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On nonhomogeneous geometric quadratic stochastic operators

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Abstract: In this paper, we construct a nonhomogeneous geometric quadratic stochastic operator generated by 2-partition ξ on countable state space $X = \mathbb{Z}^*$. The limiting behavior of such operator is studied. We have proved that such operator possesses the regular property.

Key words: Quadratic stochastic operator, countable state space, partition, limiting behavior, regular transformation

1. Introduction

Let $(X, \mathcal{F})$ be a measurable space and $S(X, \mathcal{F})$ be a set of all probability measures on $(X, \mathcal{F})$. We consider a family of functions $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ on $X \times X \times x \times \mathcal{F}$ with the following properties:

(1) For every $x, y \in X$, $P(x, y, \cdot) \in S(X, \mathcal{F})$, that is $P(x, y, \cdot)$ is a probability measure on $(X, \mathcal{F})$;

(2) For fixed $A \in \mathcal{F}$, $P(\cdot, \cdot, A)$ is a jointly measurable function;

(3) $P(x, y, A) = P(y, x, A)$ for every $x, y \in X$, and $A \in \mathcal{F}$.

The family of functions above can be used to define a nonlinear operator $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$, such that

$$(V\mu)(A) = \int_X \int_X P(x, y, A) d\mu(x) d\mu(y) \tag{1.1}$$

for every $\mu \in S(X, \mathcal{F})$ and $A \in \mathcal{F}$. Note that, this operator is called a quadratic stochastic operator. A quadratic stochastic operator is also known as an evolutionary operator of free population which being studied in many publications, see ([1–18]).

Definition 1.1 *A quadratic stochastic operator V is called a regular if for any initial point $\mu \in S(X, \mathcal{F})$ the limit*

$$\lim_{n\to\infty} V^n(\mu)$$

exists.

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The case where X is a finite set has been studied in numerous publications, for example see ([2, 3, 9–11, 13–18]). In this paper, we consider the case where X is a countable set, where $X = \mathbb{Z}^*$ is a set of all nonnegative integers. On such a state space, one can consider a probability distribution. Below, we will consider a geometric distribution. Recall that the following distribution,

$$G_r(k) = (1 - r)r^k,$$

for any $k \in \mathbb{Z}^*$, where $0 < r < 1$ is called geometric distribution. In the case where $X = \mathbb{Z}^*$, one can define a measure $\{k\}$ as $P(i, j, \{k\})$, and by additivity of this measure we have

$$P(i, j, A) = \sum_{k \in A} P(i, j, \{k\}).$$

Below, we assume

$$P(i, j, \{k\}) = P_{ij,k}.$$

Then, a family of functions $\{P_{ij,k} : i, j, k \in \mathbb{Z}^*\}$ satisfies the following conditions,

(1) $P_{ij,\cdot} \in S(X, \mathcal{F})$ is the probability measure, and

(2) $P_{ij,k} = P_{ji,k}$ for any $i, j, k \in X$.

In this particular case, a qso (1.1) can be written as

$$V\mu(k) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(i, j, k)\mu(i)\mu(j), \tag{1.2}$$

where $k = \mathbb{Z}^*$ for measure $\mu \in S(X, \mathcal{F})$.

Definition 1.2 *A quadratic stochastic operator V in (1.2) is called a geometric quadratic stochastic operator if for any $i, j \in \mathbb{Z}^*$, the probability measure $P(i, j, \cdot)$ is the geometric distribution $G_{r(i,j)}$ with a real parameter $r(i, j) = r(j, i)$, $0 < r(i, j) < 1$.*

Note that if $r(i, j) = r$ for any $i, j \in \mathbb{Z}^*$, their corresponding family of distribution is called homogeneous family. It is evident that qso (1.2) generated by a homogeneous family of geometric distribution is identity transformation. Throughout this paper, we will consider a nonhomogeneous geometric qso. Assume $\xi = \{A_1, A_2\}$ is a 2-partition of state space $\mathbb{Z}^*$. Then, we define a corresponding partition $\zeta = \{B_1, B_2\}$ of space $\mathbb{Z}^* \times \mathbb{Z}^*$ where $B_1 = (A_1 \times A_1) \cup (A_2 \times A_2)$ and $B_2 = (A_1 \times A_2) \cup (A_2 \times A_1)$. Now, we define a family of functions $\{P_{ij,k} : i, j, k \in \mathbb{Z}^*\}$ as follows:

$$P_{ij,k} = \begin{cases} (1 - r_1)r_1^k & \text{if } (i, j) \in B_1 \\ (1 - r_2)r_2^k & \text{if } (i, j) \in B_2 \end{cases} \tag{1.3}$$

Then, a qso defined by this family (1.3) is called nonhomogeneous qso generated by 2-partition. The case $A_1 = 2m$ and $A_2 = 2m + 1$, where $m \in \mathbb{Z}^*$ was considered in [1]. The case where $A_1 = \{k\}$, k is a singleton that has been studied in [12]. In this paper, we consider the case $A_1 = \{k_1, k_2\}$ where $k_1, k_2 \in \mathbb{Z}^* \setminus A_1$, and $\zeta = \{B_1, B_2\}$ with $B_1 = (A_1 \times A_1) \cup (A_2 \times A_2)$ and $B_2 = (A_1 \times A_2) \cup (A_2 \times A_1)$.

2. Regularity of quadratic stochastic operator generated by 2-partition ξ of two points

Using the fact that arbitrary two points in the countable state space $X = \mathbb{Z}^*$ may be consecutive or nonconsecutive, thus we need to consider the following cases.

Case 1. Consecutive two points

Let $A_1 = \{x_1, x_1+1 : x_1 \in \mathbb{Z}^*\}$ where A_1 consists of consecutive two points and $A_2 = \mathbb{Z}^* \setminus A_1$. Then, we consider the following subcases:

(1) $x_1 = 0$, and

(2) $x_1 \neq 0$

where $x_1 \in A_1$. Note that both cases are necessary to be investigated as they yield different numbers of partitions in $\mathbb{Z}^* \times \mathbb{Z}^*$. For the subcase of $x_1 = 0$, we consider a geometric qso defined by a family of functions (1.3). For arbitrary initial measure $\mu \in S(X, \mathcal{F})$, we have

$$\begin{aligned} V\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) \\ &= \sum_{i,j=0}^{1} P_{ij,k}\mu(i)\mu(j) + \sum_{i,j=2}^{\infty} P_{ij,k}\mu(i)\mu(j) + \sum_{i=0}^{1}\sum_{j=2}^{\infty} P_{ij,k}\mu(i)\mu(j) + \sum_{i=2}^{\infty}\sum_{j=0}^{1} P_{ij,k}\mu(i)\mu(j) \\ &= (1-r_1)r_1^k\left\{[\mu(0)+\mu(1)]^2 + [1-(\mu(0)+\mu(1))]^2\right\} \\ &+ (1-r_2)r_2^k\left\{2[\mu(0)+\mu(1)][1-(\mu(0)+\mu(1))]\right\}, \text{ and} \end{aligned}$$

$$\begin{aligned} V^2\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}V\mu(i)V\mu(j) \\ &= \sum_{i,j=0}^{1} P_{ij,k}V\mu(i)V\mu(j) + \sum_{i,j=2}^{\infty} P_{ij,k}V\mu(i)V\mu(j) + \sum_{i=0}^{1}\sum_{j=2}^{\infty} P_{ij,k}V\mu(i)V\mu(j) \\ &+ \sum_{i=2}^{\infty}\sum_{j=0}^{1} P_{ij,k}V\mu(i)V\mu(j) \\ &= (1-r_1)r_1^k\left\{[V\mu(0)+V\mu(1)]^2 + [1-(V\mu(0)+V\mu(1))]^2\right\} \\ &+ (1-r_2)r_2^k\left\{2[V\mu(0)+V\mu(1)][1-(V\mu(0)+V\mu(1))]\right\}. \end{aligned}$$

Thus, by using induction on the sequence $V^n\mu(k)$, the following recurrent equation is produced

$$\begin{aligned} V^{n+1}\mu(k) = (1-r_1)r_1^k&\left\{(V^n\mu(0)+V^n\mu(1))^2 + [1-(V^n\mu(0)+V^n\mu(1))]^2\right\} \\ &+ (1-r_2)r_2^k\left\{2(V^n\mu(0)+V^n\mu(1))[1-(V^n\mu(0)+V^n\mu(1))]\right\} \end{aligned} \tag{2.1}$$

where $n = 0, 1, 2, \ldots$. One can show that the limit behavior of the recurrent equation (2.1) is fully determined by the limit behavior of recurrent equation $V^n\mu(0)$ and $V^n\mu(1)$ such that

$$\begin{aligned} V^{n+1}\mu(0) &= (1-r_1)\left\{(V^n\mu(0)+V^n\mu(1))^2+[1-(V^n\mu(0)+V^n\mu(1))]^2\right\} \\ &\quad+(1-r_2)\left\{2\left(V^n\mu(0)+V^n\mu(1)\right)[1-(V^n\mu(0)+V^n\mu(1))]\right\}, \text{ and} \\ V^{n+1}\mu(1) &= (1-r_1)r_1\left\{(V^n\mu(0)+V^n\mu(1))^2+[1-(V^n\mu(0)+V^n\mu(1))]^2\right\} \\ &\quad+(1-r_2)r_2\left\{2\left(V^n\mu(0)+V^n\mu(1)\right)[1-(V^n\mu(0)+V^n\mu(1))]\right\}. \end{aligned} \tag{2.2}$$

where $n = 0, 1, 2, \ldots$. Next, for the subcase where $x_1 \neq 0$, we have

$$\begin{aligned} V\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}\mu(i)\mu(j) \\ &= \sum_{i=0}^{x_1-1}\sum_{j=0}^{x_1-1}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1}^{x_1+1}\sum_{j=x_1}^{x_1+1}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1+2}^{\infty}\sum_{j=x_1+2}^{\infty}P_{ij,k}\mu(i)\mu(j) \\ &+\sum_{i=0}^{x_1-1}\sum_{j=x_1+2}^{\infty}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1+2}^{\infty}\sum_{j=0}^{x_1-1}P_{ij,k}\mu(i)\mu(j)+\sum_{i=0}^{x_1-1}\sum_{j=x_1}^{x_1+1}P_{ij,k}\mu(i)\mu(j) \\ &+\sum_{i=x_1}^{x_1+1}\sum_{j=0}^{x_1-1}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1}^{x_1+1}\sum_{j=x_1+2}^{\infty}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1+2}^{\infty}\sum_{j=x_1}^{x_1+1}P_{ij,k}\mu(i)\mu(j) \\ &= (1-r_1)r_1^k\left\{(\mu(x_1)+\mu(x_1+1))^2+[1-(\mu(x_1)+\mu(x_1+1))]^2\right\} \\ &+(1-r_2)r_2^k\left\{2\left(\mu(x_1)+\mu(x_1+1)\right)[1-(\mu(x_1)+\mu(x_1+1))]\right\}, \text{ and} \end{aligned}$$

$$\begin{aligned} V^2\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}V\mu(i)V\mu(j) \\ &= \sum_{i=0}^{x_1-1}\sum_{j=0}^{x_1-1}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=x_1}^{x_1+1}\sum_{j=x_1}^{x_1+1}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=x_1+2}^{\infty}\sum_{j=x_1+2}^{\infty}P_{ij,k}V\mu(i)V\mu(j) \\ &+\sum_{i=0}^{x_1-1}\sum_{j=x_1+2}^{\infty}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=x_1+2}^{\infty}\sum_{j=0}^{x_1-1}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=0}^{x_1-1}\sum_{j=x_1}^{x_1+1}P_{ij,k}V\mu(i)V\mu(j) \\ &+\sum_{i=x_1}^{x_1+1}\sum_{j=0}^{x_1-1}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=x_1}^{x_1+1}\sum_{j=x_1+2}^{\infty}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=x_1+2}^{\infty}\sum_{j=x_1}^{x_1+1}P_{ij,k}V\mu(i)V\mu(j) \\ &= (1-r_1)r_1^k\left\{(V\mu(x_1)+V\mu(x_1+1))^2+[1-(V\mu(x_1)+V\mu(x_1+1))]^2\right\} \\ &+(1-r_2)r_2^k\left\{2\left(\mu(x_1)+\mu(x_1+1)\right)[1-(\mu(x_1)+\mu(x_1+1))]\right\}. \end{aligned}$$

for any initial measure $\mu \in S(X, \mathcal{F})$. By using induction on the sequence $V^n\mu(k)$, we obtain the following recurrent equation:

$$\begin{aligned}
&V^{n+1}\mu(k) \\
&= (1-r_1)r_1^k \left\{ (V^n\mu(x_1) + V^n\mu(x_1+1))^2 + [1-(V^n\mu(x_1)+V^n\mu(x_1+1))]^2 \right\} \\
&+ (1-r_2)r_2^k \left\{ 2\left(V^n\mu(x_1)+V^n\mu(x_1+1)\right)\left[1-(V^n\mu(x_1)+V^n\mu(x_1+1))\right] \right\}
\end{aligned} \tag{2.3}$$

where $n = 0, 1, 2, \ldots$. From this, it is easy to identify that the limit behavior of the recurrent equation (2.3) is fully determined by the limit behavior of recurrent equation $V^n\mu(x_1)$ and $V^n\mu(x_1+1)$ such that

$$\begin{aligned}
&V^{n+1}\mu(x_1) \\
&= (1-r_1)r_1^{x_1} \left\{ (V^n\mu(x_1) + V^n\mu(x_1+1))^2 + [1-(V^n\mu(x_1)+V^n\mu(x_1+1))]^2 \right\} \\
&+ (1-r_2)r_2^{x_1} \left\{ 2\left(V^n\mu(x_1)+V^n\mu(x_1+1)\right)\left[1-(V^n\mu(x_1)+V^n\mu(x_1+1))\right] \right\}, \text{ and} \\
&V^{n+1}\mu(x_1+1) \\
&= (1-r_1)r_1^{x_1+1} \left\{ (V^n\mu(x_1) + V^n\mu(x_1+1))^2 + [1-(V^n\mu(x_1)+V^n\mu(x_1+1))]^2 \right\} \\
&+ (1-r_2)r_2^{x_1+1} \left\{ 2\left(V^n\mu(x_1)+V^n\mu(x_1+1)\right)\left[1-(V^n\mu(x_1)+V^n\mu(x_1+1))\right] \right\}.
\end{aligned} \tag{2.4}$$

where $n = 0, 1, 2, \ldots$. Generally, for both considered subcases, $x_1 = 0$ and $x_1 \neq 0$ of the consecutive two points case, we can say that for any $x_1 \in A_1$ where $A_1 \subset \mathbb{Z}^*$, the recurrent equation (2.3) is fully determined by the recurrent equation (2.4). Now, we shall consider A_1 that consists of nonconsecutive two points.

Case 2. Nonconsecutive two points

Let $A_1 = \{x_1, x_1+2 : x_1 \in \mathbb{Z}^*\}$ and $A_2 = \mathbb{Z}^* \setminus A_1$. It is always necessary to present the two subcases like in the case of consecutive two points, i.e. $x_1 = 0$ and $x_1 \neq 0$, as they vary in the number of partitions in $\mathbb{Z}^* \times \mathbb{Z}^*$, where $x_1 \in A_1$. Similar to the *Case 1*, for $x_1 = 0$, we consider a nonhomogeneous geometric quadratic stochastic operator defined by a family of functions (1.3). Accordingly, we have the following for any

$\mu \in S\left(X, \mathcal{F}\right),$

$$
\begin{aligned}
V\mu\left(k\right) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) \\
&= P_{00,k}\mu(0)\mu(0) + P_{11,k}\mu(1)\mu(1) + P_{22,k}\mu(2)\mu(2) + P_{02,k}\mu(0)\mu(2) + P_{20,k}\mu(2)\mu(0) \\
&+ \sum_{j=3}^{\infty} P_{1j,k}\mu(1)\mu(j) + \sum_{i=3}^{\infty} P_{i1,k}\mu(i)\mu(1) + \sum_{i=3}^{\infty}\sum_{j=3}^{\infty} P_{ij,k}\mu(i)\mu(j) \\
&+ P_{01,k}\mu(0)\mu(1) + P_{10,k}\mu(1)\mu(0) + P_{21,k}\mu(2)\mu(1) + P_{12,k}\mu(1)\mu(2) \\
&+ \sum_{j=3}^{\infty} P_{0j,k}\mu(0)\mu(j) + \sum_{i=3}^{\infty} P_{i0,k}\mu(i)\mu(0) + \sum_{j=3}^{\infty} P_{2j,k}\mu(2)\mu(j) + \sum_{i=3}^{\infty} P_{i2,k}\mu(i)\mu(2) \\
&= (1-r_1)r_1^k\left\{\left(\mu(0)+\mu(2)\right)^2 + \left[1-\left(\mu(0)+\mu(2)\right)\right]^2\right\} \\
&+ (1-r_2)r_2^k\left\{2\left(\mu(0)+\mu(2)\right)\left[1-\left(\mu(0)+\mu(2)\right)\right]\right\}, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
V^2\mu\left(k\right) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}V\mu(i)V\mu(j) \\
&= P_{00,k}V\mu(0)V\mu(0) + P_{11,k}V\mu(1)V\mu(1) + P_{22,k}V\mu(2)V\mu(2) + P_{02,k}V\mu(0)V\mu(2) \\
&+ P_{20,k}V\mu(2)V\mu(0) + \sum_{j=3}^{\infty} P_{1j,k}V\mu(1)V\mu(j) + \sum_{i=3}^{\infty} P_{i1,k}V\mu(i)V\mu(1) \\
&+ \sum_{i=3}^{\infty}\sum_{j=3}^{\infty} P_{ij,k}V\mu(i)V\mu(j) + P_{01,k}V\mu(0)V\mu(1) + P_{10,k}V\mu(1)V\mu(0) \\
&+ P_{21,k}V\mu(2)V\mu(1) + P_{12,k}V\mu(1)V\mu(2) + \sum_{j=3}^{\infty} P_{0j,k}V\mu(0)V\mu(j) \\
&+ \sum_{i=3}^{\infty} P_{i0,k}V\mu(i)V\mu(0) + \sum_{j=3}^{\infty} P_{2j,k}V\mu(2)V\mu(j) + \sum_{i=3}^{\infty} P_{i2,k}V\mu(i)V\mu(2) \\
&= (1-r_1)r_1^k\left\{\left(V\mu(0)+V\mu(2)\right)^2 + \left[1-\left(V\mu(0)+V\mu(2)\right)\right]^2\right\} \\
&+ (1-r_2)r_2^k\left\{2\left(V\mu(0)+V\mu(2)\right)\left[1-\left(V\mu(0)+V\mu(2)\right)\right]\right\}.
\end{aligned}
$$

Hence, by mathematical induction, we attain the following recurrent equation

$$
\begin{aligned}
V^{n+1}\mu(k) &= (1-r_1)r_1^k\left\{\left(V^n\mu(0)+V^n\mu(2)\right)^2 + \left[1-\left(V^n\mu(0)+V^n\mu(2)\right)\right]^2\right\} \\
&\quad +(1-r_2)r_2^k\left\{2\left(V^n\mu(0)+V^n\mu(2)\right)\left[1-\left(V^n\mu(0)+V^n\mu(2)\right)\right]\right\} \qquad (2.5)
\end{aligned}
$$

where $n = 0, 1, 2, \ldots$. It is not hard to unveil that the limit behavior of the recurrent equation (2.5) is fully determined by the limit behavior of recurrent equation $V^n\mu(0)$ and $V^n\mu(2)$ such that

$$\begin{aligned}
V^{n+1}\mu(0) &= (1-r_1)\left\{(V^n\mu(0)+V^n\mu(2))^2+[1-(V^n\mu(0)+V^n\mu(2))]^2\right\} \\
&+ (1-r_2)\left\{2\left(V^n\mu(0)+V^n\mu(2)\right)[1-(V^n\mu(0)+V^n\mu(2))]\right\}, \text{ and} \\
V^{n+1}\mu(2) &= (1-r_1)r_1^2\left\{(V^n\mu(0)+V^n\mu(2))^2+[1-(V^n\mu(0)+V^n\mu(2))]^2\right\} \\
&+ (1-r_2)r_2^2\left\{2\left(V^n\mu(0)+V^n\mu(2)\right)[1-(V^n\mu(0)+V^n\mu(2))]\right\}
\end{aligned} \tag{2.6}$$

where $n = 0, 1, 2, \ldots$ Next, for the subcase of $x_1 \neq 0$, we have such that for any initial measure $\mu \in S(X, \mathcal{F})$,

$$\begin{aligned}
V\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) \\
&= \sum_{i\in A_1}\sum_{j\in A_1} P_{ij,k}\mu(i)\mu(j) + \sum_{i\in A_2}\sum_{j\in A_2} P_{ij,k}\mu(i)\mu(j) \\
&+ \sum_{i\in A_1}\sum_{j\in A_2} P_{ij,k}\mu(i)\mu(j) + \sum_{i\in A_2}\sum_{j\in A_1} P_{ij,k}\mu(i)\mu(j) \\
&= (1-r_1)r_1^k\left\{(\mu(x_1)+\mu(x_1+2))^2+[1-(\mu(x_1)+\mu(x_1+2))]^2\right\} \\
&+ (1-r_2)r_2^k\left\{2\left(\mu(x_1)+\mu(x_1+2)\right)[1-(\mu(x_1)+\mu(x_1+2))]\right\}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
V^2\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}V\mu(i)V\mu(j) \\
&= \sum_{i\in A_1}\sum_{j\in A_1} P_{ij,k}V\mu(i)V\mu(j) + \sum_{i\in A_2}\sum_{j\in A_2} P_{ij,k}V\mu(i)V\mu(j) \\
&+ \sum_{i\in A_1}\sum_{j\in A_2} P_{ij,k}V\mu(i)V\mu(j) + \sum_{i\in A_2}\sum_{j\in A_1} P_{ij,k}V\mu(i)V\mu(j) \\
&= (1-r_1)r_1^k\left\{(V\mu(x_1)+V\mu(x_1+2))^2+[1-(V\mu(x_1)+V\mu(x_1+2))]^2\right\} \\
&+ (1-r_2)r_2^k\left\{(V\mu(x_1)+V\mu(x_1+2))[1-(V\mu(x_1)+V\mu(x_1+2))]\right\}.
\end{aligned}$$

By using induction on the sequence $V^n\mu(k)$, the following recurrent equation is obtained

$$\begin{aligned}
V^{n+1}\mu(k) &= (1-r_1)r_1^k\left\{(V^n\mu(x_1)+V^n\mu(x_1+2))^2+[1-(V^n\mu(x_1)+V^n\mu(x_1+2))]^2\right\} \\
&+ (1-r_2)r_2^k\left\{2\left(V^n\mu(x_1)+V^n\mu(x_1+2)\right)[1-(V^n\mu(x_1)+V^n\mu(x_1+2))]\right\}
\end{aligned} \tag{2.7}$$

where $n = 0, 1, 2, \ldots$. One can see that the limit behaviour of the recurrent equation (2.7) is fully determined by the limit behaviour of recurrent equation $V^n\mu(x_1)$ and $V^n\mu(x_1+2)$ such that

$$\begin{aligned} V^{n+1}\mu(x_1) &= (1-r_1)r_1^{x_1}\left\{(V^n\mu(x_1)+V^n\mu(x_1+2))^2+[1-(V^n\mu(x_1)+V^n\mu(x_1+2))]^2\right\} \\ &+ (1-r_2)r_2^{x_1}\left\{2\left(V^n\mu(x_1)+V^n\mu(x_1+2)\right)[1-(V^n\mu(x_1)+V^n\mu(x_1+2))]\right\}, \text{ and} \\ V^{n+1}\mu(x_1+2) &= (1-r_1)r_1^{x_1+2}\left\{(V^n\mu(x_1)+V^n\mu(x_1+2))^2+[1-(V^n\mu(x_1)+V^n\mu(x_1+2))]^2\right\} \\ &+ (1-r_2)r_2^{x_1+2}\left\{2\left(V^n\mu(x_1)+V^n\mu(x_1+2)\right)[1-(V^n\mu(x_1)+V^n\mu(x_1+2))]\right\} \end{aligned} \tag{2.8}$$

where $n = 0, 1, 2, \ldots$. Therefore, for both subcases, $x_1 = 0$ and $x_1 \neq 0$ of the nonconsecutive two points case, we can see that for any $x_1 \in A_1$ where $A_1 \subset \mathbb{Z}^*$, the recurrent equation in (2.7) is fully determined by the recurrent equation (2.8). Now, as we have considered and investigated both cases, *Case 1* and *Case 2*, we can generalize the case of two points as follows. Let $\xi = \{A_1, A_2\}$ be a measurable 2-partition of the state space $\mathbb{Z}^*$ where $A_1 = \{x_1, x_2 : x_1, x_2 \in \mathbb{Z}^*\}$, and $A_2 = \mathbb{Z}^* \setminus A_1$. Next, we consider a nonhomogeneous geometric quadratic stochastic operator with a family of functions (1.3). Hence, it is given that

$$\begin{aligned} V\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}\mu(i)\mu(j) \\ &= \sum_{i\in A_1}\sum_{j\in A_1} P_{ij,k}\mu(i)\mu(j) + \sum_{i\in A_2}\sum_{j\in A_2} P_{ij,k}\mu(i)\mu(j) \\ &+ \sum_{i\in A_1}\sum_{j\in A_2} P_{ij,k}\mu(i)\mu(j) + \sum_{i\in A_2}\sum_{j\in A_1} P_{ij,k}\mu(i)\mu(j) \\ &= (1-r_1)r_1^k\left\{(\mu(x_1)+\mu(x_2))^2+[1-(\mu(x_1)+\mu(x_2))]^2\right\} \\ &+ (1-r_2)r_2^k\left\{2\left(\mu(x_1)+\mu(x_2)\right)[1-(\mu(x_1)+\mu(x_2))]\right\}, \text{ and} \end{aligned}$$

$$\begin{aligned} V^2\mu(k) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P_{ij,k}V\mu(i)V\mu(j) \\ &= \sum_{i\in A_1}\sum_{j\in A_1} P_{ij,k}V\mu(i)V\mu(j) + \sum_{i\in A_2}\sum_{j\in A_2} P_{ij,k}V\mu(i)V\mu(j) \\ &+ \sum_{i\in A_1}\sum_{j\in A_2} P_{ij,k}V\mu(i)V\mu(j) + \sum_{i\in A_2}\sum_{j\in A_1} P_{ij,k}V\mu(i)V\mu(j) \\ &= (1-r_1)r_1^k\left\{(V\mu(x_1)+V\mu(x_2))^2+[1-(V\mu(x_1)+V\mu(x_2))]^2\right\} \\ &+ (1-r_2)r_2^k\left\{2\left(V\mu(x_1)+V\mu(x_2)\right)[1-(V\mu(x_1)+V\mu(x_2))]\right\} \end{aligned}$$

We are bound to use mathematical induction on the sequence $V^n\mu(k)$. Therefore, we have the following recurrent equation

$$\begin{aligned} V^{n+1}\mu(k) &= (1-r_1)r_1^k\left\{(V^n\mu(x_1)+V^n\mu(x_2))^2+[1-(V^n\mu(x_1)+V^n\mu(x_2))]^2\right\} \\ &+ (1-r_2)r_2^k\left\{2\left(V^n\mu(x_1)+V^n\mu(x_2)\right)[1-(V^n\mu(x_1)+V^n\mu(x_2))]\right\} \end{aligned} \tag{2.9}$$

where $n=0,1,2,\ldots$. Based on the recurrent equation 2.9, it follows that such recurrent equation is determined by the limit behavior of recurrent equation $V^n\mu(x_1)$ and $V^n\mu(x_1+2)$ such that

$$\begin{aligned} V^{n+1}\mu(x_1) &= (1-r_1)r_1^{x_1}\left\{(V^n\mu(x_1)+V^n\mu(x_2))^2+[1-(V^n\mu(x_1)+V^n\mu(x_2))]^2\right\} \\ &+ (1-r_2)r_2^{x_1}\left\{2\left(V^n\mu(x_1)+V^n\mu(x_2)\right)[1-(V^n\mu(x_1)+V^n\mu(x_2))]\right\}, \text{ and} \\ V^{n+1}\mu(x_2) &= (1-r_1)r_1^{x_2}\left\{(V^n\mu(x_1)+V^n\mu(x_2))^2+[1-(V^n\mu(x_1)+V^n\mu(x_2))]^2\right\} \\ &+ (1-r_2)r_2^{x_2}\left\{2\left(V^n\mu(x_1)+V^n\mu(x_2)\right)[1-(V^n\mu(x_1)+V^n\mu(x_2))]\right\} \end{aligned} \tag{2.10}$$

where $n=0,1,2,\ldots$. As $n\to\infty$, then, the recurrent equation (2.10) can be written as follows:

$$\begin{aligned} \bar{x} &= (1-r_1)r_1^{x_1}\left\{(\bar{x}+\bar{y})^2+[1-(\bar{x}+\bar{y})]^2\right\} \\ &+ (1-r_2)r_2^{x_1}\left\{2(\bar{x}+\bar{y})[1-(\bar{x}+\bar{y})]\right\}, \text{ and} \\ \bar{y} &= (1-r_1)r_1^{x_2}\left\{(\bar{x}+\bar{y})^2+[1-(\bar{x}+\bar{y})]^2\right\} \\ &+ (1-r_2)r_2^{x_2}\left\{2(\bar{x}+\bar{y})[1-(\bar{x}+\bar{y})]\right\}. \end{aligned} \tag{2.11}$$

Based on the equation (2.11), one can clearly see that both equations are fully determined by $\bar{x}+\bar{y}$. Here, by using simple calculus, let $x=\bar{x}+\bar{y}$, then we obtain

$$\begin{aligned} x &= [(1-r_1)r_1^{x_1}+(1-r_1)r_1^{x_2}]\left[x^2+(1-x)^2\right] \\ &+[(1-r_2)r_2^{x_1}+(1-r_2)r_2^{x_2}][2x(1-x)]. \end{aligned} \tag{2.12}$$

where $x_1,x_2\in A_1$. Notice that the above equation is a quadratic equation. Hence, we can rewrite the right-hand side equation as follows:

$$\begin{aligned} y &= 2\left\{[(1-r_1)r_1^{x_1}+(1-r_1)r_1^{x_2}]-[(1-r_2)r_2^{x_1}+(1-r_2)r_2^{x_2}]\right\}x^2 \\ &-2\left\{[(1-r_1)r_1^{x_1}+(1-r_1)r_1^{x_2}]-[(1-r_2)r_2^{x_1}+(1-r_2)r_2^{x_2}]\right\}x \\ &+[(1-r_1)r_1^{x_1}+(1-r_1)r_1^{x_2}], \end{aligned} \tag{2.13}$$

where $x_1,x_2\in A_1$, and $0<r_1,r_2<1$. One can see that a function (2.13) maps the segment $[0,1]$ into itself with

$$y|_{x=0}=y|_{x=1}=(1-r_2)r_2^{x_1}+(1-r_2)r_2^{x_2}.$$

Without loss of generality, we assume that $0<r_1<1$. Therefore, the validity of following statements is established.

Theorem 2.1 *A fixed point of the transformation (2.13) is unique and belongs to open interval* $(0,1)$.

Proof Here, we have the equation

$$x = 2\left[(a_1+a_2)-(b_1+b_2)\right]x^2 - 2\left[(a_1+a_2)-(b_1+b_2)\right]x + (a_1+a_2), \tag{2.14}$$

where we denote $a_1 = (1-r_1)r_1^{x_1}, a_2 = (1-r_1)r_1^{x_2}, b_1 = (1-r_2)r_2^{x_1}$, and $b_2 = (1-r_2)r_2^{x_2}$. We may consider three cases, where the equation has a root in the interval $(1,\infty)$ when $2\left[(a_1+a_2)-(b_1+b_2)\right] > 0$, has a root in the interval $(-\infty,0)$ when $2\left[(a_1+a_2)-(b_1+b_2)\right] < 0$, and it became a linear when $2\left[(a_1+a_2)-(b_1+b_2)\right] = 0$ with $a_1+a_2 = (1-r_1)r_1^{x_1} > 0$. Thus, for all cases, a root in $[0,1]$ is unique. It is evident that this root differs from 0 to 1. This completes the proof. □

Now, we are going to consider the discriminant of the quadratic equation (2.14) to investigate the local character of the fixed point, where

$$\Delta = 4\left[1-(a_1+a_2)\right](a_1+a_2) + \left[1-2(b_1+b_2)\right]^2. \tag{2.15}$$

Simply, we have $0 < \Delta < 2$, and Δ takes all value in this interval. Given the discriminant of the quadratic equation (2.14), then we have the following theorem.

Theorem 2.2 *The fixed point of the transformation (2.13) is attractive.*

Proof Let ζ be a fixed point in the open interval $(0,1)$, where

$$\zeta = \frac{2\left[(a_1+a_2)-(b_1+b_2)\right]+1-\sqrt{\Delta}}{4\left[(a_1+a_2)-(b_1+b_2)\right]}. \tag{2.16}$$

Recall that the character of such fixed point is defined by $f'(\zeta)$, where $f(x)$ is the right hand side of the equation (2.14), and $f'(x)$ is its derivative. Let $\lambda = f'(\zeta)$, where

$$\lambda = 4\left[(a_1+a_2)-(b_1+b_2)\right]\zeta - 2\left[(a_1+a_2)-(b_1+b_2)\right]. \tag{2.17}$$

By substituting 2.16 into 2.17, we have

$$\lambda = 1-\sqrt{\Delta} \tag{2.18}$$

for a fixed point in the open interval $(0,1)$. Since $0 < \Delta < 2$, then we obtain $1-\sqrt{2} < \lambda < 1$. Note that, if $|\lambda| < 1$, then ξ is an attractive point, and if $|\lambda| > 1$, then ζ is a repelling point. Therefore, it implies that any unique fixed point in the open interval $(0,1)$ is attractive and the statement of the Theorem 2.2 follows from the equality in (2.18). Thus, the proof is completed. □

It is shown that the trajectory behavior of the quadratic stochastic operator in (2.10) converges to a fixed point in the open interval $(0,1)$. By Definition 1.2, the convergence of the trajectory indicates that the limit exists. Hence, it is regular.

3. Conclusion

A nonhomogeneous geometric quadratic stochastic operator generated by 2-partition $\xi = \{A_1, A_2\}$ with $A_1 = |2|$ is a regular transformation.

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References

[1] Ganikhodjaev NN, Hamzah NZA. Geometric quadratic stochastic operator on countable infinite set. AIP Conference Proceedings 2015; 1643: 706-712.

[2] Ganikhodjaev NN, Ganikhodjaev RN, Jamilov U. Quadratic stochastic operators and zero-sum game dynamics. Ergodic Theory and Dynamical Systems 2015; 35 (5): 1443-1473.

[3] Ganikhodjaev RN, Mukhamedov F, Rozikov U. Quadratic stochastic operators and processes: results and open problems. Infinite Dimensional Analysis, Quantum Probability and Related Topics 2011; 14 (2): 279-335.

[4] Ganikhodjaev N, Hamzah NZA. On Gaussian nonlinear transformations. AIP Conference Proceedings 2014; 1682: 040009.

[5] Ganikhodjaev N, Hamzah NZA. On Poisson nonlinear transformations. The Scientific World Journal 2014; 2014.

[6] Ganikhodjaev N, Hamzah NZA. Quadratic stochastic operators on segment [0,1] and their limit behavior. Indian Journal of Science and Technology 2015; 8 (30).

[7] Ganikhodjaev N, Hamzah NZA. Nonhomogeneous Poisson nonlinear transformations on countable infinite set. Malaysian Journal of Mathematical Sciences 2016; 10: 143-155.

[8] Ganikhodjaev N, Hamzah NZA. On (3,3)-Gaussian quadratic stochastic operators. Journal of Physics: Conference Series 2017; 819.

[9] Ganikhodjaev N, Muhitdinov R, Saburov M. On Lebesgue nonlinear transformations. Bulletin of Korean Mathematical Society 2017; 54 (2): 607-618.

[10] Ganikhodjaev NN, Jamilov UU, Ramazon TM. On non-ergodic transformations on S^3. Journal of Physics: Conference Series 2013; 435: 012005.

[11] Ganikhodjaev N, Saburov M, Jamilov U. Mendelian and non-Mendelian quadratic operators. Applied Mathematics and Information Sciences 2013; 7 (5): 1721-1729.

[12] Karim SN, Hamzah NZA, Ganikhodjaev N. A class of Geometric quadratic stochastic operator on countable state space and its regularity. Malaysian Journal of Fundamental and Applied Sciences 2019; 15 (6): 872-877.

[13] Mukhamedov F, Embong AF. On b-bistochastic quadratic stochastic operators. Journal of Inequalities and Applications 2015; 2015 (1).

[14] Mukhamedov F, Saburov M, Mohd Jamal AH. On dynamics of ξ^{as}-quadratic stochastic operators. International Journal of Modern Physics: Conference Series 2012; 9: 299–307.

[15] Mukhitdinov R. Constrained quadratic stochastic operators. Journal of Pure and Applied Mathematics 2015; 6 (2): 233-237.

[16] Rozikov UA, Zhamilov UU. F-quadratic stochastic operators. Mathematical Notes 2008; 83 (3-4): 554-559.

[17] Saburov M, Yusof NA. P-majorizing quadratic stochastic operators. Theory of Stochastic Processes 2017; 22 (1): 80-87.

[18] Saburov M, Yusof NA. On uniqueness of fixed points of quadratic stochastic operators on a 2D simplex. Methods of Functional Analysis and Topology 2018; 24 (3): 255-264.