \\ n'_i, & \text{otherwise} \end{cases}$$

where $\{m'_i\}$ and $\{n'_i\}$ are subsequences of $\{m_i\}$ and $\{n_i\}$, respectively, with $m'_l > n'_j$, $n'_l > m'_j$ for any $l > j$. Then $\{p_i\}$ is an increasing sequence of positive integers. Let $(x, y) \in DR(f, \{m_i\}) \cap AR(f, \{n_i\})$. Then $(x, y) \in DR(f, \{m_i\})$ implies there exists $U \in \mathcal{U}$ such that $F^{m_i}(x, y) \notin U$, for all $i \in \mathbb{N}$. Thus

$$\begin{aligned} F_{xy}(U, \{p_i\}) &= \liminf_{n\to\infty} \frac{1}{n}|\{0 \leqslant i < n \mid F^{p_i}(x, y) \in U\}| \\ &\leqslant \lim_{i\to\infty} \frac{1}{j_i}|\{0 \leqslant k < j_i \mid F^{p_k}(x, y) \in U\}| \quad \Big(\text{where } j_i = \sum_{h=1}^{2i+1} b_h\Big) \\ &\leqslant \lim_{i\to\infty} \frac{b_1 + b_2 + \ldots + b_{2i}}{j_i} \\ &= \lim_{i\to\infty} \frac{b_1 + b_2 + \ldots + b_{2i}}{b_1 + b_2 + \ldots + b_{2i} + 2^{b_1+b_2+\ldots+b_{2i}}} = 0. \end{aligned}$$

Further $(x, y) \in AR(f, \{n_i\})$ implies that for any $U \in \mathcal{U}$ there exists a positive integer $N > 0$ such that $F^{n_i}(x, y) \in U$, for all $i > N$. Thus

$$\begin{aligned} F^*_{xy}(U, \{p_i\}) &= \limsup_{n\to\infty} \frac{1}{n}|\{0 \leqslant i < n \mid F^{p_i}(x, y) \in U\}| \\ &\geq \lim_{i\to\infty} \frac{1}{l_i}|\{0 \leqslant j < l_i \mid F^{p_j}(x, y) \in U\}| \quad \Big(\text{where } l_i = \sum_{h=1}^{2i} b_h\Big) \\ &\geq \lim_{i\to\infty} \frac{b_{2i}}{l_i} \\ &= \lim_{i\to\infty} \frac{2^{b_1+b_2+\ldots+b_{2i-1}}}{b_1 + b_2 + \ldots + b_{2i-1} + 2^{b_1+b_2+\ldots+b_{2i-1}}} = 1 \end{aligned}$$

□

Lemma 4.7 [5] *Let $\{S_i\}$ be a sequence of increasing sequences of positive integers. Then there exists an increasing sequence Q of positive integers such that $\bar{d}(S_i \cap Q \mid Q) = 1$ for all $i \geq 1$.*

Lemma 4.8 *Let $(X,\mathcal{U})$ be a uniform Hausdorff space and let f be a uniformly continuous self-map on X. If S is countable Li-Yorke scrambled set, then there exists an increasing sequence Q of positive integers such that S is topologically distributionally scrambled set in the sequence Q.*

Proof For any pair of distinct points $x, y \in S$, by definition there exist sequences $\{m_i\}$ and $\{n_i\}$ such that $(x, y) \in DR(f, \{m_i\}) \cap AR(f, \{n_i\})$. Using Lemma 4.6, there exists an increasing sequence $\{p_i^{(x,y)}\}$ of positive integers such that $(x, y) \in DCR(f, \{p_i^{(x,y)}\})$. Hence by Lemma 4.7, there exists a sequence Q such that S is topologically distributionally scrambled set in the sequence Q. □

Using Lemma 3.1 and 3.2 in [6] one can obtain the proof of the following lemma.

Lemma 4.9 *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space without isolated points. If* R *is a symmetric relation on* X *which contains a dense* G_δ *subset of* $X \times X$*. Then there is an uncountable dense subset* B *of* X *such that* $B \times B \setminus \triangle \subset R$

Theorem 4.10 *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space without isolated points and let* f *be a uniformly continuous self-map defined on* X*. Then* f *is chaotic in the sense of Li-Yorke if and only if* f *is topologically distributionally chaotic in a sequence.*

Proof If f is chaotic in the sense of Li-Yorke, then f has an uncountable scrambled set $D \subset X$. Since X is second countable, then so is D, hence we can choose a countable dense subset S of D. By Lemma 4.8, there exists an increasing sequence Q of positive integers such that S is topologically distributionally scrambled set in the sequence Q. Let E be the collection of all topologically distributionally scrambled pairs in the sequence Q, then by Lemma 4.5, E is a G_δ subset of $X \times X$. Since $S \times S \setminus \triangle \subset E$ and S is dense in D. By Lemma 4.9, there exists an uncountable dense set $K \subset D$ such that $K \times K \setminus \triangle \subset E$. Thus (X, f) is topologically distributionally chaotic in a sequence. Conversely if f is topologically distributionally chaotic in a sequence $Q = \{p_i\}$, then f has an uncountable scrambled set D such that for any $x, y \in D$ with $x \neq y$ we have $F^*_{xy}(U, \{p_i\}) = 1$, for all $U \in \mathcal{U}$ and $F_{xy}(U, \{p_i\}) = 0$, for some $U \in \mathcal{U}$. This implies that for each $U \in \mathcal{U}$, there exist some $j \in \mathbb{N}$ such that $F^{p_j}(x, y) \in U$. Thus $(x, y) \in PR$. Note that $(x, y) \notin AR$. For if $(x, y) \in AR$, then for any $U \in \mathcal{U}$, there exists a positive integer $N > 0$ such that $F^{p_i}(x, y) \in U$, for all $p_i > N$. This implies that $F_{xy}(U, \{p_i\}) = F^*_{xy}(U, \{p_i\}) = 1$, for each $U \in \mathcal{U}$, which contradicts that f is topologically distributionally chaotic in a sequence $\{p_i\}$. Thus $(x, y) \in PR \setminus AR$, for all $(x, y) \in D$. Hence f is Li-Yorke chaotic. $\square$

The following result follows from Theorem 3.2 and Theorem 4.10:

Corollary 4.11 *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space without isolated points and* f *a uniformly continuous self-map defined on* X*. If* f *is weakly mixing then* f *is topologically distributionally chaotic in a sequence.*

Recall that, a map $f : X \to X$ is said to be Devaney chaotic if f is transitive, the set of periodic points of f is dense in X and f has sensitive dependence on initial conditions.

Corollary 4.12 *Let* $(X,\mathcal{U})$ *be a second countable Baire uniform Hausdorff space without isolated points and* f *a uniform self homeomorphism on* X*. If* f *is Devaney chaotic then* f *is topologically distributionally chaotic in a sequence.*

Proof From [2, Theorem 1.2], it follows that f is Li-Yorke chaotic. That f is topologically distributionally chaotic in a sequence follows from Theorem 4.10. $\square$

As a consequence of the results obtained in the paper for a uniform self homeomorphism f defined on a second countable Baire uniform Hausdorff space X without isolated points, we have the following implications:

$$\begin{array}{rcl} \text{Weakly mixing} & \Rightarrow & \text{Li-Yorke chaos} \\ & & \Updownarrow \\ \text{Devaney chaos} & \Rightarrow & \text{Topological distributional chaos in a sequence} \end{array}$$

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