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# Representation variety of free or surface groups and Reidemeister torsion

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**Abstract:** For $G \in \{\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})\}$, we consider $G-$valued representations of free or surface group with genus $> 1$. We establish a formula for computing Reidemeister torsion of such representations in terms of Atiyah-Bott-Goldman symplectic form for $G$. Furthermore, we apply the obtained results to hyperbolic 3-manifolds.

**Key words:** Reidemeister torsion, representation varieties, Atiyah-Bott-Goldman symplectic form, symplectic chain complex, hyperbolic 3-manifolds.

## 1. Introduction

The importance of character varieties in many branches of mathematics and physics is well known. For example, for a compact Riemann surface $\Sigma$ of genus at least 2, Teichmüller space $\mathrm{Teich}(\Sigma)$ of $\Sigma$ is the space of deformation classes of complex structures on it. With the help of the Uniformization Theorem, one can also interpret it as the space of isotopy classes of Riemanian metrics on $\Sigma$ of constant curvature $(-1)$, namely, hyperbolic metrics. Moreover, $\mathrm{Teich}(\Sigma)$ can be identified as the space of conjugacy classes of discrete faithful representations from the fundamental group $\pi_1(\Sigma)$ of the surface to $\mathrm{PSL}(2,\mathbb{R})$. More generally, other geometric structures on $\Sigma$ can also be interpreted as certain surface group variety, see e.g. [5, 7, 14, 16, 21, 24] and the references therein.

Character varieties have several applications in many branches of mathematics and physics such as in 3-manifold topology (in Bass-Culler-Shalen theory [8, 31, 32], in A-polynamial [12, 25], in hyperbolic geometry [17, 34], in Casson invariant theory [1–3]), in Yang-Mills and Chern-Simons quantum field theories [15, 46, 47], in skein theory of quantum invariants of 3-manifolds [4, 10, 38], in the moduli spaces of flat connections, holomorphic bundles, and Higgs bundles [11, 20, 26, 40].

The topological invariant Reidemeister torsion was first introduced by K. Reidemeister in [36]. With the help of this invariant, he classified 3-dimensional lens spaces. By extending Reidemeister torsion, W. Franz classified the higher dimensional lens spaces [9]. Reidemeister torsion has several applications in many branches of mathematics and theoretical phsyics such as topology [9, 28, 29, 36], differential geometry [6, 33, 35], representation spaces [48] dynamical systems [23], 3-dimensional Seiberg-Witten theory [27], algebraic K-theory [30], Chern-Simon theory [47], knot theory [30], theoretical physics and quantum field theory [37, 47]. See Refs. [34] and [45] and the references therein for more information.

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In [48], E. Witten introduced the real symplectic chain complex. Using Reidemeister torsion and the real symplectic chain complex, he computed the volume of several moduli space of $\mathrm{Rep}(\Sigma, G)$, which is the set of all conjugacy classes of homomorphisms from the fundamental group $\pi_1(\Sigma)$ of a Riemann surface $\Sigma$ to the compact gauge group $G \in \{\mathrm{SU}(2), \mathrm{SO}(3)\}$.

In the present article, we consider the smooth part of the representation variety $\mathrm{Rep}(\Gamma, G)$ consisting of conjugacy classes of homomorphisms from $\Gamma$ to $G$, where $G \in \{\mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C})\}$ and $\Gamma$ is a free group or fundamental group of closed orientable surface of genus at least $g \geq 2$.

Combining the symplectic chain complex and the topological invariant Reidemeister torsion, we investigate $G-$valued representation spaces $\mathrm{Rep}(\Gamma, G)$. We prove the well definiteness (Theorem 3.1) of the Reidemeister torsion for such representations. Moreover, we establish a formula for computing the Reidemeister torsion of such representations (Theorem 3.2) expressed in terms of the well-known Atiyah-Bott-Goldman symplectic form for $G$. In addition, we apply the obtained results to good surface group representations (Theorem 4.1), to free group representations (Theorem 4.2), and to complete orientable hyperbolic 3-manifolds with boundary consisting orientable surfaces with genus at least 2 (Theorem 4.3).

## 2. Reidemeister torsion and symplectic chain complex

This section provides the necessary definition and basic facts about the topological invariant Reidemeister torsion and the symplectic chain complex. For more information the reader is referred to [34, 41, 45, 48] and the references therein.

Suppose $C_* = \left(0 \to C_n \stackrel{\partial_n}{\to} C_{n-1} \to \cdots \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0\right)$ is a chain complex of finite dimensional vector spaces over the field $\mathbb{C}$ of complex numbers. Let $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$ denote the kernel of $\partial_p$, the image of $\partial_{p+1}$, and the $p$th homology group of the chain complex $C_*$, respectively, $p = 0, \dots, n$. Let us note that the definition of $Z_p(C_*)$, $B_p(C_*)$, and $H_p(C_*)$ yields the short-exact sequences:

$$0 \longrightarrow Z_p(C_*) \hookrightarrow C_p \twoheadrightarrow B_{p-1}(C_*) \longrightarrow 0$$

and

$$0 \longrightarrow B_p(C_*) \hookrightarrow Z_p(C_*) \twoheadrightarrow H_p(C_*) \longrightarrow 0.$$

For $p = 0, \dots, n$, let $\mathbf{c}_p$, $\mathbf{b}_p$, and $\mathbf{h}_p$ be bases of $C_p$, $B_p(C_*)$, and $H_p(C_*)$, respectively. If $\ell_p : H_p(C_*) \to Z_p(C_*)$, $s_p : B_{p-1}(C_*) \to C_p$ are sections of $Z_p(C_*) \to H_p(C_*)$, $C_p \to B_{p-1}(C_*)$, respectively, then using the above short-exact sequences the basis $\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1})$ of $C_p$ is obtained, where $\sqcup$ is the disjoint union.

If $\mathbf{c}_p$, $\mathbf{b}_p$, $\mathbf{h}_p$, $\ell_p$, and $s_p$ are as above, then Reidemeister torsion of the chain complex $C_*$ with respect to bases $\{\mathbf{c}_p\}_{p=0}^n$, $\{\mathbf{h}_p\}_{p=0}^n$ is the alternating product

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right) = \prod_{p=0}^{n} \left[\mathbf{b}_p \sqcup \ell_p(\mathbf{h}_p) \sqcup s_p(\mathbf{b}_{p-1}), \mathbf{c}_p\right]^{(-1)^{(p+1)}}.$$

Here, $[\mathbf{e}_p, \mathbf{f}_p]$ is determinant of the change-base-matrix from basis $\mathbf{f}_p$ to $\mathbf{e}_p$ of $C_p$.

Reidemeister torsion is independent of the bases $\mathbf{b}_p$ and sections $s_p, \ell_p$ [30].

Assume $\mathbf{c}'_p$, $\mathbf{h}'_p$ are also bases of $C_p$, $H_p(C_*)$, respectively. Then, the following change-base-formula holds [30]:

$$\mathbb{T}\left(C_*, \left\{\mathbf{c}'_p\right\}_0^n, \left\{\mathbf{h}'_p\right\}_0^n\right) = \prod_{p=0}^{n}\left(\frac{\left[\mathbf{c}'_p, \mathbf{c}_p\right]}{\left[\mathbf{h}'_p, \mathbf{h}_p\right]}\right)^{(-1)^p} \mathbb{T}\left(C_*, \left\{\mathbf{c}_p\right\}_0^n, \left\{\mathbf{h}_p\right\}_0^n\right). \tag{2.1}$$

Suppose

$$0 \longrightarrow A_* \xrightarrow{\imath} B_* \xrightarrow{j} D_* \longrightarrow 0 \tag{2.2}$$

is a short-exact sequence of chain complexes. Let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ be bases of $A_p$, $B_p$, $D_p$, $H_p(A_*)$, $H_p(B_*)$, and $H_p(D_*)$, respectively. Consider the Mayer-Vietoris long-exact sequence of vector spaces

$$C_*: \cdots \longrightarrow H_p(A_*) \xrightarrow{\imath_p} H_p(B_*) \xrightarrow{j_p} H_p(D_*) \xrightarrow{\delta_p} H_{p-1}(A_*) \longrightarrow \cdots$$

associated to short-exact sequence (2.2). Since $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$, and $C_{3p+2} = H_p(B_*)$, the bases $\mathbf{h}_p^D$, $\mathbf{h}_p^A$, and $\mathbf{h}_p^B$ can be considered bases of $C_{3p}$, $C_{3p+1}$, and $C_{3p+2}$, respectively.

**Theorem 2.1** *([30]) Let $\mathbf{c}_p^A$, $\mathbf{c}_p^B$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, $\mathbf{h}_p^B$, and $\mathbf{h}_p^D$ be as above. If, moreover, $\left[\mathbf{c}_p^B, \mathbf{c}_p^A \oplus \widetilde{\mathbf{c}_p^D}\right] = \pm 1$, where $j\left(\widetilde{\mathbf{c}_p^D}\right) = \mathbf{c}_p^D$, then*

$$\begin{aligned}\mathbb{T}\left(B_*, \left\{\mathbf{c}_p^B\right\}_0^n, \left\{\mathbf{h}_p^B\right\}_0^n\right) &= \mathbb{T}\left(A_*, \left\{\mathbf{c}_p^A\right\}_0^n, \left\{\mathbf{h}_p^A\right\}_0^n\right) \\ \times \quad \mathbb{T}\left(D_*, \left\{\mathbf{c}_p^D\right\}_{p=0}^n, \left\{\mathbf{h}_p^D\right\}_0^n\right) &\mathbb{T}\left(C_*, \left\{\mathbf{c}_{3p}\right\}_0^{3n+2}, \left\{0\right\}_0^{3n+2}\right).\end{aligned}$$

From Theorem 2.1, it follows the following the sum-lemma.

**Lemma 2.2** *Let $A_*$, $D_*$ be chain complexes of vector spaces and $\mathbf{c}_p^A$, $\mathbf{c}_p^D$, $\mathbf{h}_p^A$, and let $\mathbf{h}_p^D$ be bases of $A_p$, $D_p$, $H_p(A_*)$, and $H_p(D_*)$, respectively. Then, the following equality holds:*

$$\mathbb{T}(A_* \oplus D_*, \{\mathbf{c}_p^A \sqcup \mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^A \sqcup \mathbf{h}_p^D\}_0^n) = \mathbb{T}(A_*, \{\mathbf{c}_p^A\}_0^n, \{\mathbf{h}_p^A\}_0^n)\mathbb{T}(D_*, \{\mathbf{c}_p^D\}_0^n, \{\mathbf{h}_p^D\}_0^n).$$

The reader can also find a proof of Lemma 2.2 in [42].

A triple $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is called a $\mathbb{C}-$symplectic chain complex of length $q$, if the following conditions hold:

1. $C_* : 0 \to C_q \overset{\partial_q}{\to} C_{q-1} \to \cdots \to C_{q/2} \to \cdots \to C_1 \overset{\partial_1}{\to} C_0 \to 0$ is a chain complex of length $q$, where $q \equiv 2 \pmod 4$,
2. for $p = 0, \ldots, q$, $\omega_{p,q-p} : C_p \times C_{q-p} \to \mathbb{C}$ is a $\partial-$compatible nondegenerate antisymmetric bilinear form. More precisely,
$$\omega_{p,q-p}\left(\partial_{p+1}a, b\right) = (-1)^{p+1}\omega_{p+1,q-(p+1)}\left(a, \partial_{q-p}b\right)$$
and
$$\omega_{p,q-p}(a, b) = (-1)^{p(q-p)}\omega_{q-p,p}(b, a).$$

Note that the fact $q \equiv 2 \pmod 4$ follows that $\omega_{p,q-p}(a,b)$ is $(-1)^p \omega_{q-p,p}(b,a)$. $\partial-$compatibility of the nondegenerate antisymmetric bilinear maps $\omega_{p,q-p}$ yields the nondegenerate pairing $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{C}$.

For the rest, once the $\mathbb{C}-$symplectic chain complex $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is clear, $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})$ will denote the determinant of the matrix of the nondegenerate pairing $[\omega_{p,q-p}] : H_p(C_*) \times H_{q-p}(C_*) \to \mathbb{C}$ in the bases $\mathbf{h}_p,\ \mathbf{h}_{q-p}$.

Let $C_*$ be a $\mathbb{C}-$symplectic chain complex of length $q$ and let $\mathbf{c}_p,\ \mathbf{c}_{q-p}$ be bases of $C_p,\ C_{q-p}$, respectively. These bases are said to be $\omega-$compatible, if the matrix of $\omega_{p,q-p}$ in bases $\mathbf{c}_p,\ \mathbf{c}_{q-p}$ is the $k \times k$ identity matrix $\mathrm{Id}_{k\times k}$ when $p \neq q/2$ and $\begin{pmatrix} 0_{l\times l} & \mathrm{Id}_{l\times l} \\ -\mathrm{Id}_{l\times l} & 0_{l\times l} \end{pmatrix}$ when $p = q/2$, where $k$ is $\dim C_p = \dim C_{q-p}$ and $2l$ is $\dim C_{q/2}$.

The following result suggests a formula for computing Reidemeister torsion in terms of intersections pairings. More precisely,

**Theorem 2.3** *[44] Assume $(C_*, \partial_*, \{\omega_{*,q-*}\})$ is a $\mathbb{C}-$symplectic chain complex with the $\omega-$compatible bases $\mathbf{c}_p,\ p = 0, \ldots, q$. If $\mathbf{h}_p$ is a basis of $H_p(C_*),\ p = 0, \ldots, q$, then the following formula is valid:*

$$\left|\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^q, \{\mathbf{h}_p\}_0^q\right)\right| = \prod_{p=0}^{(q/2)-1} |\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})|^{(-1)^p} \sqrt{\left|\Delta\left(\mathbf{h}_{q/2}, \mathbf{h}_{q/2}\right)\right|}^{(-1)^{q/2}}. \tag{2.3}$$

Note that in formula (2.3), if $\mathbf{h}_p = \mathbf{h}_{q-p} = 0$, then the convention $0 = 1.0$ is used and thus $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p}) = 1$.

**Remark 2.4** *Before stating our main results, let us note that formula (2.3) can be improved as follows: For a $\mathbb{C}-$symplectic chain complex with the $\omega-$compatible bases $\mathbf{c}_p,\ p = 0, \ldots, q$, there exist bases $\mathbf{h}_p^0$ so that $\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^q, \left\{\mathbf{h}_p^0\right\}_0^q\right) = 1$. From this and change-base-formula (2.1) we have*

$$\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right) = \prod_{p=0}^{q} \left[\mathbf{h}_p^0, \mathbf{h}_p\right]^{(-1)^p}$$

*and thus*

$$\left|\mathbb{T}\left(C_*, \{\mathbf{c}_p\}_0^n, \{\mathbf{h}_p\}_0^n\right)\right| = \prod_{p=0}^{q} \left|\left[\mathbf{h}_p^0, \mathbf{h}_p\right]\right|^{(-1)^p}. \tag{2.4}$$

*Let $\left[\mathbf{h}_p^0, \mathbf{h}_p\right] = \left|\left[\mathbf{h}_p^0, \mathbf{h}_p\right]\right| e^{i\theta_p}$, where $-\pi < \theta_p \leq \pi$. From this and the fact that $\Delta(\mathbf{h}_p, \mathbf{h}_{q-p}) = [\mathbf{h}_p^0, \mathbf{h}_p][\mathbf{h}_{q-p}^0, \mathbf{h}_{q-p}]$, it follows*

$$|\Delta(\mathbf{h}_p, \mathbf{h}_{q-p})| = e^{-i\theta_p} e^{-i\theta_{q-p}} \Delta(\mathbf{h}_p, \mathbf{h}_{q-p})$$

*and*

$$\left|\Delta(\mathbf{h}_{q/2}, \mathbf{h}_{q/2})\right| = \left|[\mathbf{h}_{q/2}^0, \mathbf{h}_{q/2}]\right| \left|[\mathbf{h}_{q/2}^0, \mathbf{h}_{q/2}]\right| = e^{-i\theta_{q/2}} e^{-i\theta_{q/2}} \Delta(\mathbf{h}_{q/2}, \mathbf{h}_{q/2}).$$

*Using these and equation (2.3), we have*

$$\begin{aligned}\left|\mathbb{T}\left(C_*,\{\mathbf{c}_p\}_0^q,\{\mathbf{h}_p\}_0^q\right)\right| &= \prod_{p=0}^{(q/2)-1}\left(e^{-i\theta_p}e^{-i\theta_{q-p}}\Delta(\mathbf{h}_p,\mathbf{h}_{q-p})\right)^{(-1)^p} \\ &\quad\times\sqrt{e^{-i\theta_{q/2}}e^{-i\theta_{q/2}}\Delta(\mathbf{h}_{q/2},\mathbf{h}_{q/2})}^{(-1)^{q/2}} \\ &= e^{-i\sum\limits_{p=0}^{q}(-1)^p\theta_p}\prod_{p=0}^{(q/2)-1}\Delta(\mathbf{h}_p,\mathbf{h}_{q-p})^{(-1)^p}\sqrt{\Delta(\mathbf{h}_{q/2},\mathbf{h}_{q/2})}^{(-1)^{q/2}}.\end{aligned} \tag{2.5}$$

*From (2.4), it follows that*

$$\left|\mathbb{T}\left(C_*,\{\mathbf{c}_p\}_0^n,\{\mathbf{h}_p\}_0^n\right)\right| = \prod_{p=0}^{q}\left[\mathbf{h}_p^0,\mathbf{h}_p\right]^{(-1)^p}\prod_{p=0}^{q}e^{-i\theta_p(-1)^p} = e^{-i\sum\limits_{p=0}^{q}(-1)^p\theta_p}\prod_{p=0}^{q}\left[\mathbf{h}_p^0,\mathbf{h}_p\right]^{(-1)^p}. \tag{2.6}$$

*Combining equations (2.5) and (2.6), we get*

$$\mathbb{T}\left(C_*,\{\mathbf{c}_p\}_0^q,\{\mathbf{h}_p\}_0^q\right) = \prod_{p=0}^{(q/2)-1}\Delta(\mathbf{h}_p,\mathbf{h}_{q-p})^{(-1)^p}\sqrt{\Delta\left(\mathbf{h}_{q/2},\mathbf{h}_{q/2}\right)}^{(-1)^{q/2}}.$$

For further applications of Theorem 2.3, we refer the reader to [18, 19, 42].

**3. Main results**

Throughout this section, $\Sigma$ denotes a closed orientable surface of genus at least 2 with the universal covering $\widetilde{\Sigma}$. $G$ is one of the Lie groups $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$. $\mathcal{G}$ denotes the Lie algebra of $G$ with the nondegenerate symmetric bilinear form $B$. For $\mathrm{GL}(n,\mathbb{C})$, we consider $B:\mathcal{G}\times\mathcal{G}\to\mathbb{C}$ as $B(u,v)=\mathrm{Trace}(uv^t)$, where $t$ is the transpose. For $\mathrm{SL}(n,\mathbb{C})$, we consider $B$ is the Killing form.

Let $\varrho:\pi_1(\Sigma)\to G$ be a homomorphism from the fundamental group of the surface to $G$ and $E_\varrho = \widetilde{\Sigma}\times\mathcal{G}/\sim$ be the corresponding adjoint bundle over $\Sigma$. Here, $(x_1,t_1)\sim(x_2,t_2)$, if $(x_2,t_2)=(\gamma\cdot x_1,\gamma\cdot t_1)$ for some $\gamma\in\pi_1(\Sigma)$, $\gamma$ acts in the first component by deck transformation $(\gamma\cdot x_1=\gamma(x_1))$ and in the second component by the adjoint action $(\gamma\cdot t_1=\mathrm{Ad}_{\varrho(\gamma)}(t_1)=\varrho(\gamma)t_1\varrho(\gamma)^{-1})$.

Suppose $K$ is a cell-decomposition of $\Sigma$ so that the adjoint bundle $E_\varrho$ is trivial over each cell and $\widetilde{K}$ is the lift of $K$ to the universal covering $\widetilde{\Sigma}$. Let $\mathbb{Z}[\pi_1(\Sigma)]$ be the integral group ring and let $C_*\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right) = C_*\left(\widetilde{K};\mathbb{Z}\right)\otimes\mathcal{G}/\sim$. Here, for all $\gamma\in\pi_1(\Sigma)$, $\sigma\otimes t\sim\gamma\cdot\sigma\otimes\gamma\cdot t$, $\gamma$ acts in the first component by deck transformation and in the second by adjoint action. Hence, there is the following chain complex:

$$0\longrightarrow C_2\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)\xrightarrow{\partial_2\otimes\mathrm{id}}C_1\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)\xrightarrow{\partial_1\otimes\mathrm{id}}C_0\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)\longrightarrow 0, \tag{3.1}$$

where $\partial_p$ is the usual boundary operator. Let $H_*\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $H^*\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ be respectively the homologies and cohomologies of the chain complex (3.1). Here, $C^*\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ denotes the set of $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms from $C_*\left(\widetilde{K};\mathbb{Z}\right)$ to $\mathcal{G}$. For details, we refer to [34].

Note that if $\varrho, \varrho' : \pi_1(\Sigma) \to G$ are conjugate; that is, $\varrho'(.) = A\varrho(.)A^{-1}$ for some $A \in G$, then chains $C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_*\left(K; \mathcal{G}_{\mathrm{Ad}_{\varrho'}}\right)$ are isomorphic. Likewise, the corresponding cochains $C^*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C^*\left(K; \mathcal{G}_{\mathrm{Ad}_{\varrho'}}\right)$ are isomorphic.

Let us consider chain complex (3.1). Suppose $\{e_j^p\}_{j=1}^{m_p}$ is a basis of $C_p(K;\mathbb{Z})$. For $j = 1, \ldots, m_p$, let us fix a lift $\widetilde{e}_j^p$ of $e_j^p$. Then, $c_p = \{\widetilde{e}_j^p\}_{j=1}^{m_p}$ of $C_p\left(\widetilde{K};\mathbb{Z}\right)$ becomes a $\mathbb{Z}[\pi_1(\Sigma)]-$basis. Let $\mathcal{A} = \{a_k\}_{k=1}^{\dim \mathcal{G}}$ be a $B-$*orthonormal basis* of the Lie algebra $\mathcal{G}$. To be more precise, the matrix of the form $B$ is equal to the identity matrix of size $\dim \mathcal{G}$. In this way, we get a $\mathbb{C}-$basis $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ of $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$. Such a basis will be called a *geometric basis* for $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$.

Let $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ and $\mathbf{h_p}$ be respectively the geometric basis of $C_p\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and a basis of $H_p\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$. Then, $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ is called the Reidemeister torsion of the triple $K$, $Ad_\varrho$, and $\{\mathbf{h}_p\}_{p=0}^2$.

Arguments as in [41, Lemma 1.4.2 and Lemma 2.0.5] enable one to conclude that the definition is independent of basis $\mathcal{A}$, conjugacy class of $\varrho$, lifts $\widetilde{e}_j^p$, and of the cell-decomposition. For the sake of completeness, we will explain the independence of $\mathcal{A}$, lifts $\widetilde{e}_j^p$, and conjugacy class of $\varrho$. We refer [41, Lemma 2.0.5] for the independence of the cell-decomposition.

**Theorem 3.1** *Let us consider that* $\Sigma$, $K$, $\varrho$, $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$, *and* $\mathbf{h}_p$, $p = 0, 1, 2$, *are as above. Then,* $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ *does not depend on the basis* $\mathcal{A}$, *lifts* $\widetilde{e}_j^p$, *conjugacy class of* $\varrho$, *and the cell-decomposition* $K$.

**Proof** Let us start with the independence of the torsion from the basis $\mathcal{A}$. If $\mathcal{A}'$ is also a $B-$orthonormal basis of $\mathcal{G}$, then by the change-base-formula (2.1) of Reidemeister torsion, we have

$$\frac{\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{\mathbf{c}'_p\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)}{\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{\mathbf{c}_p\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)} = [\mathcal{A}, \mathcal{A}']^{-\chi(\Sigma)},$$

where $\mathbf{c}'_p = c_p \otimes_\varrho \mathcal{A}'$ and $\chi$ is the Euler characteristic. The fact that $\mathcal{A}$ and $\mathcal{A}'$ are $B-$orthonormal bases of $\mathcal{G}$ yields the independence of Reidemeister torsion from the basis $\mathcal{A}$.

Let us note that for conjugate representations, the corresponding twisted chains and cochains are isomorfic. Thus, Reidemeister torsion is independent of conjugacy class of $\varrho$.

Let us now show that the torsion is independent of the lifts $\widetilde{e}_j^p$. To do that, let us fix $\gamma \in \pi_1(\Sigma)$ and consider the lift $c'_p = \left\{\widetilde{e}_1^p \cdot \gamma, \widetilde{e}_2^p, \ldots, \widetilde{e}_{m_p}^p\right\}$ of $\left\{e_1^p, \ldots, e_{m_p}^p\right\}$. Clearly, we have $\widetilde{e}_1^p \cdot \gamma \otimes t = \widetilde{e}_1^p \otimes \gamma \cdot t$. Note that $\gamma$ acts on the left hand side by deck transformation and right hand side by adjoint action.

Change-base-formula (2.1) yields

$$\frac{\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{\mathbf{c}'_p\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)}{\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{\mathbf{c}_p\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)} = \det(T),$$

where $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$, $\mathbf{c}'_p = c'_p \otimes_\varrho \mathcal{A}$, and $T$ is the matrix of the linear map $Ad_{\varrho(\gamma)} : \mathcal{G} \to \mathcal{G}$ with respect to basis $\mathcal{A}$.

Let us consider $G = \mathrm{GL}(n, \mathbb{C})$ case. Recall that for some $Q \in \mathrm{GL}(n, \mathbb{C})$, we have $\varrho(\gamma) = Q^{-1} U Q$, where $U = [u_{i,j}] \in \mathrm{GL}(n, \mathbb{C})$ is an upper triangular matrix. Then, we have

$$Ad_{\varrho(\gamma)} = (Ad_Q)^{-1} \circ Ad_U \circ Ad_Q.$$

Let us consider the ordered basis $\mathcal{B}_{\mathfrak{gl}(n,\mathbb{C})}$

$$\{E_{i,n-j+1},\ 1 \le j \le n,\ 1 \le i \le n-j+1\} \cup \{E_{i,n-j},\ 1 \le j \le n-1,\ n-j+1 \le i \le n\}$$

for Lie algebra $\mathfrak{gl}(n, \mathbb{C})$.

Recall that the $j^{th}$ column of the $n \times n-$matrix $ME_{ij}$ is the $i^{th}$ column of the $n \times n-$matrix $M$. Recall also that the $i^{th}$ row of $E_{ij}M$ is the $j^{th}$ row of $M$. From this it follows that the matrix of $Ad_U$ in the ordered basis $\mathcal{B}_{\mathfrak{gl}(n,\mathbb{C})}$ is an upper triangular matrix with diagonal elements are

$$\left( \frac{U_{j,j}}{U_{i,i}},\ 1 \le i \le j \le n;\ \frac{U_{j,j}}{U_{i,i}},\ 1 \le j < i \le n \right).$$

Clearly, determinant of the $Ad_U$ in the basis $\mathcal{B}_{\mathfrak{gl}(n,\mathbb{C})}$ is 1.

See Appendix 4.1 for the case $n = 3$.

Note that for the case $\mathrm{SL}(n, \mathbb{C})$ the matrix $Q \in \mathrm{GL}(n, \mathbb{C})$ can be chosen in $\mathrm{SL}(n, \mathbb{C})$. As in arguments above, consider the ordered basis $\mathcal{B}_{\mathfrak{sl}(n,\mathbb{C})}$

$$\{E_{i,n-j+1},\ 1 \le j \le n-1,\ 1 \le i \le n-j\} \cup \{E_{i,n-j},\ 1 \le j \le n-1,\ n-j+1 \le i \le n\}$$

$$\cup \{E_{i,i} - E_{i+1,i+1},\ 1 \le 1 \le n-1\}$$

for Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

The matrix of $Ad_U$ in the this ordered basis $\mathcal{B}_{\mathfrak{sl}(n,\mathbb{C})}$ is an upper triangular matrix with determinant 1.

Thus, $\det T = 1$ for $G = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$. Hence, we conclude the independence of Reidemeister torsion from the lifts $\widetilde{e}_j^p$. This ends the proof of Theorem 3.1. □

Since Theorem 3.1 proves the well definiteness of Reidemeister torsion of such representations, rather than $\mathbb{T}\left(C_*\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right), \{c_p \otimes_\varrho \mathcal{A}\}_{p=0}^2, \{\mathbf{h}_p\}_{p=0}^2\right)$ we will write $\mathbb{T}(\Sigma, \{\mathbf{h}_p\}_{p=0}^2)$. Let us also note that if $\varrho : \pi_1(\Sigma) \to G$, where $G = G_1 \times \cdots \times G_d$ and $G_i, i = 1, \ldots, d$, are one of the Lie groups from the above list, then by Theorem 3.1, Reidemeister torsion of such representation is well defined, too.

With the help of $\mathbb{C}-$symplectic chain complex, we will establish a formula for computing Reidemeister torsion of representations in terms of the well-known symplectic structure on $\mathrm{Rep}(\Sigma, G)$, more explicitly, Atiyah-Bott-Goldman symplectic form for the Lie group $G$.

Let $\Sigma$, $K$, $G$, $\mathcal{G}$, $\varrho$, $\mathbf{c}_p = c_p \otimes_\varrho \mathcal{A}$ be as above. Consider the dual cell-decomposition $K'$ of $\Sigma$ corresponding to the cell-decomposition $K$. Let us consider the lifts $\widetilde{K}$ and $\widetilde{K'}$ of $K$ and $K'$, respectively. For $i = 0, 1, 2$, let us consider the intersection form

$$(\cdot, \cdot)_{i,2-i} : C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow \mathbb{C} \tag{3.2}$$

defined by $(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{i,2-i} \;= \sum_{\gamma\in\pi_1(\Sigma)} \sigma_1.\left(\gamma \bullet \sigma_2\right)\; B\left(t_1, \gamma \bullet t_2\right)$. Here, "." is the intersection number pairing, $\gamma$ acts on $\sigma_2$ by deck transformation and on $t_2$ by the adjoint action.

Clearly, the intersection form $(\cdot,\cdot)_{i,2-i}$ is antisymmetric, $\partial-$compatible. It can naturally be extended to twisted homologies and yield the nondegenerate antisymmetric form

$$[\cdot,\cdot]_{i,2-i} : H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow \mathbb{C}. \tag{3.3}$$

Let us consider $D_i = C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \oplus C_i\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and the bilinear form $\omega_{i,2-i} : D_i \times D_{2-i} \to \mathbb{C}$ defined by extending the intersection form (3.2) zero on $C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_i\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_{2-i}\left(K'; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$. Following arguments as in [43, Theorem 4.1], $D_*$ is a $\mathbb{C}-$symplectic chain complex. Clearly, the bases $c_i$ of $C_i\left(\widetilde{K}; \mathbb{Z}\right)$ and $c_i'$ of $C_i\left(\widetilde{K'}; \mathbb{Z}\right)$ corresponding to $c_i$ yield an $\omega-$compatible basis for $D_*$.

Recall that Kronecker pairing is the nondegenerate form $\langle\cdot,\cdot\rangle : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C_i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to \mathbb{C}$ defined by $\langle \theta, \sigma \otimes_\varrho t\rangle = B\left(t, \theta\left(\sigma\right)\right)$. It can be extended to $\langle\cdot,\cdot\rangle : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to \mathbb{C}$.

The cup product $\cup : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C^j\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \to C^{i+j}\left(\widetilde{\Sigma}; \mathbb{C}\right)$ is defined by $\left(\theta_i \cup \theta_j\right)\left(\sigma_{i+j}\right) = B\left(\theta_i\left(\left(\sigma_{i+j}\right)_{\text{front}}\right), \theta_j\left(\left(\sigma_{i+j}\right)_{\text{back}}\right)\right)$. Here, $\sigma_{i+j}$ is in $C_{i+j}(\widetilde{K}; \mathbb{Z})$ and $\widetilde{K}$ is the lift of $K$ to the universal covering $\widetilde{\Sigma}$ of $\Sigma$, $\theta_i : C_i\left(\widetilde{K}; \mathbb{Z}\right) \to \mathcal{G}$, $\theta_j : C_j\left(\widetilde{K}; \mathbb{Z}\right) \to \mathcal{G}$ are $\mathbb{Z}[\pi_1(\Sigma)]$-module homomorphisms. We have the cup product

$$\smile_B : C^i\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times C^j\left(K; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow C^{i+j}\left(K; \mathbb{C}\right).$$

Clearly, $\smile_B$ can be extended to twisted cohomologies

$$\smile_B : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H^j\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \longrightarrow H^{i+j}\left(\Sigma; \mathbb{C}\right).$$

Here, $[\theta_i] \smile_B [\theta_j]$ equals $[\theta_i \smile_B \theta_j]$.

Combining the isomorphisms induced by intersection form (3.3) and the Kronecker pairing, we have the Poincare duality isomorphisms

$$\mathrm{PD} : H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \cong H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)^* \cong H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right).$$

The following commutative diagram exists for $i = 0, 1, 2$

$$\begin{array}{ccccccc} H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \times & H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \xrightarrow{\smile_B} & H^2\left(\Sigma; \mathbb{C}\right) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ H_i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \times & H_{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) & \xrightarrow{[\cdot,\cdot]_{i,2-i}} & \mathbb{C}, \end{array} \tag{3.4}$$

where the isomorphism $\mathbb{C} \to H^2\left(\Sigma; \mathbb{C}\right)$ sends $1 \in \mathbb{C}$ to the fundamental class of $H^2(\Sigma; \mathbb{C})$ and the inverse of this isomorphism is integration over $\Sigma$.

Note that commutative diagram (3.4) yields the pairing

$$\Omega_{i,2-i} : H^i\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \times H^{2-i}\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right) \xrightarrow{\smile_B} H^2\left(\Sigma; \mathbb{C}\right) \xrightarrow{\int_\Sigma} \mathbb{C}. \tag{3.5}$$

Note also that $\Omega_{1,1}$ is Atiyah-Bott-Goldman symplectic form for the Lie group $G$ on $\mathrm{Rep}(\Sigma, G)$.

Let us state one of our main results where we establish a formula for computing Reidemeister torsion of representations in terms of $\Omega_{1,1}$ Atiyah-Bott-Goldman symplectic form for the Lie group $G$. We have:

**Theorem 3.2** *Suppose $\Sigma,\ K,\ K',\ \varrho$ are as above. Suppose also $\mathbf{c}_p$ and $\mathbf{c}'_p$ are the corresponding geometric bases of $C_p\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and $C_p\left(K';\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, respectively, $p=0,1,2$. Let $\mathbf{h}_p$ be a basis of $H_p\left(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, $p=0,1,2$. Then, the following formulas hold:*

1. $\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right) = ie^{\frac{i\theta}{2}}\dfrac{\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)}{\sqrt{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}}$,

2. $\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right) = ie^{\frac{i\theta}{2}}\dfrac{\sqrt{\delta\left(\mathbf{h}^1,\mathbf{h}^1\right)}}{\delta\left(\mathbf{h}^2,\mathbf{h}^0\right)}$.

*Here, $\Delta\left(\mathbf{h}_p,\mathbf{h}_{2-p}\right)$ denotes the determinant of the matrix of the intersection pairing (3.3) in the bases $\mathbf{h}_p$ and $\mathbf{h}_{2-p}$, $\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)=\left|\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)\right|e^{i\theta}$, where $i=\sqrt{-1}$ and $-\pi<\theta\leq\pi$. $\delta\left(\mathbf{h}^{2-p},\mathbf{h}^p\right)$ denotes the determinant of the matrix of the pairing (3.5) in the bases $\mathbf{h}^p$ and $\mathbf{h}^{2-p}$, and $\mathbf{h}^p$ denotes the Poincare dual basis of $H^p(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho})$ associated to the basis $\mathbf{h}_p$ of $H_p(\Sigma;\mathcal{G}_{\mathrm{Ad}_\varrho})$, $p=0,1,2$.*

**Proof** $D_*=C_*\left(K;\mathcal{G}_{\mathrm{Ad}_\varrho}\right)\oplus C_*\left(K';\mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ is a $\mathbb{C}-$symplectic chain complex with $\omega-$compatible basis obtained by the geometric bases $\mathbf{c}_p$, $\mathbf{c}'_p$. We have $\mathbb{T}\left(D_*,\left\{\mathbf{c}_p\oplus\mathbf{c}'_p\right\}_{p=0}^2,\left\{\mathbf{h}_{p,p}^0\right\}_{p=0}^2\right)=1$ for $p=0,1,2$, $\mathbf{h}_{p,p}^0=[\mathbf{c}_p]\oplus[\mathbf{c}'_p]$. By using Theorem 2.3 and Remark 2.4, the following holds:

$$\mathbb{T}\left(D_*,\left\{\mathbf{c}_i\oplus\mathbf{c}'_p\right\}_{p=0}^2,\left\{\mathbf{h}_p\oplus\mathbf{h}_p\right\}_{p=0}^2\right)=\frac{\Delta\left(\mathbf{h}_0\oplus\mathbf{h}_0,\mathbf{h}_2\oplus\mathbf{h}_2\right)}{\sqrt{\Delta\left(\mathbf{h}_1\oplus\mathbf{h}_1,\mathbf{h}_1\oplus\mathbf{h}_1\right)}}. \tag{3.6}$$

Note that in the middle dimension we have an nondegenerate antisymmetric bilinear form. Then, $\Delta\left(\mathbf{h}_1\oplus\mathbf{h}_1\right)$ is positive (See e.g. [22, Theorem 6]) and thus $\sqrt{\Delta\left(\mathbf{h}_1\oplus\mathbf{h}_1,\mathbf{h}_1\oplus\mathbf{h}_1\right)}=\Delta\left(\mathbf{h}_1\oplus\mathbf{h}_1\right)$.

Clearly, we have

$$\Delta\left(\mathbf{h}_0\oplus\mathbf{h}_0,\mathbf{h}_2\oplus\mathbf{h}_2\right)=-\Delta(\mathbf{h}_0,\mathbf{h}_2)^2. \tag{3.7}$$

Moreover, we get

$$\Delta\left(\mathbf{h}_1\oplus\mathbf{h}_1,\mathbf{h}_1\oplus\mathbf{h}_1\right)=\Delta(\mathbf{h}_1,\mathbf{h}_1)^2. \tag{3.8}$$

Eqs. (3.6)−(3.8) and the fact that $[\cdot,\cdot]_{1,1}$ is antisymmetric yield

$$\mathbb{T}\left(D_*,\left\{\mathbf{c}_p\oplus\mathbf{c}'_p\right\}_{p=0}^2,\left\{\mathbf{h}_p\oplus\mathbf{h}_p\right\}_{p=0}^2\right)=-\frac{\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)^2}{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}. \tag{3.9}$$

Let $\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)=\left|\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)\right|e^{i\theta}$, where $i=\sqrt{-1}$ and $-\pi<\theta\leq\pi$.

By Lemma 2.2 and equation (3.9), we obtain

$$\mathbb{T}\left(\Sigma,\{\mathbf{h}_p\}_{p=0}^2\right)=ie^{\frac{i\theta}{2}}\frac{\Delta\left(\mathbf{h}_0,\mathbf{h}_2\right)}{\sqrt{\Delta\left(\mathbf{h}_1,\mathbf{h}_1\right)}}. \tag{3.10}$$

From commutative diagram (3.4), it follows $\delta\left(\mathbf{h}^{2-p}, \mathbf{h}^{p}\right) \cdot \Delta\left(\mathbf{h}_{p}, \mathbf{h}_{2-p}\right) = 1$.

This and Eq. (3.10) end the proof of Theorem 3.2. □

Let us also note that in case $H_0\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ and hence $H_2\left(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ are zero, then by Theorem 3.2, we have

$$\mathbb{T}\left(\Sigma, \{0, \mathbf{h}_1, 0\}\right) = i\ \sqrt{\Delta\left(\mathbf{h}_1, \mathbf{h}_1\right)}^{\,(-1)} = i\ \sqrt{\delta\left(\mathbf{h}^1, \mathbf{h}^1\right)}.$$

Here, we use the convention $0 = 1.0$ and thus $\Delta(0,0) = 1$.

## 4. Applications

### 4.1. Good free or surface group representations

In this section, we will apply our result (Theorem 3.2) to good free or surface group representations. For more information and unexplained subjects, we refer the reader to [39] and the references therein.

Let $\Gamma$ be a finitely generated group and $G$ be the complex reductive algebraic groups $\mathrm{GL}\left(n, \mathbb{C}\right),\ \mathrm{SL}(n, \mathbb{C})$ or their quotients. Recall that a representation $\varrho : \Gamma \to G$ is said to be irreducible, if $\varrho(\Gamma)$ is not contained in any proper parabolic subgroup of $G$.

Let $\mathrm{Hom}\left(\Gamma, G\right)$ be the set of all homomorphisms from $\Gamma$ to $G$ and $\mathrm{Hom}^{irr}(\Gamma, G)$ denote the set of irreducible $G-$representations of $\Gamma$. The space $\mathrm{Hom}^{irr}(\Gamma, G)$ is a Zariski open subset of $\mathrm{Hom}(\Gamma, G)$ [39, Proposition 27]. The set $\mathrm{Hom}^{irr}(\Gamma, G)$ is invariant under the $G$ action by conjugation. Each orbit in $\mathrm{Hom}^{irr}(\Gamma, G)$ is closed [39, Theorem 30] and each equivalence class in a categorical quotient contains a unique closed orbit. Thus, the categorical quotient, $\mathrm{Hom}^{irr}(\Gamma, G)//G$ coincides with the set-theoretic quotient. Let us denote this by $\chi^{irr}(\Gamma, G)$.

Recall that if $\Gamma = F_n$ denotes a free group with generator $n$ and $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$, then $\chi^{irr}(\Gamma, G)$ is a manifold dimension $(n-1)\dim G$. If $\Gamma = \pi_1(\Sigma_g)$ denotes a surface group with genius $g \geq 2$ and $G = \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$, then $\chi^{irr}(\Gamma, G)$ is a manifold dimension $(2g-2)\dim G$ [13], [39, Propositions 5 and 49]. Recall also that an irreducible $\varrho : \Gamma \to G$ is said to be good, if the stabilizer of its image coincides with the center of $G$. The set of such homomorphisms is a Zariski open subset of $\mathrm{Hom}(\Gamma, G)$ [39, Proposition 33]. Let $\chi^{good}(\Gamma, G) = \mathrm{Hom}^{good}(\Gamma, G)/G$. This is an open subset of $\chi^{irr}(\Gamma, G)$ and is a smooth manifold for free groups and surface groups $\Gamma$ [39, Corollary 50].

Let $\Sigma_g$ be a closed orientable surface of genus $g \geq 2$. Let $\Gamma$ denote the fundamental group $\pi_1(\Sigma_g)$ of the surface $\Sigma_g$. Let $\varrho : \Gamma \to G$ be a good representation. For the reductive group $G = \mathrm{GL}\left(n, \mathbb{C}\right),\ \mathrm{SL}(n, \mathbb{C})$, the corresponding Lie algebra $\mathcal{G}$ has the nondegenerate symmetric bilinear form $B$. More precisely, for $\mathrm{GL}(n, \mathbb{C})$, $B : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ as $B\left(u, v\right) = \mathrm{Trace}\left(uv^t\right)$, where $t$ is the transpose. For $\mathrm{SL}(n, \mathbb{C})$, we consider $B$ is the Killing form. From these, we have for $i = 0, 2,\ H_i\left(\Sigma_g; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$, and $H^i\left(\Sigma_g; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ vanish [13].

Combining these and applying Theorem 3.2, we have:

**Theorem 4.1** *Let $\Sigma_g$ be a closed orientable surface of genus $g \geq 2$, $\Gamma$ be the fundamental group $\pi_1\left(\Sigma_g\right)$ of $\Sigma_g$, and $G$ be $\mathrm{GL}\left(n, \mathbb{C}\right)$ or $\mathrm{SL}(n, \mathbb{C})$. For $\varrho \in \chi^{good}(\Gamma, G)$, we have*

1. $\mathbb{T}\left(\Sigma_g, \{\mathbf{h}_1\}\right) = \dfrac{i}{\sqrt{\Delta\left(\mathbf{h}_1, \mathbf{h}_1\right)}}$,

*2.* $\mathbb{T}\left(\Sigma_g, \{\mathbf{h}_1\}\right) = i\,\sqrt{\delta\left(\mathbf{h}^1, \mathbf{h}^1\right)}$.

*Here,* $i = \sqrt{-1}$, $\Delta\left(\mathbf{h}_1, \mathbf{h}_1\right)$ *is the determinant of the matrix of the intersection pairing (3.3) in the basis* $\mathbf{h}_1$, *and* $\delta\left(\mathbf{h}^1, \mathbf{h}^1\right)$ *is the determinant of the matrix of the pairing (3.5) in the basis* $\mathbf{h}^1$, *and* $\mathbf{h}^1$ *is the Poincare dual basis of* $H^1(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho})$ *associated to the basis* $\mathbf{h}_1$ *of* $H_1(\Sigma; \mathcal{G}_{\mathrm{Ad}_\varrho})$.

Let $\mathrm{H}_g$ be handlebody with $g \geq 2$, $\Sigma_g$ be the boundary surface of $\mathrm{H}_g$, and M be the double of $\mathrm{H}_g$. Let $G$ be $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$ with Lie algebra $\mathcal{G}$. Let $\varrho : \pi_1(\mathrm{H}_g) \to G$ be a homomorphism so that $\varrho \circ r : \pi_1(\Sigma_g) \to G$ is a good homomorphism, where $r : \pi_1(\Sigma_g) \to \pi_1(\mathrm{H}_g)$ is the homomorphism obtained by the embedding $\partial \mathrm{H}_g \hookrightarrow \mathrm{H}_g$. Note that $\varrho : \pi_1(\mathrm{H}_g) \to G$ is also a good representation [39, Remark 65].

Let us consider the following short-exact sequence

$$0 \to C_*\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right) \to C_*\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \oplus C_*\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \to C_*\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right) \to 0 \tag{4.1}$$

of chain complexes and the associated Mayer-Vietoris long exact sequence $\mathcal{H}_*$ :

$$\begin{array}{c}
0 \to H_3\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right) \to H_3\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \oplus H_3\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \to H_3\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right) \\
\to H_2\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right) \to H_2\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \oplus H_2\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \to H_2\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right) \\
\to H_1\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right) \to H_1\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \oplus H_1\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \to H_1\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right) \\
\to H_0\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right) \to H_0\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \oplus H_0\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \to H_0\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right) \to 0.
\end{array} \tag{4.2}$$

**Theorem 4.2** *Let* $\Sigma_g$, $\mathrm{H}_g$, M, $G$, $\mathcal{G}$, $\varrho$, *and* $r$ *be as above. Consider the short-exact sequence (4.1) and the corresponding Mayer-Vietoris long exact sequence (4.2). Let* $\mathbf{h}_i^{\mathrm{H}_g}$ *be a basis for* $H_i\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right)$, $i = 0, 1, 2, 3$. *Then, there exist bases* $\mathbf{h}_j^{\mathrm{M}}$ *and* $\mathbf{h}_k^{\Sigma_g}$ *of* $H_j\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right)$ *and* $H_k\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right)$, $j = 0, 1, 2, 3$, $k = 0, 1, 2$, *respectively so that Reidemeister torsion of sequence (4.2) in these bases becomes* 1. *Moreover, the following formulas hold:*

*1.* $\mathbb{T}\left(\mathrm{H}_g, \left\{\mathbf{h}_i^{\mathrm{H}_g}\right\}_0^3\right) = \dfrac{i^{\left(1+\dim H_0\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right)\right)/2}}{\sqrt[4]{\Delta\left(\mathbf{h}_1^{\Sigma_g}, \mathbf{h}_1^{\Sigma_g}\right)}}$,

*2.* $\mathbb{T}\left(\mathrm{H}_g, \left\{\mathbf{h}_i^{\mathrm{H}_g}\right\}_0^3\right) = i^{\left(1+\dim H_0\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right)\right)/2}\,\sqrt[4]{\delta\left(\mathbf{h}^1, \mathbf{h}^1\right)}$.

*Here,* $\mathbf{h}^1$ *is the Poincare dual basis of* $H^1\left(\Sigma_g; \mathcal{G}_{\mathrm{Ad}_\varrho}\right)$ *corresponding to* $\mathbf{h}_1^{\Sigma_g}$.

**Proof** First, since $\varrho \circ r$ is good, $H_0\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right)$ and thus $H_2\left(\Sigma_g; \mathcal{G}_{Ad_{\varrho \circ r}}\right)$ vanish. From this and (4.2), we have

$$0 \longrightarrow H_3\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \oplus H_3\left(\mathrm{H}_g; \mathcal{G}_{Ad_\varrho}\right) \longrightarrow H_3\left(\mathrm{M}; \mathcal{G}_{Ad_\varrho}\right) \longrightarrow 0, \tag{4.3}$$

$$\begin{array}{c} 0 \longrightarrow H_2\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \oplus H_2\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \longrightarrow H_2\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right) \\ \downarrow \\ H_1\left(\Sigma_g;\mathcal{G}_{Ad_{\varrho\circ r}}\right) \to H_1\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \oplus H_1\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \to H_1\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right) \to 0, \end{array} \tag{4.4}$$

and

$$0 \longrightarrow H_0\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \oplus H_0\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \longrightarrow H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right) \longrightarrow 0. \tag{4.5}$$

For $j = 0, 3,$ let us consider the basis for $H_j\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$, which is obtained by using the isomorphisms from the short-exact sequences (4.3), (4.5), and the basis for $\mathbf{h}_i^{\mathrm{H}_g}$ of $H_i\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right)$.

By the exactness of (4.4), and Poincare duality, we have

$$2\dim H_2\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) = 2\dim H_1\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) - \dim\ H_1\left(\Sigma_g;\mathcal{G}_{Ad_{\varrho\circ r}}\right).$$

From this and [39, Theorem 61], we get $H_2\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) = 0$. For the sake of simplicity, let $U$, $V$, $W$, and $T$ denote respectively $H_2\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$, $H_1\left(\Sigma_g;\mathcal{G}_{Ad_{\varrho\circ r}}\right)$, $H_1\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right) \oplus H_1\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right)$, and $H_1\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$. The short-exact sequence (4.4) becomes

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \xrightarrow{\gamma} T \longrightarrow 0. \tag{4.6}$$

Let us denote by $\{w_1, \ldots, w_d\}$ the basis of the vector space $W$ which is obtained by considering the basis $\mathbf{h}_1^{\mathrm{H}_g}$ of $H_1\left(\mathrm{H}_g;\mathcal{G}_{Ad_\varrho}\right)$. We will consider the basis on $T$ as $\mathbf{h}_1^{\mathrm{M}} = \{\gamma(w_{i_1}), \ldots, \gamma(w_{i_k})\}$. Let us take any basis $\mathbf{h}_2^{\mathrm{M}}$ of $U$.

The sequence (4.6) and The First Isomorphism Theorem yield

$$0 \longrightarrow B_0 \hookrightarrow T \twoheadrightarrow B_{-1} \longrightarrow 0,$$

where $B_0 = \mathrm{Im}\gamma = T$ and $B_{-1} = 0$. Considering the basis $\mathbf{h}_1^{\mathrm{M}}$ of $T$ as the basis on $B_0$, the determinant of the change-base-matrix for the bases of $T$ becomes 1.

By the sequence (4.6), we also have $0 \longrightarrow B_1 \hookrightarrow W \overset{\gamma}{\twoheadrightarrow} B_0 \longrightarrow 0$, where $B_1 = \mathrm{Im}\beta$ and $B_0 = \mathrm{Im}\gamma = T$. Let us consider the inverse of the isomorphism obtained from $W/\mathrm{Ker}\gamma \cong \mathrm{Im}\gamma$ as section of $W \overset{\gamma}{\to} B_0$. Let us also consider the basis $\left\{w_{i_j}; i_j \in \{1, \ldots, d\} \setminus \{i_1, i_2, \ldots, i_k\}\right\}$ of $B_1$. Hence, the determinant of the change-base-matrix for the bases of $W$ equals 1.

Finally, from (4.6), it follows $0 \to B_2 \hookrightarrow V \overset{\beta}{\twoheadrightarrow} B_1 \to 0$, where $B_2 = \mathrm{Im}\alpha,\ B_1 = \mathrm{Im}\beta$. We consider $\alpha\left(\mathbf{h}_2^{\mathrm{M}}\right)$ as the basis of $B_2$. Using the basis of $B_1$ obtained in the previous step and the inverse of the isomorphism obtained from $V/\mathrm{Ker}\beta \cong \mathrm{Im}\beta$ as section of $V \overset{\beta}{\to} B_1$, we get a basis for $V$. Therefore, by letting this basis as the basis $\mathbf{h}_1^{\Sigma_g}$ of $V$, the determinant of the change-base-matrix for the bases of $V$ is 1.

Combining all the above, Reidemeister torsion of the sequence $\mathcal{H}_*$ in these bases is equal to 1.

Theorem 2.1 and Lemma 2.2 yield

$$\mathbb{T}\left(\mathrm{H}_g, \left\{\mathbf{h}_i^{\mathrm{H}_g}\right\}_0^3\right)^2 = \mathbb{T}\left(\Sigma_g, \left\{\mathbf{h}_k^{\Sigma_g}\right\}_0^2\right)\mathbb{T}\left(\mathrm{M}, \left\{\mathbf{h}_j^{\mathrm{M}}\right\}_0^3\right). \tag{4.7}$$

We also have

$$\mathbb{T}\left(\mathrm{M},\left\{\mathbf{h}_j^{\mathrm{M}}\right\}_0^3\right)=i^{\,\dim H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)}. \tag{4.8}$$

Equations (4.7), (4.8), and Theorem 4.1 finish the proof of Theorem 4.2. □

Let H be a compact hyperbolic 3-manifold with boundary $\partial$H consisting of surfaces $\Sigma_{g_1},\ldots,\Sigma_{g_\ell}$ of genus at least 2. Let M be the double of H. Let $G$ be the Lie groups $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$ or their quotients. Let $\varrho:\pi_1(\mathrm{H})\to G$ be a homomorphism so that $\varrho\circ r_i:\pi_1\left(\Sigma_{g_i}\right)\to G$ is a good homomorphism for $i=1,\ldots,\ell$. Here, $r_i:\pi_1\left(\Sigma_{g_i}\right)\to\pi_1(\mathrm{H})$ is the homomorphism obtained by the embedding $\Sigma_{g_i}\hookrightarrow$ H. Clearly, for $\varrho\circ r_i$ being good, $H_j\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$ vanishes $i=1,\ldots,\ell,\ j=0,2$.

We consider the following short-exact sequence

$$0\to\bigoplus_{i=1}^{\ell}C_*\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)\to C_*\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)\oplus C_*\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)\to C_*\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)\to 0 \tag{4.9}$$

of chain complexes and the associated Mayer-Vietoris long exact sequence of vector spaces $\mathcal{H}_*$.

Combining these and following the arguments as in Theorem 4.2, we have

**Theorem 4.3** *Let $\Sigma_{g_i}$, H, M, $G$, $\mathcal{G}$, $\varrho$, and $r_i$ be as above. Considering the short-exact sequence (4.9) and the corresponding Mayer-Vietoris $\mathcal{H}_*$, let $\mathbf{h}_k^{\mathrm{H}}$ denote a basis for $H_k\left(\mathrm{H};\mathcal{G}_{Ad_\varrho}\right)$, $k=0,1,2,3$. Then, there exist bases $\mathbf{h}_k^{\mathrm{M}}$ and $\mathbf{h}_1^{\Sigma_{g_i}}$ of $H_k\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$ and $H_1\left(\Sigma_{g_i};\mathcal{G}_{Ad_{\varrho\circ r_i}}\right)$, $i=1,\ldots,\ell$, $k=0,1,2,3$, respectively so that Reidemeister torsion of the long exact sequence $\mathcal{H}_*$ in these bases becomes 1. Furthermore, the following formulas*

1. $\mathbb{T}\left(\mathrm{H},\left\{\mathbf{h}_k^{\mathrm{H}}\right\}_0^3\right)=i^{\left(l+\dim H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)\right)/2}\prod_{i=1}^{\ell}\Delta\left(\mathbf{h}_1^{\Sigma_{g_i}},\mathbf{h}_1^{\Sigma_{g_i}}\right)^{-1/4}$,

2. $\mathbb{T}\left(\mathrm{H},\left\{\mathbf{h}_k^{\mathrm{H}}\right\}_0^3\right)=i^{\left(l+\dim H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)\right)/2}\prod_{i=1}^{\ell}\sqrt[4]{\delta\left(\mathbf{h}^{1,i},\mathbf{h}^{1,i}\right)}$

*are valid, where $\beta_0=H_0\left(\mathrm{M};\mathcal{G}_{Ad_\varrho}\right)$.*

## A. Programming in Matlab for the Lie group $\mathrm{GL}(3,\mathbb{C})$

```
syms a b c d e f

A1 = [a b c; 0 d e; 0 0 f];

X1 = zeros(3); X1(1,3) = 1;     X2 = zeros(3); X2(2,3) = 1;
X3 = zeros(3); X3(3,3) = 1;     X4 = zeros(3); X4(1,2) = 1;
X5 = zeros(3); X5(2,2) = 1;     X6 = zeros(3); X6(1,1) = 1;
X7 = zeros(3); X7(3,2) = 1;     X8 = zeros(3); X8(2,1) = 1;
X9 = zeros(3); X9(3,1) = 1;
```

```
B1 = inv(A1);

T1=B1*X1*A1;    T2=B1*X2*A1;    T3=B1*X3*A1;    T4=B1*X4*A1;
T5=B1*X5*A1;    T6=B1*X6*A1;    T7=B1*X7*A1;    T8=B1*X8*A1;
T9=B1*X9*A1;

K1=[T1(1,3);T1(2,3);T1(3,3);T1(1,2);T1(2,2);T1(1,1);T1(3,2);T1(2,1);T1(3,1)];
K2=[T2(1,3);T2(2,3);T2(3,3);T2(1,2);T2(2,2);T2(1,1);T2(3,2);T2(2,1);T2(3,1)];
K3=[T3(1,3);T3(2,3);T3(3,3);T3(1,2);T3(2,2);T3(1,1);T3(3,2);T3(2,1);T3(3,1)];
K4=[T4(1,3);T4(2,3);T4(3,3);T4(1,2);T4(2,2);T4(1,1);T4(3,2);T4(2,1);T4(3,1)];
K5=[T5(1,3);T5(2,3);T5(3,3);T5(1,2);T5(2,2);T5(1,1);T5(3,2);T5(2,1);T5(3,1)];
K6=[T6(1,3);T6(2,3);T6(3,3);T6(1,2);T6(2,2);T6(1,1);T6(3,2);T6(2,1);T6(3,1)];
K7=[T7(1,3);T7(2,3);T7(3,3);T7(1,2);T7(2,2);T7(1,1);T7(3,2);T7(2,1);T7(3,1)];
K8=[T8(1,3);T8(2,3);T8(3,3);T8(1,2);T8(2,2);T8(1,1);T8(3,2);T8(2,1);T8(3,1)];
K9=[T9(1,3);T9(2,3);T9(3,3);T9(1,2);T9(2,2);T9(1,1);T9(3,2);T9(2,1);T9(3,1)];

K=[K1 K2 K3 K4 K5 K6 K7 K8 K9]

det(K)
```

```
K =

[ f/a, -(b*f)/(a*d), (b*e - c*d)/(a*d), e/a, -(b*e)/(a*d), c/a, (e*(b*e - c*d))/(a*d*f), -(b*c)/(a*d), (c*(b*e - c*d))/(a*d*f)]
[   0,          f/d,             -e/d,   0,          e/d,   0,            -e^2/(d*f),          c/d,             -(c*e)/(d*f)]
[   0,            0,                1,   0,            0,   0,                  e/f,            0,                      c/f]
[   0,            0,                0, d/a,         -b/a, b/a,     (b*e - c*d)/(a*f), -b^2/(a*d), (b*(b*e - c*d))/(a*d*f)]
[   0,            0,                0,   0,            1,   0,                 -e/f,          b/d,             -(b*e)/(d*f)]
[   0,            0,                0,   0,            0,   1,                    0,         -b/d,      (b*e - c*d)/(d*f)]
[   0,            0,                0,   0,            0,   0,                  d/f,            0,                      b/f]
[   0,            0,                0,   0,            0,   0,                    0,          a/d,             -(a*e)/(d*f)]
[   0,            0,                0,   0,            0,   0,                    0,            0,                      a/f]


ans =

1
```

**Figure 1**. Command window for the Lie group $GL(3,\mathbb{C})$.

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