Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 1153 – 1176
© TÜBİTAK
doi:10.3906/mat-2105-93

On Hom-F-manifold algebras and quantization

Abdelkader BEN HASSINE[1,2,*], **Taoufik CHTIOUI**[2], **Mohamed Ali MAALAOUI**[2], **Sami MABROUK**[3]

[1]Department of Mathematics, Faculty of Science and Arts at Belqarn, University of Bisha, Sabt Al-Alaya, Saudi Arabia
[2]Faculty of Sciences, University of Sfax, Sfax, Tunisia
[3]Faculty of Sciences, University of Gafsa, Gafsa, Tunisia

Received: 25.05.2021 • **Accepted/Published Online:** 23.02.2022 • **Final Version:** 05.05.2022

Abstract: The notion of a F-manifold algebras is an algebraic description of a F-manifold. In this paper, we introduce the notion of Hom-F-manifold algebras which is generalisation of F-manifold algebras and Hom-Poisson algebras. We develop the representation theory of Hom-F-manifold algebras and generalize the notion of Hom-pre-Poisson algebras by introducing the Hom-pre-F-manifold algebras which give rise to a Hom-F-manifold algebra through the subadjacent commutative Hom-associative algebra and the subadjacent Hom-Lie algebra. Using $\mathcal{O}$-operators on a Hom-F-manifold algebras we construct a Hom-pre-F-manifold algebras on a module. Then, we study Hom-pre-Lie formal deformations of commutative Hom-associative algebra and we prove that Hom-F-manifold algebras are the corresponding semiclassical limits. Finally, we study Hom-Lie infinitesimal deformations and extension of Hom-pre-Lie n-deformation to Hom-pre-Lie $(n+1)$-deformation of a commutative Hom-associative algebra via cohomology theory.

Key words: Hom-F-manifold algebra, Hom-pre-F-manifold algebra, representation theory, quantization, $\mathcal{O}$-operators.

1. Introduction

Algebras of Hom-type appeared in the physics literature of the 1990's, in the context of quantum deformations of some algebras of vector fields, such as the Witt and Virasoro algebras, in connection with oscillator algebras ([1, 24]). A quantum deformation consists of replacing the usual derivation by a σ-derivation. It turns out that the algebras obtained in this way do not satisfy the Jacobi identity anymore, but instead they satisfy a modified version involving a homomorphism. These kind of algebras were called Hom-Lie algebras and studied by Hartwig, Larsson and Silvestrov in [21, 26]. The corresponding associative type objects, called Hom-associative algebras were introduced by Makhlouf and Silvestrov in [31]. Hom-alternative and Hom-flexible algebras were introduced first in [32]. Hom-Jordan, Hom-Malcev and Hom-Poisson algebras were studied first in [42, 43]. Hom-Lie algebras are widely studied in the following aspects: representation and cohomology theory [3, 6, 39, 41], deformation theory [32], categorification theory [40].

Rota–Baxter (associative) algebras, originated from the work of G. Baxter[7] in probability and populated by the work of Cartier and Rota [10, 37, 38], have also been studied in connection with many areas of mathematics

*Correspondence: benhassine.abdelkader@yahoo.fr

2010 *AMS Mathematics Subject Classification:* 17A30, 53D55, 17B38

and physics, including combinatorics, number theory, operators and quantum field theory [2, 4, 16–18]. In particular Rota–Baxter algebras have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory of Connes and Kreimer [9, 14, 15], as well as in the application of the renormalization method in solving divergent problems in number theory [19, 34]. Furthermore, Rota–Baxter operators on a Lie algebra are an operator form of the classical Yang–Baxter equations and contribute to the study of integrable systems [4, 5].

The notion of Frobenius manifolds was invented by Dubrovin [11] in order to give a geometrical expression of the Witten–Dijikgraaf–Verlinde equations. In 1999, Hertling and Manin [23] introduced the concept of F-manifolds as a relaxation of the conditions of Frobenius manifolds. Inspired by the investigation of algebraic structures of F-manifolds, the notion of an F-manifold algebra is given by Dotsenko [13] in 2019 to relate the operad F-manifold algebras to the operad pre-Lie algebras. F-manifolds appear in many fields of mathematics such as singularity theory [22], quantum K-theory [27], integrable systems [11, 12, 28], operad [35] and so on. Recently, the concept of pre-F-manifold algebras, which gives rise to F-manifold algebras, and the notion of representations of F-manifold algebras, are introduced in [30]. The notion of F-manifold color algebras and theirs proprieties is given by Ming, Chen and Li ([36]).

The paper is organized as follows. In Section 2, we give a backround of Hom-associative algebras, Hom-Lie algebras, Hom-pre-Lie algebras, Hom-Lie-admissible algebras, F-manifolds, F-manifolds algebras. In Section 3, we define Hom-F-manifold-algebras and introduce the notion of Hom-F -manifold admissible algebras, which give rise to Hom-F-manifold algebras. In Section 4, first we develop (dual) representations of a Hom-F-manifold algebras. Then we introduce the notions of Hom-pre-F-manifold algebras and $\mathcal{O}$-operators on a Hom-F-manifold algebra. We show that on one hand, an $\mathcal{O}$-operator on a Hom-F-manifold algebra gives a Hom-pre-F-manifold algebra. On the other hand, a Hom-pre-F-manifold algebra naturally gives an $\mathcal{O}$-operator on the subadjacent Hom-F-manifold algebra. Section 5 is devoted to the study of the Hom-pre-Lie formal deformations of commutative Hom-associative algebras and prove that Hom-F-manifold algebras are the corresponding semiclassical limits. Furthermore, we study extensions of Hom-pre-Lie n-deformations to Hom-pre-Lie $(n+1)$-deformations of a commutative Hom-associative algebra.
In this paper, all the vector spaces are over an algebraically closed field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

In this section, we recall Hom-associative algebras, Hom-Lie algebras, Hom-pre-Lie algebras, Hom-Lie-admissible algebras, F-manifolds, F-manifolds algebras (for further details we can refer to [23, 32, 42, 43]).

A Hom-associative algebra is a triple $(A, \cdot, \alpha)$, where A is a vector space $\cdot : A \otimes A \longrightarrow A$ and $\alpha : A \to A$ are a two linear maps satisfying that for all $x, y, z \in A$, the Hom-associator $as_\alpha(x, y, z) = (x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y * z) = 0$, i.e.

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z).$$

Furthermore, if $x \cdot y = y \cdot x$ for all $x, y \in A$, then $(A, \cdot, \alpha)$ is called a commutative Hom-associative algebra.

A representation of Hom-associative commutative algebra $(A, \cdot, \alpha)$ on the vector space V with respect to $\varphi \in \mathfrak{gl}(V)$ is a linear map $\mu : A \longrightarrow \mathfrak{gl}(V)$, such that for any $x, y \in A$, the following equalities are satisfied:

$$\mu(\alpha(x))\varphi = \varphi\mu(x); \quad (2.1)$$

$$\mu(x \cdot y)\varphi = \mu(\alpha(x))\mu(y). \quad (2.2)$$

Let $(V; \mu, \varphi)$ be a representation of a commutative Hom-associative algebra $(A, \cdot, \alpha)$. In the sequel, we always assume that φ is invertible. Define $\mu^* : A \longrightarrow \mathfrak{gl}(V^*)$ and $\varphi^* : V^* \longrightarrow V^*$ by

$$\langle \mu^*(x)\xi, v\rangle = -\langle \xi, \mu(x)v\rangle, \quad \forall\ x \in A, \xi \in V^*, v \in V.$$

However, in general μ^* is not a representation of A anymore. Define $\mu^\star : A \longrightarrow \mathfrak{gl}(V^*)$ by

$$\mu^\star(x)(\xi) := \mu^*(\alpha(x))\big((\varphi^{-2})^*(\xi)\big), \quad \forall x \in A, \xi \in V^*. \quad (2.3)$$

More precisely, we have

$$\langle \mu^\star(x)(\xi), u\rangle = -\langle \xi, \mu(\alpha^{-1}(x))(\varphi^{-2}(u))\rangle, \quad \forall x \in A, u \in V, \xi \in V^*. \quad (2.4)$$

Lemma 2.1 *Under the above notation,* $(V^*, \mu^\star, (\varphi^{-1})^*)$ *is a representation of* $(A, \cdot, \alpha)$ *which is called the dual representation of the representation* $(V; \mu, \varphi)$.

A Hom-Zinbiel algebra is a pair $(A, \diamond, , \alpha)$, where A is a vector space, $\diamond : A \otimes A \longrightarrow A$ is a bilinear multiplication , and $\alpha : A \to A$ be a linear map satisfying that for all $x, y, z \in A$,

$$\alpha(x) \diamond (y \diamond z) = (y \diamond x) \diamond \alpha(z) + (x \diamond y) \diamond \alpha(z). \quad (2.5)$$

Lemma 2.2 *Let* $(A, \diamond, \alpha)$ *be a Hom-Zinbiel algebra. Then* $(A, \cdot, \alpha)$ *is a commutative Hom-associative algebra, where* $x \cdot y = x \diamond y + y \diamond x$. *Moreover, for* $x \in A$, *define* $\mathrm{ad}^L_\diamond(x) : A \longrightarrow \mathfrak{gl}(A)$ *by*

$$\mathrm{ad}^L_\diamond(x)(y) = x \diamond y, \quad \forall\ y \in A. \quad (2.6)$$

Then $(A; \mathrm{ad}^L_\diamond, \alpha)$ *is a representation of the commutative Hom-associative algebra* $(A, \cdot, \alpha)$.

A Hom-Lie algebra is a triple $(A, [\cdot, \cdot], \alpha)$ consisting of vector space A, a linear map α and a bilinear mapping $[\cdot, \cdot]$ defined respectively by:
$\alpha : A \to A,\ [\cdot, \cdot] : A \times A \to A$ such that for $x, y, z \in A$ we have

$$[x, y] = -[y, x], \quad (2.7)$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \ \ \text{(Jacobi condition)}. \quad (2.8)$$

where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

A representation of Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$ on the vector space V with respect to $\varphi \in \mathfrak{gl}(V)$ is a linear map $\rho : A \longrightarrow \mathfrak{gl}(V)$, such that for any $x, y \in A$, the following equalities are satisfied:

$$\rho(\alpha(x))\varphi = \varphi\rho(x); \quad (2.9)$$

$$\rho([x, y])\varphi = \rho(\alpha(x))\rho(y) - \rho(\alpha(y))\rho(x). \quad (2.10)$$

Let $(V;\rho,\varphi)$ be a representation of a Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$. Assume that φ is invertible. For all $x\in A, u\in V, \xi\in V^*$, define $\rho^*: A\longrightarrow \mathfrak{gl}(V^*)$ as usual by

$$\langle \rho^*(x)(\xi),u\rangle = -\langle \xi,\rho(x)(u)\rangle.$$

Then define $\rho^\star: A\longrightarrow \mathfrak{gl}(V^*)$

$$\rho^\star(x)(\xi) := \rho^*(\alpha(x))\big((\varphi^{-2})^*(\xi)\big). \tag{2.11}$$

Lemma 2.3 *[8] Let $(V;\rho,\varphi)$ be a representation of a Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$. Then $(V^*;\rho^\star,(\varphi^{-1})^*)$ is a representation of $(A,[\cdot,\cdot],\alpha)$, which is called the dual representation of $(V;\rho,\varphi)$.*

A Hom-pre-Lie algebra is a triple $(A,*,\alpha)$, where A is a vector space $*: A\otimes A\longrightarrow A$ and $\alpha: A\to A$ are a two linear maps satisfying that for all $x,y,z\in A$, the Hom-associator $as_\alpha(x,y,z)=(x*y)*\alpha(z)-\alpha(x)*(y*z)$ is symmetric in x,y, i.e.

$$as_\alpha(x,y,z)=as_\alpha(y,x,z),$$

or equivalently

$$(x*y)*\alpha(z)-\alpha(x)*(y*z)=(y*x)*\alpha(z)-\alpha(y)*(x*z).$$

It is obvious that any Hom-associative algebra is a Hom-pre-Lie algebra. In addition A commutative Hom-pre-Lie algebra is Hom-associative.

Lemma 2.4 *Let $(A,*,\alpha)$ be a Hom-pre-Lie algebra. Define the commutator $[x,y]=x*y-y*x$, then $(A,[\cdot,\cdot],\alpha)$ is a Hom-Lie algebra, which is called the subadjacent Hom-Lie algebra of $(A,*,\alpha)$ and denoted by A^c. Furthermore, $L: A\to\mathfrak{gl}(A)$ defined by*

$$L_x y = x*y,\quad \forall x,y\in A \tag{2.12}$$

gives a representation of A^c on A.

A Hom-Lie admissible algebra is a nonassociative Hom-algebra $(A,*,\alpha)$ whose commutator algebra is a Hom-Lie algebra. More precisely, it is equivalent to the following condition:

$$\circlearrowleft_{x,y,z} as_\alpha(x,y,z)-as_\alpha(y,x,z)=0,\quad \forall x,y,z\in A. \tag{2.13}$$

Obviously, a Hom-pre-Lie algebra is a Hom-Lie-admissible algebra.

Let $(A,*,\alpha)$ be a Hom-pre-Lie algebra and V a vector space. A representation of A on V with respect to $\varphi\in gl(V)$ consists of a pair (ρ,μ), where $\rho: A\longrightarrow\mathfrak{gl}(V)$ is a representation of the Hom-Lie algebra A^c on V with respect φ and $\mu: A\longrightarrow\mathfrak{gl}(V)$ is a linear map satisfying

$$\varphi\mu(x)=\mu(\alpha(x))\varphi, \tag{2.14}$$

$$\rho(\alpha(x))\mu(y)-\mu(\alpha(y))\rho(x)=\mu(x*y)\varphi-\mu(\alpha(y))\mu(x),\quad \forall\ x,y\in A,\ u\in V. \tag{2.15}$$

We denote such a representation by $(V;\rho,\mu,\varphi)$. Let $R: A\to\mathfrak{gl}(A)$ be a linear map with $x\longrightarrow R_x$, where the linear map $R_x: A\longrightarrow A$ is defined by $R_x(y)=y*x$, for all $x,y\in A$. Then $(A;\rho=L,\mu=R,\varphi=\alpha)$ is a representation, which we call the adjoint representation.

Let $(V;\rho,\mu,\varphi)$ be a representation of a Hom-pre-Lie algebra $(A,\cdot,\alpha)$. Assume that φ is invertible. For all $x\in A, u\in V, \xi\in V^*$, define $\rho^*: A\longrightarrow \mathfrak{gl}(V^*)$ and $\mu^*: A\longrightarrow \mathfrak{gl}(V^*)$ as usual by

$$\langle \rho^*(x)(\xi),u\rangle = -\langle \xi,\rho(x)(u)\rangle,\quad \langle \mu^*(x)(\xi),u\rangle = -\langle \xi,\mu(x)(u)\rangle.$$

Then define $\rho^\star: A\longrightarrow \mathfrak{gl}(V^*)$ and $\mu^\star: A\longrightarrow \mathfrak{gl}(V^*)$ by

$$\rho^\star(x)(\xi) := \rho^*(\alpha(x))\big((\varphi^{-2})^*(\xi)\big), \tag{2.16}$$

$$\mu^\star(x)(\xi) := \mu^*(\alpha(x))\big((\varphi^{-2})^*(\xi)\big). \tag{2.17}$$

Lemma 2.5 *[29] Let $(V;\rho,\mu,\varphi)$ be a representation of a Hom-pre-Lie algebra $(A,\cdot,\alpha)$. Then $(V^*;\rho^\star-\mu^\star,-\mu^\star,(\varphi^{-1})^*)$ is a representation of $(A,\cdot,\alpha)$, which is called the dual representation of $(V;\rho,\mu,\varphi)$.*

Let (V,ρ,μ,φ) be a representation of a Hom-pre-Lie algebra $(A,\cdot,\alpha)$. The set of $(n+1)$-cochains is given by

$$C^{n+1}(A;V) = \mathrm{Hom}(\wedge^n A\otimes A, V),\quad \forall n\geq 0. \tag{2.18}$$

For all $f\in C^n(A;V)$, $x_1,\dots,x_{n+1}\in A$, define the operator $\partial: C^n(A;V)\longrightarrow C^{n+1}(A;V)$ by

$$\begin{aligned}
&(\partial f)(x_1,\dots,x_{n+1}) \qquad (2.19)\\
=\ &\sum_{i=1}^{n}(-1)^{i+1}\rho(x_i)f(\alpha^{-1}(x_1),\dots,\widehat{\alpha^{-1}(x_i)},\dots,\alpha^{-1}(x_{n+1}))\\
&+\sum_{i=1}^{n}(-1)^{i+1}\mu(x_{n+1})f(\alpha^{-1}(x_1),\dots,\widehat{\alpha^{-1}(x_i)},\dots,\alpha^{-1}(x_n),\alpha^{-1}(x_i))\\
&-\sum_{i=1}^{n}(-1)^{i+1}\varphi f(\alpha^{-1}(x_1),\dots,\widehat{\alpha^{-1}(x_i)}\dots,\alpha^{-1}(x_n),\alpha^{-2}(x_i)\cdot\alpha^{-2}(x_{n+1}))\\
&+\sum_{1\leq i<j\leq n}(-1)^{i+j}\varphi f([\alpha^{-2}(x_i),\alpha^{-2}(x_j)]_C,\alpha^{-1}(x_1),\dots,\widehat{\alpha^{-1}(x_i)},\dots,\widehat{\alpha^{-1}(x_j)},\dots,\alpha^{-1}(x_{n+1})).
\end{aligned}$$

Theorem 2.6 *The operator $\partial: C^n(A;V)\longrightarrow C^{n+1}(A;V)$ defined as above satisfies $\partial\circ\partial=0$.*

Denote the set of closed n-cochains by $Z^n(A;V)$ and the set of exact n-cochains by $B^n(A;V)$. We denote by $H^n(A;V)=Z^n(A;V)/B^n(A;V)$ the corresponding cohomology groups of the Hom-pre-Lie algebra $(A,\cdot,\alpha)$ with the coefficient in the representation (V,φ,ρ,μ). There is a close relation between the cohomologies of Hom-pre-Lie algebras and the corresponding subadjacent Hom-Lie algebras.

In [33], the authors defined a Hom-Poisson algebra as a tuple $(A,[\cdot,\cdot],\mu,\alpha)$ consists of

1. a Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$ and

2. a commutative Hom-associative algebra (A,μ,α)

such that the Hom-Leibniz identity

$$[\alpha(x),y\cdot z] = [x,y]\cdot\alpha(z)+\alpha(y)\cdot[x,z] \tag{2.20}$$

is satisfied.

The notion of F-manifold is introduced by C. Hertling and Y. I. Manin ([23]). It is a weak version of Frobenius manifolds.

An F-manifold is a pair $(M, \circ)$, where M is a manifold, $\circ$ is a smooth bilinear commutative, associative multiplication on the tangent sheaf TM, such that the Hertling–Manin relation holds

$$P_{X_1 \circ X_2}(X_3, X_4) = X_1 \circ P_{X_2}(X_3, X_4) + X_2 \circ P_{X_1}(X_3, X_4),$$

where $P_{X_1}(X_2, X_3) = [X_1, X_2 \circ X_3] - [X_1, X_2] \circ X_3 - X_2 \circ [X_1, X_3]$ measures to what extent the product $\circ$ and the usual Lie bracket of vector fields fail the Poisson algebra axioms.

In [13], the author gives an algebraic description of F-manifold constructing in this way F-manifold algebras. By a definition, an F-manifold algebra is a triple $(A, \cdot, [\cdot, \cdot])$, where $(A, \cdot)$ is a commutative associative algebra and $(A, [\cdot, \cdot])$ is a Lie algebra, such that for all $x, y, z, w \in A$, the Hertling–Manin relation holds :

$$\mathcal{L}(x \cdot y, z, w) = x \cdot \mathcal{L}(y, z, w) + y \cdot \mathcal{L}(x, z, w), \tag{2.21}$$

where $\mathcal{L}(x, y, z)$ is the Leibnizator define by

$$\mathcal{L}(x, y, z) = [x, y \cdot z] - [x, y] \cdot z - y \cdot [x, z]. \tag{2.22}$$

In the sequel, $\mathcal{L}(x, y, z)$ is denoted also $P_x(y, z)$.

3. Hom-F-manifold algebras: definitions and constructions

In this section, we introduce the notion of Hom-F-manifold algebras and various examples are given.

Definition 3.1 *A Hom-F-manifold algebra is a tuple $(A, \cdot, [\cdot, \cdot], \alpha)$, where $(A, \cdot, \alpha)$ is a commutative Hom-associative algebra and $(A, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra, such that for all $x, y, z, w \in A$:*

$$\mathcal{L}(x \cdot y, \alpha(z), \alpha(w)) = \alpha^2(x) \cdot \mathcal{L}(y, z, w) + \alpha^2(y) \cdot \mathcal{L}(x, z, w), \tag{3.1}$$

where $\mathcal{L}(x, y, z)$ is the Hom-Leibnizator define by

$$\mathcal{L}(x, y, z) = [\alpha(x), y \cdot z] - [x, y] \cdot \alpha(z) - \alpha(y) \cdot [x, z]. \tag{3.2}$$

The identity Eq. (3.1) is called the Hom-Hertling–Manin relation.

Let $x, y \in A$ such that $\alpha(x) = x, \alpha(y) = y$ then the map $\mathcal{L}(\cdot, x, y) : A \to A$ is α^2-derivation in the Hom-associative algebra $(A, \cdot, \alpha)$.

Example 3.2 *Any Hom-Poisson algebra is a Hom-F-manifold algebra.*

Definition 3.3 *Let $(A, \cdot_A, [\cdot, \cdot]_A, \alpha_A)$ and $(B, \cdot_B, [\cdot, \cdot]_B, \alpha_B)$ be two Hom-F-manifold algebras. A homomorphism between A and B is a linear map $\varphi : A \to B$ such that*

$$\alpha_A \varphi = \varphi \alpha_B, \tag{3.3}$$

$$\varphi(x \cdot_A y) = \varphi(x) \cdot_B \varphi(y), \tag{3.4}$$

$$\varphi[x, y]_A = [\varphi(x), \varphi(y)]_B, \quad \forall\, x, y \in A. \tag{3.5}$$

If Eq. (3.4) is not satisfied, then we call φ is a weak homomorphism.

Example 3.4 *Let $(A,\cdot,[\cdot,\cdot])$ be a F-manifold algebra and $\alpha : A \to A$ be a F-manifold algebra morphism. Then $(A,\cdot_\alpha,[\cdot,\cdot]_\alpha,\alpha)$ is a Hom-F-manifold algebra, where for all $x,y \in A$*

$$x \cdot_\alpha y = \alpha(x)\cdot\alpha(y), \qquad [x,y]_\alpha = [\alpha(x),\alpha(y)].$$

Example 3.5 Let $(A,\cdot_A,[\cdot,\cdot]_A,\alpha_A)$ and $(B,\cdot_B,[\cdot,\cdot]_B,\alpha_B)$ be two Hom-F-manifold algebras. Then $(A\oplus B,\cdot_{A\oplus B},[\cdot,\cdot]_{A\oplus B},\alpha_{A\oplus B})$ is a Hom-F-manifold algebra, where the product $\cdot_{A\oplus B}$, the bracket $[\cdot,\cdot]_{A\oplus B}$ and the twist map $\alpha_{A\oplus B}$ are given by

$$\begin{aligned}
(x_1+x_2)\cdot_{A\oplus B}(y_1+y_2) &= x_1\cdot_A y_1 + x_2\cdot_B y_2,\\
[(x_1+x_2),(y_1+y_2)]_{A\oplus B} &= [x_1,y_1]_A + [x_2,y_2]_B,\\
\alpha_{A\oplus B} &= \alpha_A + \alpha_B
\end{aligned}$$

for all $x_1,y_1 \in A, x_2,y_2 \in B$.

Now, we introduce the notion of Hom-F-manifold admissible algebras.

Definition 3.6 *A Hom-F-manifold admissible algebra is a tuple $(A,\cdot,*,\alpha)$ such that $(A,\cdot,\alpha)$ is a commutative Hom-associative algebra and $(A,*,\alpha)$ is a Hom-Lie admissible algebra satisfying for all $x,y,z\in A$,*

$$\alpha(x)*(y\cdot z)-(x*y)\cdot\alpha(z)-\alpha(y)\cdot(x*z)=\alpha(y)*(x\cdot z)-(y*x)\cdot\alpha(z)-\alpha(x)\cdot(y*z). \tag{3.6}$$

Proposition 3.7 *Let $(A,\cdot,*,\alpha)$ be a Hom-F-manifold admissible algebra. Then $(A,\cdot,[\cdot,\cdot],\alpha)$ is a Hom-F manifold algebra, where the bracket $[\cdot,\cdot]$ is given by*

$$[x,y]=x*y-y*x, \quad \forall\ x,y\in A. \tag{3.7}$$

Proof Let $x,y,z,w\in A$. Using Eq. (3.6), we have

$$\begin{aligned}
\mathcal{L}(x,y,z) &= [\alpha(x),y\cdot z]-[x,y]\cdot\alpha(z)-\alpha(y)\cdot[x,z]\\
&= \alpha(x)*(y\cdot z)-(y\cdot z)*\alpha(x)-(x*y)\cdot\alpha(z)+(y*x)\cdot\alpha(z)\\
&\quad -\alpha(y)\cdot(x*z)+\alpha(y)\cdot(z*x)\\
&= \alpha(x)*(y\cdot z)-(x*y)\cdot\alpha(z)-\alpha(y)\cdot(x*z)-(y\cdot z)*\alpha(x)\\
&\quad +(y*x)\cdot\alpha(z)+\alpha(y)\cdot(z*x)\\
&= \alpha(y)*(x\cdot z)-(y*x)\cdot\alpha(z)-\alpha(x)\cdot(y*z)-(y\cdot z)*\alpha(x)\\
&\quad +(y*x)\cdot\alpha(z)+\alpha(y)\cdot(z*x).
\end{aligned}$$

By this formula and (3.6), we have

$$
\begin{aligned}
& \mathcal{L}(x \cdot y, \alpha(z), \alpha(w)) \\
= \; & \alpha^2(z) * ((x \cdot y) \cdot \alpha(w)) - (\alpha(z) * (x \cdot y)) \cdot \alpha^2(w) \\
& -(\alpha(x) \cdot \alpha(y)) \cdot (\alpha(z) * \alpha(w)) - (\alpha(z) \cdot \alpha(w)) * (\alpha(x) \cdot \alpha(y)) \\
& +(\alpha(z) * (x \cdot y)) \cdot \alpha^2(w) + \alpha^2(z) \cdot (\alpha(w) * (x \cdot y)) \\
= \; & \alpha^2(z) * (\alpha(x) \cdot (y \cdot w)) - \big((z * x) \cdot \alpha(y) - \alpha(x) \cdot (z * y) + \alpha(x) * (z \cdot y) - (x * z) \cdot \alpha(y) \\
& -\alpha(z) \cdot (x * y)\big) \cdot \alpha^2(w) - (\alpha(x) \cdot \alpha(y)) \cdot (\alpha(z) * \alpha(w)) \\
& -\big(((z \cdot w) * \alpha(x)) \cdot \alpha^2(y) + \alpha^2(x) \cdot ((z \cdot w) * \alpha(y)) \\
& +\alpha^2(x) * ((z \cdot w) \cdot \alpha(y)) - (\alpha(x) * (z \cdot w) \cdot \alpha^2(y)) - (\alpha(z) \cdot \alpha(w)) \cdot (\alpha(x) * \alpha(y))\big) \\
& +\big((z * x) \cdot \alpha(y) + \alpha(x) \cdot (z * y) + \alpha(x) * (z \cdot y) - (x * z) \cdot \alpha(y) - \alpha(z) \cdot (x * y))\big) \cdot \alpha^2(w) \\
& +\big((w * x) \cdot \alpha(y) + \alpha(x) \cdot (w * y) + \alpha(x) * (w \cdot y) - (x * w) \cdot \alpha(y) - \alpha(w) \cdot (x * y)\big) \cdot \alpha^2(z) \\
= \; & \alpha^2(y) \cdot \mathcal{L}(x, z, w) + \alpha^2(x) \cdot \big(- \alpha(w) \cdot (z * y) - \alpha(y) \cdot (z * w) - (z \cdot w) * \alpha(y) + (z * y) \cdot \alpha(w) \\
& +(w * y) \cdot \alpha(z)\big) + \big(\alpha^2(z) * (\alpha(x) \cdot (y \cdot w)) - (\alpha(z) * \alpha(x)) \cdot (\alpha(y) \cdot \alpha(w)) \\
& -\alpha^2(x) * (\alpha(z) \cdot (y \cdot w)) + (\alpha(x) * \alpha(z)) \cdot (\alpha(y) \cdot \alpha(w)) + \alpha^2(z) \cdot (\alpha(x) * (y \cdot w))\big) \\
= \; & \alpha^2(y) \cdot \mathcal{L}(x, z, w) + \alpha^2(x) \cdot \big(- \alpha(w) \cdot (z * y) - \alpha(y) \cdot (z * w) - (z \cdot w) * \alpha(y) + (z * y) \cdot \alpha(w) \\
& +(w * y) \cdot \alpha(z) + \big) + \alpha^2(x) \cdot (\alpha(z) * (y \cdot w)) \\
= \; & \alpha^2(y) \cdot \mathcal{L}(x, z, w) + \alpha^2(x) \cdot \mathcal{L}(y, z, w).
\end{aligned}
$$

Then, the Hom-Hertling–Manin identity (3.1) holds.

□

As a special case of Hom-F-manifold admissible algebras, we define Hom-pre-Lie commutative algebra.

Definition 3.8 *A Hom-pre-Lie commutative algebra is a tuple* $(A, \cdot, *, \alpha)$*, where* $(A, \cdot, \alpha)$ *is a commutative Hom-associative algebra and* $(A, *, \alpha)$ *is a Hom-pre-Lie algebra satisfying*

$$\alpha(x) * (y \cdot z) - (x * y) \cdot \alpha(z) - \alpha(y) \cdot (x * z) = 0, \quad \forall\ x, y, z \in A. \tag{3.8}$$

Using Proposition 3.7, it is obvious to obtain the following result.

Corollary 3.9 *Let* $(A, \cdot, *, \alpha)$ *be a Hom-pre-Lie commutative algebra. Then* $(A, \cdot, [\cdot, \cdot], \alpha)$ *is a Hom-F-manifold algebra, where* $[\cdot, \cdot]$ *is given by* (3.7).

Recall that a derivation of commutative Hom-associative algebra $(A, \cdot, \alpha)$ is a linear map $D : A \to A$ such that

$$D\alpha = \alpha D$$

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y), \forall x, y \in A.$$

Proposition 3.10 *Let $(A, \cdot, \alpha)$ be a commutative Hom-associative algebra with a derivation D. Then $(A, \cdot, *, \alpha)$ being a Hom-F-manifold admissible algebra where the new product $*$ is defined by*

$$x * y \quad = \quad x \cdot D(y) + \lambda x \cdot y, \quad \forall\ x, y \in A$$

for any fixed $\lambda \in \mathbb{K}$. In particular, for $\lambda = 0$, $(A, \cdot, *)$ is a Hom-pre-Lie commutative algebra.

Proof It is easy to show that $(A, *, \alpha)$ is a Hom-pre-Lie algebra. Furthermore, by the fact that $(A, \cdot, \alpha)$ is a commutative Hom-associative algebra and D is a derivation on it, we have

$$\begin{aligned} & \alpha(x) * (y \cdot z) - (x * y) \cdot \alpha(z) - \alpha(y) \cdot (x * z) \\ = \quad & \alpha(x) \cdot D(y \cdot z) + \lambda \alpha(x) \cdot (y \cdot z) - (x \cdot D(y) + \lambda x \cdot y) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot D(z) + \lambda x \cdot z) \\ = \quad & -\lambda \alpha(x) \cdot (y \cdot z). \end{aligned}$$

Similarly, we have

$$\alpha(y) * (x \cdot z) - (y * x) \cdot \alpha(z) - \alpha(x) \cdot (y * z) = -\lambda \alpha(x) \cdot (y \cdot z).$$

Thus

$$\alpha(x) * (y \cdot z) - (x * y) \cdot \alpha(z) - \alpha(y) \cdot (x * z) = \alpha(y) * (x \cdot z) - (y * x) \cdot \alpha(z) - \alpha(x) \cdot (y * z).$$

Therefore, $(A, \cdot, *, \alpha)$ is a Hom-F-manifold admissible algebra. When $\lambda = 0$, it is obvious that $(A, \cdot, *, \alpha)$ is a Hom-pre-Lie commutative algebra. □

Example 3.11 *Let A be a 2-dimensional vector space with basis $\{e_1, e_2\}$. Define the nonzero multiplication by*

$$e_1 \cdot e_1 \quad = \quad e_1, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = b e_2, \quad b \in \mathbb{K}$$

and the linear map $\alpha : A \to A$ by

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = b e_2.$$

Then $(A, \cdot, \alpha)$ is commutative Hom-associative algebra. It is straightforward to check that the linear map $D : A \to A$ given by

$$D(e_2) = a e_2, \quad a \in \mathbb{K}$$

is a derivation on $(A, \cdot, \alpha)$. Thus by Proposition 3.10, $(A, \cdot, [\cdot, \cdot], \alpha)$ is a Hom-F-manifold algebra, where $[e_1, e_2] = a b e_2$.

Example 3.12 Let A be a 3-dimensional vector space with basis $\{e_1, e_2, e_3\}$. Define the nonzero multiplication by

$$e_2 \cdot e_3 = e_3 \cdot e_2 = b^3 e_1, \quad e_3 \cdot e_3 = b^2 e_2, \quad a \in \mathbb{K}$$

and the linear map $\alpha : A \to A$ by

$$\alpha(e_1) = b^3 e_1, \quad \alpha(e_2) = b^2 e_2, \quad \alpha(e_3) = b e_3.$$

Then $(A, \cdot, \alpha)$ is commutative Hom-associative algebra. It is straightforward to check that the linear map $D : A \to A$ given by

$$D(e_1) = 3ae_1, \quad D(e_2) = 2ae_2, \quad D(e_3) = ae_3, \quad a \in \mathbb{K}.$$

is a derivation on $(A, \cdot, \alpha)$. Thus by Proposition 3.10, $(A, \cdot, [\cdot, \cdot], \alpha)$ is a Hom-F-manifold algebra, where $[e_2, e_3] = -ab^3e_1$.

4. Representations of Hom-F-manifold algebras and Hom-pre-F-manifold algebras

In this section, first we study representations of an Hom-F-manifold algebra. Then we introduce the notions of Hom-pre-F-manifold algebras and $\mathcal{O}$-operators on an Hom-F-manifold algebra.

4.1. Representations of Hom-F-manifold algebras

In this section, we introduce the notion of representations of Hom-F-manifold algebras.

Definition 4.1 *Let $(A, \cdot, [\cdot, \cdot], \alpha)$ be a Hom-F-manifold algebra. A representation of A is a tuple $(V; \rho, \mu, \varphi)$ such that $(V; \rho, \varphi)$ is a representation of the Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$ and $(V; \mu, \varphi)$ is a representation of the commutative Hom-associative algebra $(A, \cdot, \alpha)$ satisfying*

$$R^{\alpha,\varphi}_{\rho,\mu}(x \cdot y, \alpha(z))\varphi = \mu(\alpha^2(x))R^{\alpha,\varphi}_{\rho,\mu}(y, z) + \mu(\alpha^2(y))R^{\alpha,\varphi}_{\rho,\mu}(x, z), \tag{4.1}$$

$$\mu(\mathcal{L}(x, y, z))\varphi^2 = S^{\alpha,\varphi}_{\rho,\mu}(\alpha(y), \alpha(z))\mu(x)) - \mu(\alpha^2(x))S^{\alpha,\varphi}_{\rho,\mu}(y, z), \tag{4.2}$$

where $R^{\alpha,\varphi}_{\rho,\mu}, S^{\alpha,\varphi}_{\rho,\mu} : A \otimes A \to \mathfrak{gl}(V)$ are defined by

$$R^{\alpha,\varphi}_{\rho,\mu}(x, y) = \rho(\alpha(x))\mu(y) - \mu(\alpha(y))\rho(x) - \mu([x, y])\varphi, \tag{4.3}$$

$$S^{\alpha,\varphi}_{\rho,\mu}(x, y) = \mu(\alpha(x))\rho(y) + \mu(\alpha(y))\rho(x) - \rho(x \cdot y)\varphi \tag{4.4}$$

for all $x, y, z \in A$.

Example 4.2 *Let $(V; \rho, \mu, \varphi)$ be a representation of a Hom-Poisson algebra $(P, \cdot_P, \{\cdot, \cdot\}_P, \alpha)$, i.e. $(V; \rho, \varphi)$ is a representation of the Hom-Lie algebra $(P, \{\cdot, \cdot\}_P, \alpha)$ and $(V; \mu, \varphi)$ is a representation of the commutative Hom-associative algebra $(P, \cdot_P, \alpha)$ satisfying*

$$\rho(\alpha(x)) \circ \mu(y) - \mu(\alpha(y))\rho(x) - \mu([x, y]) \circ \varphi = 0, \tag{4.5}$$

$$\mu(\alpha(x)) \circ \rho(y) + \mu(\alpha(y))\rho(x) - \rho(x \cdot y)\varphi = 0, \quad \forall\ x, y, z \in P. \tag{4.6}$$

Then $(V; \rho, \mu, \varphi)$ is also a representation of the Hom-F-manifold algebra given by this Hom-Poisson algebra P.

Example 4.3 *Let $(V; \rho, \mu)$ be a representation of a F-manifold algebra $(A, \cdot, [\cdot, \cdot])$, $\alpha : A \to A$ be an algebra morphism and $\varphi \in \mathfrak{gl}(V)$ such that for all $x \in A$*

$$\varphi\rho(x) = \rho(\alpha(x))\varphi\varphi\mu(x) = \mu(\alpha(x))\varphi.$$

Then $(V; \tilde{\rho}, \tilde{\mu}, \varphi)$ is a representation of the Hom-F-manifold $(A, \cdot_\alpha, [\cdot, \cdot]_\alpha, \alpha)$, where

$$\tilde{\rho}(x) = \rho(\alpha(x))\varphi \qquad \text{and} \qquad \tilde{\mu}(x) = \mu(\alpha(x))\varphi.$$

It obvious to obtain the following result.

Proposition 4.4 *Let $(A, \cdot, [\cdot,\cdot], \alpha)$ be a Hom-F-manifold algebra. Then $(V; \rho, \mu, \varphi)$ is a representation of A if and only if $(A \oplus V, \cdot_\mu, [\cdot,\cdot]_\rho, \alpha_\varphi)$ is a Hom-F-manifold algebra, where $(A \oplus V, \cdot_\mu, \alpha + \varphi)$ is the semidirect product commutative Hom-associative algebra $A \ltimes_\mu V$, i.e.*

$$(x_1 + v_1) \cdot_\mu (x_2 + v_2) = x_1 \cdot x_2 + \mu(x_1)v_2 + \mu(x_2)v_1, \quad \forall\ x_1, x_2 \in A,\ v_1, v_2 \in V$$

$$(\alpha + \varphi)(x_1 + v_1) = \alpha(x_1) + \varphi(v_1), \quad \forall x_1 \in A, \quad v_1 \in V,$$

and $(A \oplus V, [\cdot,\cdot]_\rho, \alpha + \varphi)$ is the semidirect product Hom-Lie algebra $A \ltimes_\rho V$, i.e.

$$[x_1 + v_1, x_2 + v_2]_\rho = [x_1, x_2] + \rho(x_1)(v_2) - \rho(x_2)(v_1), \quad \forall\ x_1, x_2 \in A,\ v_1, v_2 \in V.$$

Remark 4.5 *Let $(V; \rho, \mu, \varphi)$ be a representation of a Hom-Poisson algebra $(P, \cdot_P, \{\cdot,\cdot\}_P, \alpha)$. Then the tuple $(V^*; \rho^\star, -\mu^\star, (\varphi^*)^{-1})$ is also a representation of P. But Hom-F-manifold algebras do not have this property.*

In fact, we have

Proposition 4.6 *Let $(A, \cdot, [\cdot,\cdot], \alpha)$ be a Hom-F-manifold algebra. If the tuple $(V; \rho, \mu, \varphi)$ is representation of A satisfying the following identities*

$$R^{\alpha,\varphi}_{\rho,\mu}(x \cdot y, \alpha(z))\varphi = R^{\alpha,\varphi}_{\rho,\mu}(y, z)\mu(\alpha^2(x)) + R^{\alpha,\varphi}_{\rho,\mu}(x, z)\mu(\alpha^2(y)), \tag{4.7}$$

$$\mu(\mathcal{L}(x, y, z))\varphi = T^{\alpha,\varphi}_{\rho,\mu}(\alpha(y), \alpha(z))\mu(x) - \mu(x)T^{\alpha,\varphi}_{\rho,\mu}(\alpha(y), \alpha(z)), \tag{4.8}$$

where $R^{\alpha,\varphi}_{\rho,\mu}$ is given by Eq. (4.3) and $T^{\alpha,\varphi}_{\rho,\mu} : A \otimes A \to \mathfrak{gl}(V)$ is defined by

$$T^{\alpha,\varphi}_{\rho,\mu}(x, y) \quad = \quad \rho(\alpha(y))\mu(x) + \rho(\alpha(x))\mu(y) - \rho(x \cdot y)\varphi, \quad \forall\ x, y \in A, \tag{4.9}$$

Then $(V^*; \rho^\star, -\mu^\star, (\varphi^{-1})^*)$ is a representation of A.

Proof By direct calculations, for all $x, y \in A, v \in V, \xi \in V^*$, we have

$$\langle R_{\rho^*,-\mu^*}(x, y)(\xi), v\rangle = \langle \xi, R_{\rho,\mu}(v)\rangle, \quad \langle S_{\rho^*,-\mu^*}(x, y)(\xi), v\rangle = \langle \xi, T_{\rho,\mu}(v)\rangle.$$

Furthermore, we have

$$\begin{aligned} & \langle R_{\rho^*,-\mu^*}(x \cdot y, z)(\xi) + \mu^*(x)R_{\rho^*,-\mu^*}(y, z)(\xi) + \mu^*(y)R_{\rho^*,-\mu^*}(x, z)(\xi), v\rangle \\ = \quad & \langle \xi, R_{\rho,\mu}(x \cdot y, z)(v) - R_{\rho,\mu}(y, z)\mu(x)(v) - R_{\rho,\mu}(x, z)\mu(y)(v)\rangle \end{aligned}$$

and

$$\begin{aligned} & \langle -\mu^*(P_x(y, z))(\xi) + S_{\rho^*,-\mu^*}(y, z)\mu^*(x)(\xi) + \mu^*(x)S_{\rho^*,-\mu^*}(y, z)(\xi), v\rangle \\ = \quad & \langle \xi, \mu(P_x(y, z))(v) - T_{\rho,\mu}(y, z)\mu(x)(v) + \mu(x)T_{\rho,\mu}(y, z)(v)\rangle. \end{aligned}$$

By the hypothesis and the definition of representation, the conclusion follows immediately. □

Corollary 4.7 Let $(A, \cdot, [\cdot,\cdot], \alpha)$ be a Hom-F-manifold algebra. Then $(A; \mathrm{ad}, \mathfrak{L}, \alpha)$ is a representation of A, which is also called the regular representation. Furthermore, if the Hom-F-manifold algebra also satisfies the following relations:

$$P_{x\cdot y}(\alpha(z), \alpha(w)) = P_y(z, \alpha^2(x) \cdot w) + P_x(z, \alpha^2(y) \cdot w), \tag{4.10}$$

$$P_x(y, z) \cdot \alpha(w) = x \cdot Q(\alpha(y), \alpha(z), w) - Q(\alpha(y), \alpha(z), x \cdot w), \quad \forall\ x, y, z, w \in A, \tag{4.11}$$

where $Q : \otimes^3 \to A$ is defined by

$$Q(x, y, z) = [x \cdot y, \alpha(z)] + [y \cdot z, \alpha(x)] + [z \cdot x, \alpha(y)].$$

Then $(A^*; \mathrm{ad}^\star, -\mathfrak{L}^\star, (\alpha^{-1})^*)$ is a representation of A.

Definition 4.8 *A coherence Hom-F-manifold algebra is an Hom-F-manifold algebra such that Eqs. (4.10) and (4.11) hold.*

Proposition 4.9 *Let $(A, \cdot, [\cdot,\cdot], \alpha)$ be a Hom-F-manifold algebra. Suppose that there is a nondegenerate symmetric bilinear form $\mathfrak{B}$ such that $\mathfrak{B}$ is invariant in the following sense*

$$\mathfrak{B}(x \cdot y, \alpha(z)) = \mathfrak{B}(\alpha(x), y \cdot z), \ \ \mathfrak{B}([x, y], \alpha(z)) = \mathfrak{B}(\alpha(x), [y, z]), \ \ \forall x, y, z \in A. \tag{4.12}$$

Then $(A, \cdot, [\cdot,\cdot])$ is a coherence F-manifold algebra.

Proof By the invariance of $\mathfrak{B}$, we have

$$\mathfrak{B}(P_x(y, z), w) = \mathfrak{B}(z, P_x(y, w)) = \mathfrak{B}(x, Q(y, z, w)), \quad \forall\ x, y, z, w \in P.$$

By the above relations, for $x, y, z, w_1, w_2 \in P$, we have

$$\begin{aligned} &\mathfrak{B}(P_{x\cdot y}(z, w_1) - P_y(z, x \cdot w_1) - P_x(z, y \cdot w_1), w_2) \\ = &\mathfrak{B}(w_1, P_{x\cdot y}(z, w_2) - x \cdot P_y(z, w_2) - y \cdot P_x(z, w_2)) \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{B}(P_x(y, z) \cdot w_1 - x \cdot Q(y, z, w_1) + Q(y, z, x \cdot w), w_2) \\ = \ &\mathfrak{B}(w_1, -P_{x\cdot y}(z, w_2) + x \cdot P_y(z, w_2) + y \cdot P_x(z, w_2)). \end{aligned}$$

By the fact that A is an F-manifold algebra and $\mathfrak{B}$ is nondegenerate, we deduce that (4.10) and (4.11) hold. Thus $(A, \cdot, [\cdot,\cdot])$ is a coherence F-manifold algebra. □

4.2. Hom-pre-F-manifold algebras

Now, we introduce the notion of Hom-pre-F-manifold algebras and give some constructions.

Definition 4.10 *A Hom-pre-F-manifold algebra is a tuple $(A, \diamond, *, \alpha)$, where $(A, \diamond, \alpha)$ is a Hom-Zinbiel algebra and $(A, *, \alpha)$ is a Hom-pre-Lie algebra, such that the following compatibility conditions hold:*

$$F_1(x \cdot y, \alpha(z), \alpha(w)) = \alpha^2(x) \diamond F_1(y, z, w) + \alpha^2(y) \diamond F_1(x, z, w), \tag{4.13}$$

$$(F_1(x, y, z) + F_1(x, z, y) + F_2(y, z, x)) \diamond \alpha^2(w) = F_2(\alpha(y), \alpha(z), x \diamond w) - \alpha^2(x) \diamond F_2(y, z, w) \tag{4.14}$$

where $F_1, F_2 : \otimes^3 A \longrightarrow A$ *are defined by*

$$
\begin{aligned}
F_1(x,y,z) &= \alpha(x)*(y\diamond z)-\alpha(y)\diamond(x*z)-[x,y]\diamond\alpha(z), && (4.15)\\
F_2(x,y,z) &= \alpha(x)\diamond(y*z)+\alpha(y)\diamond(x*z)-(x\cdot y)*\alpha(z) && (4.16)
\end{aligned}
$$

and the operation $\cdot$ *and bracket* $[\cdot,\cdot]$ *are defined by*

$$x\cdot y = x\diamond y + y\diamond x, \quad [x,y]=x*y-y*x, \tag{4.17}$$

for all $x,y,z,w\in A$.

Remark 4.11 *If* $F_1=F_2=0$ *in the above definition of a Hom-pre-*F*-manifold algebra* $(A,\diamond,*,\alpha)$, *then we obtain a Hom-pre-Poisson algebra (see [2, 20] for more details).*

Theorem 4.12 *Let* $(A,\diamond,*,\alpha)$ *be a Hom-pre-*F*-manifold algebra. Then*

(i) $(A,\cdot,[\cdot,\cdot],\alpha)$ *is a Hom-*F*-manifold algebra, where the operation* $\cdot$ *and bracket* $[\cdot,\cdot]$ *are given by Eq. (4.17), which is called the subadjacent Hom-*F*-manifold algebra of* $(A,\diamond,*,\alpha)$ *and denoted by* A^c.

(ii) $(A;L,\mathrm{ad}^L_\diamond,\alpha)$ *is a representation of the subadjacent Hom-*F*-manifold algebras* A^c, *where* L *and* $\mathrm{ad}^L_\diamond$ *are given by Eqs. (2.12) and (2.6), respectively.*

Proof

(i) By Lemma 2.2 and Lemma 2.4, we deduce that $(A,\cdot,\alpha)$ is a commutative Hom-associative algebra and $(A,[\cdot,\cdot],\alpha)$ is a Hom-Lie algebra. By direct computation, we obtain

$$\mathcal{L}(x,y,z)=F_1(x,y,z)+F_1(x,z,y)+F_2(y,z,x), \quad \forall\, x,y,z\in A. \tag{4.18}$$

According to Eqs. (4.13), (4.14), and (4.18), we get

$$
\begin{aligned}
& \mathcal{L}(x\cdot y,\alpha(z),\alpha(w))-\alpha^2(x)\cdot\mathcal{L}(y,z,w)-\alpha^2(y)\cdot\mathcal{L}(x,z,w)\\
= {} & F_1(x\cdot y,\alpha(z),\alpha(w))+F_1(x\cdot y,\alpha(w),\alpha(z))\\
& +F_2(\alpha(z),\alpha(w),x\cdot y)-\alpha^2(x)\cdot(F_1(y,z,w)+F_1(y,w,z)+F_2(z,w,y))\\
& -\alpha^2(y)\cdot(F_1(x,z,w)+F_1(x,w,z)+F_2(z,w,x))\\
= {} & \big(F_1(x\cdot y,\alpha(z),\alpha(w))-\alpha^2(x)\diamond F_1(y,z,w)-\alpha^2(y)\diamond F_1(x,z,w)\big)\\
& +\big(F_1(x\cdot y,\alpha(w),\alpha(z))-\alpha^2(x)\diamond F_1(y,w,z)-\alpha^2(y)\diamond F_1(x,w,z)\big)\\
& +\big(F_2(\alpha(z),\alpha(w),x\diamond y)-F_1(x,z,w)\diamond\alpha^2(y)\\
& +F_1(x,w,z)\diamond\alpha^2(y)+F_2(z,w,x)\diamond\alpha^2(y)+\alpha^2(x)\diamond F_2(z,w,y)\big)\\
& +\big(F_2(\alpha(z),\alpha(w),y\diamond x)-F_1(y,z,w)\diamond\alpha^2(x)\\
& +F_1(y,w,z)\diamond\alpha^2(x)+F_2(z,w,y)\diamond\alpha^2(x)+\alpha^2(y)\diamond F_2(z,w,x)\big)\\
= {} & 0.
\end{aligned}
$$

Thus $(A,\cdot,[\cdot,\cdot],\alpha)$ is a Hom-F-manifold algebra.

(ii) Thanks to Lemma 2.2 and Lemma 2.4, $(A; \mathrm{ad}^L, \alpha)$ is a representation of the commutative Hom-associative algebra $(A, \cdot, \alpha)$ as well as $(A; L, \alpha)$ is a representation of the subadjacent Hom-Lie algebra A^c. Moreover, note that

$$F_1(x, y, z) = R^{\alpha,\alpha}_{L,\mathrm{ad}^L_\diamond}(x, y)(z) \quad \text{and} \quad F_2(x, y, z) = S^{\alpha,\alpha}_{L,\mathrm{ad}^L_\diamond}(x, y)(z).$$

Thus Eq. (4.13) implies that Eq. (4.1) holds and, by Eq. (4.18), Eq. (4.14) implies that Eq. (4.2) holds. Therefore, $(A; L, \mathrm{ad}^L_\diamond, \alpha)$ is a representation of the subadjacent Hom-F-manifold algebra A^c.

□

Proposition 4.13 *Let $(A, \cdot, [\cdot, \cdot], \alpha)$ be a Hom-F-manifold algebra. If $\omega \in \wedge^2 A^*$ is nondegenerate, both a Connes cocycle on the commutative Hom-associative $(A, \cdot, \alpha)$ and a symplectic structure on the Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$, i.e.*

$$\omega\alpha^{\otimes 2} = \omega,$$
$$\omega(x \cdot y, \alpha(z)) + \omega(y \cdot z, \alpha(x)) + \omega(z \cdot x, \alpha(y)) = 0,$$
$$\omega([x, y], \alpha(z)) + \omega([y, z], \alpha(x)) + \omega([z, x], \alpha(y)) = 0, \qquad \forall\ x, y, z \in A.$$

Then $(A, \diamond, *, \alpha)$ is a Hom-pre-F-manifold algebra, where $\diamond$ and $*$ are determined by

$$\omega(x \diamond y, \alpha(z)) = \omega(\alpha(y), z \cdot x), \quad \omega(x * y, \alpha(z)) = -\omega(\alpha(y), [x, z]), \quad \forall\ x, y, z \in A.$$

4.3. $\mathcal{O}$-operators of Hom-F-manifold algebras

The notion of an $\mathcal{O}$-operator was first given for Lie algebras by Kupershmidt in [25] as a natural generalization of the classical Yang–Baxter equation and then defined by analogy in other various (associative, alternative, Jordan, etc.).

A linear map $T : V \longrightarrow A$ is called an $\mathcal{O}$-operator on a commutative Hom-associative algebra $(A, \cdot, \alpha)$ with respect to a representation $(V; \mu, \varphi)$ if T satisfies

$$T\varphi = \alpha T, \tag{4.19}$$

$$T(u) \cdot T(v) = T(\mu(T(u))v + \mu(T(v))u), \quad \forall\ u, v \in V. \tag{4.20}$$

In particular, an $\mathcal{O}$-operator on a commutative Hom-associative algebra $(A, \cdot, \alpha)$ with respect to the adjoint representation is called a Rota–Baxter operator of weight zero or briefly a Rota–Baxter operator on A.

It is obvious to obtain the following result.

Proposition 4.14 *Let $(A, \cdot, \alpha)$ be a commutative Hom-associative algebra and $(V; \mu, \varphi)$ a representation. Let $T : V \to A$ be an $\mathcal{O}$-operator on $(A, \cdot, \alpha)$ with respect to $(V; \mu, \varphi)$. Then there exists a Hom-Zinbiel algebra structure on V given by*

$$u \diamond v = \mu(T(u))v, \quad \forall\ u, v \in V.$$

A linear map $T : V \longrightarrow A$ is called an $\mathcal{O}$-operator on a Hom-Lie algebra $(A, [\cdot, \cdot], \alpha)$ with respect to a representation $(V; \rho, \varphi)$ if T satisfies

$$T\varphi = \alpha T, \tag{4.21}$$

$$[T(u), T(v)] = T\Big(\rho(T(u))(v) - \rho(T(v))(u)\Big), \quad \forall\ u, v \in V. \tag{4.22}$$

In particular, an $\mathcal{O}$-operator on a Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ with respect to the adjoint representation is called a Rota–Baxter operator of weight zero or briefly a Rota–Baxter operator on A.

Lemma 4.15 ([29]) *Let $T : V \to A$ be an $\mathcal{O}$-operator on a Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ with respect to a representation $(V; \rho, \varphi)$. Define a multiplication $*$ on V by*

$$u * v = \rho(T(u))(v), \quad \forall\ u, v \in V. \tag{4.23}$$

Then $(V, *, \alpha)$ is a Hom-pre-Lie algebra.

Let $(V; \rho, \mu, \varphi)$ be a representation of a Hom-F-manifold algebra $(A, \cdot, [\cdot,\cdot], \alpha)$.

Definition 4.16 *A linear operator $T : V \longrightarrow A$ is called an $\mathcal{O}$-operator on $(A, \cdot, [\cdot,\cdot], \alpha)$ if T is both an $\mathcal{O}$-operator on the commutative Hom-associative algebra $(A, \cdot, \alpha)$ and an $\mathcal{O}$-operator on the Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$.*

In particular, a linear operator $\mathcal{R} : A \longrightarrow A$ is called a Rota–Baxter operator of weight zero or briefly a Rota–Baxter operator on A, if R is both a Rota–Baxter operator on the commutative Hom-associative algebra $(A, \cdot, \alpha)$ and a Rota–Baxter operator on the Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$.

$$\begin{tikzcd}
\textbf{Hom-Poisson algebra} \arrow[r, "\text{generalization}"] \arrow[d, shift right, "\text{Rota-Baxter operator}"'] & \textbf{Hom-}F\textbf{-manifold algebra} \arrow[d, shift right, "\text{Rota-Baxter operator}"'] \\
\textbf{Hom-pre-Poisson algebra} \arrow[r, "\text{generalization}"] \arrow[u, shift right, "\text{commutator}"'] & \textbf{Hom-pre-}F\textbf{-manifold algebra} \arrow[u, shift right, "\text{commutator}"']
\end{tikzcd}$$

We show that on one hand, an $\mathcal{O}$-operator on an Hom-F-manifold algebra gives a Hom-pre-F-manifold algebra, and on the other hand, a Hom-pre-F-manifold algebra naturally gives an $\mathcal{O}$-operator on the subadjacent Hom-F-manifold algebra. More examples on Hom-pre-F-manifold algebras and Hom-F-manifold algebras are given.

A $\mathcal{O}$-operator on a Hom-F-manifold algebra gives a Hom-pre-F-manifold algebra.

Theorem 4.17 *Let $(A, \cdot, [\cdot,\cdot], \alpha)$ be a Hom-F-manifold algebra and $T : V \longrightarrow A$ an $\mathcal{O}$-operator on A with respect to the representation $(V; \rho, \mu, \varphi)$. Define new operations $\diamond$ and $*$ on V by*

$$u \diamond v = \mu(T(u))v, \quad u * v = \rho(T(u))v,\ \forall\ u, v \in V.$$

Then $(V, \diamond, *, \varphi)$ is a Hom-pre-F-manifold algebra and T is a homomorphism from V^c to $(A, \cdot, [\cdot,\cdot], \alpha)$.

Proof Since T is an $\mathcal{O}$-operator on the commutative Hom-associative algebra $(A, \cdot, \alpha)$ as well as an $\mathcal{O}$-operator on the Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ with respect to the representations $(V; \mu, \varphi)$ and $(V; \rho, \varphi)$, respectively. We deduce that $(V, \diamond, \varphi)$ is a Hom-Zinbiel algebra and $(V, *, \varphi)$ is a Hom-pre-Lie algebra. Put

$$[u, v]_T := u * v - v * u = \rho(T(u))v - \rho(T(v))u \quad \text{and} \quad u \cdot_T v := u \diamond v + v \diamond u = \mu(T(u))v + \mu(T(v))u.$$

By these facts and Eq. (4.1), for $v_1, v_2, v_3, v_4 \in V$, one has

$$\begin{aligned}
& F_1(v_1 \cdot_T v_2, \varphi(v_3), \varphi(v_4)) - \varphi^2(v_1) \diamond F_1(v_2, v_3, v_4) - \varphi^2(v_2) \diamond F_1(v_1, v_3, v_4) \\
= \;& \varphi(v_1 \cdot_T v_2) * (\varphi(v_3) \diamond \varphi(v_4)) - \varphi(\varphi(v_3)) \diamond ((v_1 \cdot_T v_2) * \varphi(v_4)) - [v_1 \cdot_T v_2, \varphi(v_3)]_T \diamond \varphi(\varphi(v_4)) \\
& -\varphi^2(v_1) \diamond (\varphi(v_2) * (v_3 \diamond v_4)) + \varphi^2(v_1) \diamond (\varphi(v_3) \diamond (v_2 * v_4)) + \varphi^2(v_1) \diamond ([v_2, v_3]_T \diamond \varphi(v_4)) \\
& -\varphi^2(v_2) \diamond (\varphi(v_1) * (v_3 \diamond v_4)) + \varphi^2(v_2) \diamond (\varphi(v_3) \diamond (v_1 * v_4)) + \varphi^2(v_2) \diamond ([v_1, v_3]_T \diamond \varphi(v_4)) \\
= \;& \rho(T(\varphi(v_1 \cdot_T v_2))(\mu(T(\varphi(v_3)))\varphi(v_4)) - \mu(T(\varphi(\varphi(v_3)))(\rho(T(v_1 \cdot_T v_2))\varphi(v_4)) \\
& -\mu(T[v_1 \cdot_T v_2, \varphi(v_3)]_T)\varphi(\varphi(v_4)) - \mu(T(\varphi^2(v_1))(\rho(T(\varphi(v_2))\mu(T(v_3)v_4) \\
& +\mu(T(\varphi^2(v_1))(\mu(T(\varphi(v_3))(\rho(T(v_2))v_4)) + \mu(T(\varphi^2(v_1))\mu(T[v_2, v_3]_T)\varphi(v_4)) \\
& -\mu(T(\varphi^2(v_2))(\rho(T(\varphi(v_1))(\mu(T(v_3)v_4)) + \mu(T(\varphi^2(v_2))(\mu(T(\varphi(v_3))(\rho(T(v_1)v_4)) \\
& +\mu(T(\varphi^2(v_2))(\mu(T[v_1, v_3]_T)\varphi(v_4)) \\
= \;& R^{\alpha,\varphi}_{\rho,\mu}(T(v_1) \cdot T(v_2), \alpha(T(v_3)))(\varphi(v_4)) - \mu(\alpha^2(T(v_1)))R^{\alpha,\varphi}_{\rho,\mu}(T(v_2), T(v_3))(v_4) \\
& -\mu(\alpha^2(T(v_2)))R^{\alpha,\varphi}_{\rho,\mu}(T(v_1), T(v_3))(v_4) = 0,
\end{aligned}$$

which implies that Eq. (4.13) holds.

Similarly, according to Eq. (4.2) we can check Eq. (4.14). Thus, $(V, \diamond, *, \varphi)$ is a Hom-pre-F-manifold algebra. Furthermore, for all $u, v \in V$ we have

$$[T(u), T(v)] = T\Big(\rho(T(u))(v) - \rho(T(v))(u)\Big) = T(u * v - v * u) = T([u, v]_T)$$

Similarly, it is obvious to check that $T(u) \cdot T(v) = T(u \cdot_T v)$. Thus, T is a homomorphism from V^c to $(A, \cdot, [\cdot, \cdot], \alpha)$. □

Corollary 4.18 *Let $(A, \cdot, [\cdot, \cdot], \alpha)$ be a Hom-F-manifold algebra and $T : V \longrightarrow A$ an $\mathcal{O}$-operator on A with respect to the representation $(V; \rho, \mu, \varphi)$. Then $T(V) = \{T(v) \mid v \in V\} \subset A$ is a subalgebra of A and there is an induced Hom-F-manifold algebra structure on $T(V)$ given by*

$$T(u) \diamond T(v) = T(u \diamond v), \quad T(u) * T(v) = T(u * v)$$

for all $u, v \in V$.

Corollary 4.19 *Let $(A, \cdot, [\cdot, \cdot], \alpha)$ be an Hom-F-manifold algebra and $\mathcal{R} : A \longrightarrow A$ a Rota–Baxter operator. Define new operations on A by*

$$x \diamond y = \mathcal{R}(x) \cdot y, \quad x * y = [\mathcal{R}(x), y].$$

Then $(A, \diamond, *, \alpha)$ is a Hom-pre-F-manifold algebra and $\mathcal{R}$ is a homomorphism from the subadjacent F-manifold algebras $(A, \cdot_{\mathcal{R}}, [\cdot, \cdot]_{\mathcal{R}}, \alpha)$ to $(A, \cdot, [\cdot, \cdot], \alpha)$, where $x \cdot_{\mathcal{R}} y = x \diamond y + y \diamond x$ and $[x, y]_{\mathcal{R}} = x * y - y * x$.

Corollary 4.20 *Consider the F-manifold algebra $(A, \cdot, [\cdot, \cdot])$ given by Proposition 3.10. If $\mathcal{R} : A \longrightarrow A$ is a Rota–Baxter operator on the commutative associative algebra $(A, \cdot)$ and satisfies $\mathcal{R}D = DB$, then $\mathcal{R} : A \longrightarrow A$ is a Rota–Baxter operator on the Lie algebra $(A, [\cdot, \cdot])$. Thus $(A, \diamond_{\mathcal{R}}, *_{\mathcal{R}})$ is a pre-F-manifold algebra, where*

$$x \diamond_{\mathcal{R}} y = \mathcal{R}(x) \cdot y, \quad x *_{\mathcal{R}} y = \mathcal{R}(x) \cdot D(y) - y \cdot D(\mathcal{R}(x))$$

and $(A, \cdot_{\mathcal{R}}, [\cdot,\cdot]_{\mathcal{R}})$ *is the subadjacent* F*-manifold algebra with*

$$x \cdot_{\mathcal{R}} y = x \diamond_{\mathcal{R}} y + y \diamond_{\mathcal{R}} x, \quad [x,y]_{\mathcal{R}} = x *_{\mathcal{R}} y - y *_{\mathcal{R}} x.$$

Example 4.21 We put $A = C^{\infty}([0,1])$. Then $(A, \cdot, [\cdot,\cdot], \alpha)$ is an Hom-F-manifold algebra where the multiplication, the bracket and the twist map are defined by:

$$\begin{array}{rcl} f \cdot g &=& \lambda f g, \\ [f,g] &=& \lambda(fg' - gf') \\ \alpha(f) &=& \lambda f, \quad \forall\ f,g \in A,\ \lambda \in \mathbb{R}. \end{array}$$

It is well-known that the integral operator is a Rota–Baxter operator:

$$\mathcal{R} : A \to A, \quad \mathcal{R}(f)(x) := \int_0^x f(t)dt.$$

It is easy to see that

$$\mathcal{R}\partial_x = \partial_x \mathcal{R} = \mathrm{Id}.$$

Thus $(A, \diamond_{\mathcal{R}}, *_{\mathcal{R}}, \alpha)$ is a Hom-pre-F-manifold algebra, where

$$f \diamond_{\mathcal{R}} g = g \int_0^x f(t)dt, \quad f *_{\mathcal{R}} g = g' \int_0^x f(t)dt - f \cdot g$$

and $(A, \cdot_{\mathcal{R}}, [\cdot,\cdot]_{\mathcal{R}}, \alpha)$ is the subadjacent Hom-F-manifold algebra with

$$f \cdot_{\mathcal{R}} g = f \int_0^x g(t)dt + g \int_0^x f(t)dt, \quad [f,g]_{\mathcal{R}} = f' \int_0^x g(t)dt - g' \int_0^x f(t)dt.$$

Example 4.22 Consider the F-manifold algebra $(A, \cdot, [\cdot,\cdot])$ given by Example 3.12. It is straightforward to check that $\mathcal{R}$ given by

$$\mathcal{R}(e_1) = re_1, \quad \mathcal{R}(e_2) = 2se_1 + \frac{3}{2}re_2, \quad \mathcal{R}(e_3) = te_1 + 3se_2 + 3re_3, \quad r,s,t \in \mathbb{K}$$

is a Rota–Baxter operator on the F-manifold algebra A. Thus $(A, \diamond_{\mathcal{R}}, *_{\mathcal{R}})$ is a pre-F-manifold algebra, where

$$\begin{array}{rcl} e_2 \diamond_{\mathcal{R}} e_3 &=& \frac{3}{2}re_1, \quad e_3 \diamond_{\mathcal{R}} e_2 = 3re_1, \quad e_3 \diamond_{\mathcal{R}} e_3 = 3se_1 + 3re_2; \\ e_2 *_{\mathcal{R}} e_3 &=& -\frac{3}{2}are_1, \quad e_3 *_{\mathcal{R}} e_2 = 3are_1, \quad e_3 *_{\mathcal{R}} e_3 = -3ase_1 \end{array}$$

and $(A, \cdot_{\mathcal{R}}, [\cdot,\cdot]_{\mathcal{R}})$ is the subadjacent F-manifold algebra with

$$\begin{array}{rcl} e_2 \cdot_{\mathcal{R}} e_3 &=& \frac{9}{2}re_1, \quad e_3 \cdot_{\mathcal{R}} e_3 = 6se_1 + 6re_2; \\ [e_2, e_3]_{\mathcal{R}} &=& -\frac{9}{2}are_1. \end{array}$$

At the end of this section, we give a necessary and sufficient condition on an Hom-F-manifold algebra admitting a Hom-pre-F-manifold algebra structure.

Proposition 4.23 *Let $(A,\cdot,[\cdot,\cdot],\alpha)$ be an Hom-F-manifold algebra. There is a Hom-pre-F-manifold algebra structure on A such that its subadjacent Hom-F-manifold algebra is exactly $(A,\cdot,[\cdot,\cdot],\alpha)$ if and only if there exists an invertible $\mathcal{O}$-operator on $(A,\cdot,[\cdot,\cdot],\alpha)$.*

Proof If $T:V\longrightarrow A$ is an invertible $\mathcal{O}$-operator on A with respect to the representation $(V;\rho,\mu,\varphi)$, then the compatible Hom-pre-F-manifold algebra structure on A is given by

$$x\diamond y=T(\mu(x)(T^{-1}(y))),\quad x*y=T(\rho(x)(T^{-1}(y)))$$

for all $x,y\in P$, since

$$\begin{aligned} x\diamond y+y\diamond x&=T(\mu(x)(T^{-1}(y)))+T(\mu(y)(T^{-1}(x)))\\ &=T(\mu(TT^{-1}(x))(T^{-1}(y))+\mu(TT^{-1}(y))(T^{-1}(x)))\\ &=TT^{-1}(x)\cdot TT^{-1}(y)=x\cdot y \end{aligned}$$

and

$$\begin{aligned} x*y-y*x&=T(\rho(x)(T^{-1}(y)))-T(\rho(y)(T^{-1}(x)))\\ &=T(\rho(TT^{-1}(x))(T^{-1}(y))-\rho(TT^{-1}(y))(T^{-1}(x)))\\ &=[TT^{-1}(x),TT^{-1}(y)]=[x,y]. \end{aligned}$$

Conversely, let $(A,\diamond,*,\alpha)$ be a Hom-pre-F-manifold algebra and $(A,\cdot,[\cdot,\cdot],\alpha)$ the subadjacent Hom-F-manifold algebra. Then the identity map Id is an $\mathcal{O}$-operator on A with respect to the representation $(A;\mathrm{ad},\mathcal{L},\alpha)$. □

Corollary 4.24 *Let $(A,\cdot,[\cdot,\cdot],\alpha)$ be a coherence Hom-F-manifold algebra. Let $\omega\in\wedge^2A^*$ be a cyclic 2-cocycle in the sense of Connes on the commutative Hom-associative algebra $(A,\cdot,\alpha)$, i.e.*

$$\omega(\alpha(x),\alpha(y))=\omega(x,y),\quad \forall\ x,y\in A,$$

$$\omega(x\cdot y,\alpha(z))+\omega(y\cdot z,\alpha(x))+\omega(z\cdot x,\alpha(y))=0,\quad \forall\ x,y,z\in A,$$

as well as a symplectic structure on the Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$, i.e.

$$\omega([x,y],\alpha(z))+\omega([y,z],\alpha(x))+\omega([z,x],\alpha(y))=0,\quad \forall x,y,z\in A$$

and

$$\omega(\alpha(x),\alpha(y))=\omega(x,y),\quad \forall\ x,y\in A.$$

Then there is a compatible Hom-pre-F-manifold algebra structure on A given by

$$\omega(x\diamond y,\alpha(z))=\omega(\alpha(y),x\cdot z),\quad \omega(x*y,\alpha(z))=\omega(\alpha(y),[z,x]).$$

Proof Since $(A,\cdot,[\cdot,\cdot],\alpha)$ is a coherence Hom-F-manifold algebra, $(A^*;\mathrm{ad}^\star,-\mathcal{L}^\star,(\alpha^{-1})^*)$ is a representation of A. By the fact that ω is a cyclic 2-cocycle, $(\omega^\sharp)^{-1}$ is an $\mathcal{O}$-operator on the commutative Hom-associative algebra $(A,\cdot,\alpha)$ with respect to the representation $(A^*;-\mathcal{L}^\star,(\alpha^{-1})^*)$, where $\omega^\sharp: A\longrightarrow A^*$ is defined by $\langle\omega^\sharp(x),y\rangle=\omega(x,y)$. By the fact that ω is a symplectic structure, $(\omega^\sharp)^{-1}$ is an $\mathcal{O}$-operator on the Hom-Lie algebra $(A,[\cdot,\cdot],\alpha)$ with respect to the representation $(A^*;\mathrm{ad}^\star,(\alpha^{-1})^*)$. Thus, $(\omega^\sharp)^{-1}$ is an $\mathcal{O}$-operator on the coherence Hom-F-manifold algebra $(A,\cdot,[\cdot,\cdot],\alpha)$ with respect to the representation $(A^*;\mathrm{ad}^\star,-\mathcal{L}^\star,(\alpha^{-1})^*)$. By Proposition 4.23, there is a compatible Hom-pre-F-manifold algebra structure on A given as above. □

5. Hom-pre-Lie formal deformation of commutative Hom-pre-Lie algebras

In this section, we introduce the notion of Hom-pre-Lie formal deformations of commutative Hom-associative algebras (that is, commutative Hom-pre-Lie algebras) and show that Hom-F-manifold algebras are the corresponding semiclassical limits. This illustrates that Hom-F-manifold algebras are the same to Hom-pre-Lie algebras as Hom-Poisson algebras to Hom-associative algebras. Furthermore, we show that Hom-pre-Lie infinitesimal deformations and extensions of Hom-pre-Lie n-deformations to Hom-pre-Lie $(n+1)$-deformations of a commutative Hom-associative algebra A are classified by the second and the third cohomology group of the Hom-pre-Lie algebra A (view the commutative Hom-associative algebra A as a Hom-pre-Lie algebra).

Let $(A,\cdot,\alpha)$ be a commutative Hom-associative algebra. Recall that an associative formal deformation of A is a sequence of bilinear maps $\mu_n: A\times A\to A$ for $n\geqslant 0$ with μ_0 being the commutative associative algebra product $\cdot$ on A, such that the $\mathbb{K}[[t]]$-bilinear product $\cdot_t$ on $A[[t]]$ determined by

$$x\cdot_t y=\sum_{n=0}^{\infty}t^n\mu_n(x,y),\quad \forall\ x,y\in A$$

is associative, where $A[[t]]$ is the set of formal power series of t with coefficients in A. Define

$$\{x,y\}=\mu_1(x,y)-\mu_1(y,x),\quad \forall\ x,y\in A.$$

It is well known [32] that $(A,\cdot,\{\cdot,\cdot\},\alpha)$ is a Hom-Poisson algebra, called the semiclassical limit of $(A[[t]],\cdot_t,\alpha)$.

Since a Hom-associative algebra can be regarded as a Hom-pre-Lie algebra, one may look for formal deformations of a commutative Hom-associative algebra into Hom-pre-Lie algebras, that is, in the aforementioned associative formal deformation, replace the associative product $\cdot_t$ by the Hom-pre-Lie product, and wonder what additional structure will appear on A. On the other hand, such an approach can be also seen as formal deformations of a commutative Hom-pre-Lie algebra into (noncommutative) Hom-pre-Lie algebras, which is completely parallel to the associative formal deformations of a commutative Hom-associative algebra into (noncommutative) Hom-associative algebras. Surprisingly, we find that this is the structure of the F-manifold algebra. Now we give the definition of a Hom-pre-Lie formal deformation of a commutative Hom-associative algebra.

Definition 5.1 *Let $(A,\cdot,\alpha)$ be a commutative Hom-associative algebra. A Hom-pre-Lie formal deformation of A is a sequence of bilinear maps $\mu_k: A\times A\to A$ for $k\geqslant 0$ with μ_0 being the commutative associative algebra product $\cdot$ on A, such that the $\mathbb{K}[[h]]$-bilinear product $\cdot_t$ on $A[[t]]$ determined by*

$$x\cdot_t y=\sum_{n=0}^{\infty}t^n\mu_n(x,y),\quad \forall\ x,y\in A$$

is a Hom-pre-Lie algebra product.

Note that the rule of Hom-pre-Lie algebra product $\cdot_t$ on $A[[t]]$ is equivalent to

$$\sum_{i+j=k} \big(\mu_i(\mu_j(x,y),\alpha(z)) - \mu_i(\alpha(x),\mu_j(y,z))\big) = \sum_{i+j=k} \big(\mu_i(\mu_j(y,x),\alpha(z)) - \mu_i(\alpha(y),\mu_j(x,z))\big), \quad \forall\ k \geqslant 0. \tag{5.1}$$

Theorem 5.2 *Let $(A,\cdot,\alpha)$ be a commutative Hom-associative algebra and $(A[[t]],\cdot_t,\alpha)$ a Hom-pre-Lie formal deformation of A. Define*

$$[x,y] = \mu_1(x,y) - \mu_1(y,x), \quad \forall\ x,y \in A.$$

Then $(A,\cdot,[\cdot,\cdot],\alpha)$ is an Hom-F-manifold algebra. The Hom-F-manifold algebra $(A,\cdot,[\cdot,\cdot],\alpha)$ is called the semiclassical limit of $(A[[t]],\cdot_t,\alpha)$. The Hom-pre-Lie algebra $(A[[t]],\cdot_t,\alpha)$ is called a Hom-pre-Lie deformation quantization of $(A,\cdot,\alpha)$.

Proof Define the bracket $[\cdot,\cdot]_t$ on $A[[t]]$ by

$$[x,y]_t = x \cdot_t y - y \cdot_t x = t[x,y] + t^2(\mu_2(x,y) - \mu_2(y,x)) + \cdots, \quad \forall\ x,y \in A.$$

By the fact that $(A[[t]],\cdot_t,\alpha)$ is a Hom-pre-Lie algebra, $(A[[t]],[\cdot,\cdot]_t,\alpha)$ is a Hom-Lie algebra.

The t^2-terms of the Hom-Jacobi identity for $[\cdot,\cdot]_t$ gives the Jacobi identity for $[\cdot,\cdot]$. Indeed,

$$\begin{aligned} 0 = \circlearrowleft_{x,y,z} [\alpha(x),[y,z]_t]_t &= \circlearrowleft_{x,y,z} [\alpha(x), t[y,z] + t^2(\mu_2(y,z) - \mu_2(z,y)) + \cdots]_t, \qquad \forall t \\ &= t^2 \circlearrowleft_{x,y,z} [\alpha(x),[y,z]] + \cdots, \qquad \forall t. \end{aligned}$$

Thus $(A,[\cdot,\cdot],\alpha)$ is a Hom-Lie algebra.

For $k=1$ in (5.1), by the commutativity of μ_0, we have

$$\begin{aligned} &\mu_0(\mu_1(x,y),\alpha(z)) - \mu_0(\alpha(x),\mu_1(y,z)) - \mu_1(\alpha(x),\mu_0(y,z)) \\ &= \mu_0(\mu_1(y,x),\alpha(z)) - \mu_0(\alpha(y),\mu_1(x,z)) - \mu_1(\alpha(y),\mu_0(x,z)). \end{aligned}$$

This is just the equality (3.6) with $x \cdot y = \mu_0(x,y)$ and $x * y = \mu_1(x,y)$ for $x,y \in A$. Thus $(A,\cdot,*,\alpha)$ is an Hom-F-manifold-admissible algebra. By Theorem 3.7, $(A,\cdot,[\cdot,\cdot],\alpha)$ is an Hom-F-manifold algebra. $\square$

In the sequel, we study Hom-pre-Lie n-deformations and pre-Lie infinitesimal deformations of commutative Hom-associative algebras.

Definition 5.3 *Let $(A,\cdot,\alpha)$ be a commutative Hom-associative algebra. A Hom-pre-Lie n-deformation of A is a sequence of bilinear maps $\mu_i : A \times A \to A$ for $0 \le i \le n$ with μ_0 being the commutative Hom-associative algebra product $\cdot$ on A, such that the $\mathbb{K}[[t]]/(t^{n+1})$-bilinear product $\cdot_t$ on $A[[t]]/(t^{n+1})$ determined by*

$$x \cdot_t y = \sum_{k=0}^{n} t^n \mu_k(x,y), \quad \forall\ x,y \in A$$

is a Hom-pre-Lie algebra product.

We call a Hom-pre-Lie 1-deformation of a commutative Hom-associative algebra $(A, \cdot, \alpha)$ a Hom-pre-Lie infinitesimal deformation and denote it by (A, μ_1, α).

By direct calculations, (A, μ_1, α) is a Hom-pre-Lie infinitesimal deformation of a commutative Hom-associative algebra $(A, \cdot, \alpha)$ if and only if for all $x, y, z \in A$

$$\begin{aligned} & \mu_1(x,y)\cdot\alpha(z) - \alpha(x)\cdot\mu_1(y,z) - \mu_1(x, y\cdot\alpha(z)) \\ = \; & \mu_1(y,x)\cdot\alpha(z) - \alpha(y)\cdot\mu_1(x,z) - \mu_1(\alpha(y), x\cdot z). \end{aligned} \tag{5.2}$$

Equation (5.2) means that μ_1 is a 2-cocycle for the Hom-pre-Lie algebra $(A, \cdot)$, i.e. $\delta^{\mathrm{reg}}\mu_1 = 0$.

Two Hom-pre-Lie infinitesimal deformations $A_t = (A, \mu_1, \alpha)$ and $A'_t = (A, \mu'_1, \alpha')$ of a commutative Hom-associative algebra $(A, \cdot, \alpha)$ are said to be equivalent if there exists a family of Hom-pre-Lie algebra homomorphisms $\mathrm{Id} + t\varphi : A_t \longrightarrow A'_t$ modulo t^2. A Hom-pre-Lie infinitesimal deformation is said to be trivial if there exists a family of Hom-pre-Lie algebra homomorphisms $\mathrm{Id} + t\varphi : A_t \longrightarrow (A, \cdot, \alpha)$ modulo t^2.

By direct calculations, A_t and A'_t are equivalent Hom-pre-Lie infinitesimal deformations if and only if

$$\mu_1(x,y) - \mu'_1(x,y) \;=\; x\cdot\varphi(y) + \varphi(x)\cdot y - \varphi(x\cdot y). \tag{5.3}$$

Equation (5.3) means that $\mu_1 - \mu'_1 = \delta^{\mathrm{reg}}\varphi$. Thus we have

Theorem 5.4 *There is a one-to-one correspondence between the space of equivalence classes of Hom-pre-Lie infinitesimal deformations of A and the second cohomology group $H^2_{\mathrm{reg}}(A, A)$.*

It is routine to check that.

Proposition 5.5 *Let $(A, \cdot, \alpha)$ be a commutative Hom-associative algebra such that $H^2_{\mathrm{reg}}(A, A) = 0$. Then all Hom-pre-Lie infinitesimal deformations of A are trivial.*

Definition 5.6 *Let $\{\mu_1, \cdots, \mu_n\}$ be a Hom-pre-Lie n-deformation of a commutative Hom-associative algebra $(A, \cdot, \alpha)$. A Hom-pre-Lie $(n+1)$-deformation of a commutative Hom-associative algebra $(A, \cdot, \alpha)$ given by $\{\mu_1, \cdots, \mu_n, \mu_{n+1}\}$ is called an extension of the Hom-pre-Lie n-deformation given by $\{\mu_1, \cdots, \mu_n\}$.*

Theorem 5.7 *For any Hom-pre-Lie n-deformation of a commutative Hom-associative algebra $(A, \cdot, \alpha)$, the $\Theta_n \in \mathrm{Hom}(\otimes^3\mathfrak{g}, A)$ defined by*

$$\Theta_n(x,y,z) = \sum_{i+j=n+1, i,j\geq 1} \big(\mu_i(\mu_j(x,y),\alpha(z)) - \mu_i(\alpha(x),\mu_j(y,z)) - \mu_i(\mu_j(y,x),\alpha(z)) + \mu_i(\alpha(y),\mu_j(x,z))\big) \tag{5.4}$$

is a cocycle, i.e. $\delta^{\mathrm{reg}}\Theta_n = 0$.

Moreover, the Hom-pre-Lie n-deformation $\{\mu_1, \cdots, \mu_n\}$ extends into some Hom-pre-Lie $(n+1)$-deformation if and only if $[\Theta_n] = 0$ in $H^3_{\mathrm{reg}}(A, A)$.

Proof It is obvious that

$$\Theta_n(x,y,z) = -\Theta_n(y,x,z), \quad \forall\, x,y,z \in A.$$

Thus Θ_n is an element of $C^3(A, A)$. It is straightforward to check that the cochain $\Theta_n \in C^3(A, A)$ is closed.

Assume that the Hom-pre-Lie $(n+1)$-deformation of a commutative Hom-associative algebra $(A,\cdot)$ given by $\{\mu_1,\cdots,\mu_n,\mu_{n+1}\}$ is an extension of the Hom-pre-Lie n-deformation given by $\{\mu_1,\cdots,\mu_n\}$, then we have

$$\begin{aligned}&\alpha(x)\cdot\mu_{n+1}(y,z)-\alpha(y)\cdot\mu_{n+1}(x,z)+\mu_{n+1}(y,x)\cdot\alpha(z)-\mu_{n+1}(x,y)\cdot\alpha(z)+\mu_{n+1}(y,x)\cdot\alpha(z)\\&\quad-\mu_{n+1}(x,y)\cdot\alpha(z)=\sum_{i+j=n+1,i,j\geq 1}\big(\mu_i(\mu_j(x,y),\alpha(z))-\mu_i(\alpha(x),\mu_j(y,z))-\mu_i(\mu_j(y,x),\alpha(z))\\&\quad+\mu_i(\alpha(y),\mu_j(x,z))\big).\end{aligned}$$

It is obvious that the right-hand side of the above equality is just $\Theta_n(x,y,z)$. We can rewrite the above equality as

$$\delta^{\text{reg}}\mu_{n+1}(x,y,z)=\Theta_n(x,y,z).$$

We conclude that, if a Hom-pre-Lie n-deformation of a commutative Hom-associative algebra $(A,\cdot,\alpha)$ extends to a Hom-pre-Lie $(n+1)$-deformation, then Θ_n is coboundary.

Conversely, if Θ_n is coboundary, then there exists an element $\psi\in C^2(A,A)$ such that

$$\delta^{\text{reg}}\psi(x,y,z)=\Theta_n(x,y,z).$$

It is not hard to check that $\{\mu_1,\cdots,\mu_n,\mu_{n+1}\}$ with $\mu_{n+1}=\psi$ generates a Hom-pre-Lie $(n+1)$-deformation of $(A,\cdot,\alpha)$ and thus this Hom-pre-Lie $(n+1)$-deformation is an extension of the Hom-pre-Lie n-deformation given by $\{\mu_1,\cdots,\mu_n\}$. □

References

[1] Aizawa N, Sato H. q-deformation of the Virasoro algebra with central extension. Physics Letters B 1991; 256 (2): 185-190.

[2] Aguiar M. Pre-poisson algebras. Letters in Mathematical Physics 2000; 54 (4): 263-277.

[3] Ammar F, Mabrouk S, Makhlouf A. Representations and cohomology of n-ary multiplicative Hom-Nambu-Lie algebras. Journal of Geometry and Physics 2011;61 (10):1898-913.

[4] Bai C, Guo L, Ni X. Generalizations of the classical Yang-Baxter equation and O-operators. Journal of mathematical physics 2011; 52 (6): 063515.

[5] Bai C, Guo L, Ni X. Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and Post-Lie algebras. Communications in Mathematical Physics 2010; 297(2): 553-96.

[6] Benayadi S, Makhlouf A. Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. Journal of Geometry and Physics 2014; 76: 38-60.

[7] Baxter G. An analytic problem whose solution follows from a simple algebraic identity. Pacific Journal of Mathematics 1960; 10 (3): 731-742.

[8] Cai L, Sheng Y, Purely Hom-Lie bialgebras. Science China Mathematics 2018; 61 (9): 1553-1566.

[9] Connes A, Kreimer D, Hopf algebras. renormalisation and noncommutative geometry. Quantum field theory: perspective and prospective. Springer, Dordrecht 1999: 59-109.

[10] Cartier P. On the structure of free Baxter algebras. Advances in Mathematics 1972 ; 9 (2): 253-65.

[11] David L, Strachan IAB. Dubrovin's duality for F-manifolds with eventual identities. Advances in Mathematics 2011; 22 (5): 4031-4060.

[12] David L. Strachan IAB. Compatible metrics on a manifold and nonlocal bi-Hamiltonian structures. International Mathematics Research Notices 2004; 66 :3533-3557.

[13] Dotsenko V. Algebraic structures of F-manifolds via pre-Lie algebras. Annali di Matematica Pura ed Applicata 2019; 198 (2): 517-27.

[14] Ebrahimi-Fard K, Guo L, Kreimer D. Spitzer's identity and the algebraic Birkhoff decomposition in pQFT. Journal of Physics A: Mathematical and General 2004; 37 (45): 11037-11052.

[15] Ebrahimi-Fard K, Guo L, Manchon D. Birkhoff type decompositions and the Baker-Campbell-Hausdorff recursion. Communications in mathematical physics 2006; 267 (3): 821-45.

[16] Guo L. What is a Rota-Baxter algebra. Notices of the AMS. 2009; 56 (11): 1436-1437.

[17] Guo L. Introduction to Rota-Baxter Algebra. International Press and Higher Education Press 2012.

[18] Guo L, Keigher W. Baxter algebras and shuffle products. Advances in Mathematics 2000; 150: 117-149.

[19] Guo L, Zhang B. Renormalization of multiple zeta values. Journal of Algebra 2008; 319 (9): 3770-809.

[20] Guo S, Zhang X, Wang S. Manin triples and quasitriangular structures of Hom-Poisson bialgebras. arXiv preprint arXiv 2018:1807.06412.

[21] Hartwig JT, Larsson D, Silvestrov SD. Deformations of Lie algebras using σ- derivations. Journal of Algebra 2006; 295 (2): 314-361.

[22] Hertling C. Frobenius Manifolds and Moduli Spaces for Singularities. Cambridge University Press: 2002.

[23] Hertling C, Manin YI. Weak Frobenius manifolds. International Mathematics Research Notices 1999; 6 : 277-286.

[24] Hu N. q-Witt algebras, q-Lie algebras, q-holomorph structure and representations. Algebra Colloquim 1999; 6 (1): 51-70.

[25] Kupershmidt BA. What a classical r-matrix really is. Journal of Nonlinear Mathematical Physics 1999; 6 (4): 448-488.

[26] Larsson D, Silvestrov SD. Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities. Journal of Algebra 2005; 288 (2): 321-44.

[27] Lee YP. Quantum K-theory, I: Foundations. Duke Mathematical Journal 2004; 12 1 (3): 389-424.

[28] Lorenzoni P, Pedroni M, Raimondo A. F-manifolds and integrable systems of hydrodynamic type. Archiv der Mathematik 2011; 47: 163-180.

[29] Liu S, Song L, Tang R. Representations and cohomologies of regular Hom-pre-Lie algebras. Journal of Algebra and its Applications, 19 (08): 2050149 (2020).

[30] Liu J, Sheng Y, Bai C. F-manifold algebras and deformation quantization via pre-Lie algebras. Journal of Algebra 2020; 559: 467-495.

[31] Makhlouf A, Silvestrov SD. Hom-algebras structures. Journal of Generalized Lie Theory and Applications 2008; 2 (2): 51–64.

[32] Makhlouf A, Silvestrov SD. Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras. Forum Mathematicum 2010; 22: 715-739.

[33] Makhlouf A, Silvestrov SD. Hom-algebras and Hom-coalgebras. Journal of Algebra and its Applications 2010;9(04): 553-589.

[34] Manchon D, Paycha S. Nested sums of symbols and renormalised multiple zeta values. International Mathematics Research Notices 2010; 24 : 4628-4697.

[35] Merkulov SA. Operads, deformation theory and F-manifolds. Vieweg Teubner Verlag 2004; 36 : 213-251.

[36] Ming D, Chen Z, Li J. F-manifold color algebras. arXiv 2020: 2101.00959.

[37] Rota GC. Baxter algebras and combinatorial identities I, II. Bulletin of the American Mathematical Society 1969; 75 (2): 325-334.

[38] Rota GC. Baxter operators, an introduction Gian-Carlo Rota on Combinatorics. Contemporary Mathematicians Springer 1995: 504-512.

[39] Sheng Y. Representations of hom-Lie algebras. Algebras and Representation Theory 2012; 15 (6): 1081-1098.

[40] Sheng Y, Chen D. Hom-Lie 2-algebras. Journal of Algebra 2013; 376 : 174-195.

[41] Yau D. Hom-algebras and homology. Journal of Lie Theory 2009; 19: 409-421.

[42] Yau D. Non-commutative hom-Poisson algebras. arXiv 2010 :1010.3408.

[43] Yau D. Hom-Malcev, Hom-alternative and Hom-Jordan algebras. International Journal of Algebra 2012; 11: 177-217.