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Some relations between almost paracontact metric manifolds and almost parahermitian manifolds

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Abstract: In this study, almost paracontact metric manifolds and almost para-Hermitian manifolds are considered. The relations between almost paracontact metric manifolds and almost para-Hermitian manifolds are investigated and certain results are acquired. Examples of almost para-Hermitian manifolds are presented using the determined relations.

Key words: Almost paracontact metric manifolds, almost parahermitian manifods

1. Introduction

Differential manifolds with special structures have specific forms on such manifolds. In the Riemannian case, for example, Riemannian manifolds with G_2 structure have fundamental 3-form, almost contact metric manifolds and almost hermitian manifolds have fundamental 2-forms [2–5]. In the semi-Riemannian case, almost paracontact metric manifolds, almost para-Hermitian manifolds and almost complex manifolds with B-metric are examples of manifolds with special structures [1, 6, 8, 10]. The classification of these manifolds is achieved by using the covariant derivative of their special forms.

In [9], almost contact metric manifolds are classified by using the classification of almost hermitian manifolds and correspondence between classes of almost contact and almost hermitian manifolds is given. In this study, we investigate the associations between almost paracontact and almost para-Hermitian manifolds similar to [9]. We employ the classification of almost paracontact manifolds of [10] and the classification of almost para-Hermitian manifolds of [6]. We present the relations between some classes of almost paracontact and para-Hermitian manifolds.

In this manuscript, after giving necessary preliminary information on almost paracontact and almost para-Hermitian manifolds, we obtain an almost para-Hermitian structure on the product of an almost paracontact manifold with $\mathbb{R}$. We write the covariant derivative of the metric and the fundamental 2-form of the almost para-Hermitian manifold in terms of the covariant derivative of the metric of the almost paracontact manifold. Then we state the relations between certain classes of almost paracontact and almost para-Hermitian manifolds.

2. Almost parcontact metric manifolds and almost para-Hermitian manifolds

A $(2n+1)$-dimensional smooth manifold M is called an almost paracontact metric manifod if the structure group of its tangent bundle reduces to $\mathbb{U}^{\pi}(n)\times\{1\}$, where $U^{\pi}(n)$ is paraunitary group [10]. Equivalently, if a

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smooth manifold M has a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g satisfy the following conditions

1. $\varphi^2(X) = X - \eta(X)\xi$, $\eta(\xi) = 1$,
2. There exists a distribution $\mathbb{D} : M \to T_pM$, $\mathbb{D}_p \subset T_pM$ such that $\mathbb{D}_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\}$. This distribution is called paracontact distribution generated by η,
3. $g(\varphi(X), \varphi(Y)) = -g(X,Y) + \eta(X)\eta(Y)$,

then the manifold M is called an almost paracontact metric manifold. Note that signature of g is $(n+1,n)$. The fundamental 2-form on an almost paracontact metric manifold M is defined as

$$\Phi(X,Y) := g(\varphi(X), Y).$$

This 2-form is nondegenerate on the horizontal distribution $\mathbb{D}$ and $\eta \wedge \Phi^n \neq 0$. The covariant derivative of Φ with respect to the Levi-Civita connection ∇ of g is

$$\beta(X,Y,Z) = (\nabla_X)(Y,Z) = g\left((\nabla_X \varphi)(Y), Z\right).$$

The tensor β has the following properties:

$$\beta(X,Y,Z) = -\beta(X,Z,Y),$$

$$\beta\left(X, \varphi(Y), \varphi(Z)\right) = \beta(X,Y,Z) + \eta(Y)\beta(X,Z,\xi) - \eta(Z)\beta(X,Y,\xi).$$

The following 1-forms associated with β are defined as [10]:

$$\theta(X) := g^{ij}\beta(e_i, e_j, X) + \beta(\xi,\xi,X), \qquad \theta^*(X) := g^{ij}\beta(e_i, \varphi(e_j), X),$$

where $\{e_1, e_2, \cdots, e_{2n}, \xi\}$ is an orthonormal basis of TM and (g^{ij}) is the inverse matrix of (g_{ij}).

Utilizing the properties above, the space $\mathbb{G}$ of $\otimes_3^0 TM$ is defined as follows:

$$\begin{aligned} \mathbb{G} := \big\{\beta \in \otimes_3^0 M : \beta(X,Y,Z) &= -\beta(X,Z,Y) \\ &= \beta(X,\varphi(Y),\varphi(Z)) - \eta(Y)\beta(X,Z,\xi) \\ &\quad +\eta(Z)\beta(X,Y,\xi)\} \end{aligned}$$

The space $\mathbb{G}$, therefore, decomposes into eleven subspaces $\mathbb{G} = \mathbb{G}_1 \oplus \cdots \mathbb{G}_{12}$ which are characterized by:

$$\begin{aligned} \mathbb{G}_1 := \{\beta \in \mathbb{G} : \beta(X,Y,Z) &= \tfrac{1}{2(n-1)}\{g(X,\varphi(Y))\theta_\beta(\varphi(Z)) \\ &\quad -g(X,\varphi(Z))\theta_\beta(\varphi(Y)) \\ &\quad -g(\varphi(X),\varphi(Y))\theta_\beta(\varphi^2(Z)) \\ &\quad +g(\varphi(X),\varphi(Z))\theta_\beta(\varphi^2(Y))\big\}\big\} \end{aligned}$$

$$\mathbb{G}_2 := \{\beta \in \mathbb{G} : \beta(\varphi(X),\varphi(Y),Z) = -\beta(X,Y,Z), \quad \theta_\beta = 0\}$$

$$\mathbb{G}_3 := \{\beta \in \mathbb{G} : \beta(\xi,Y,Z) = \beta(X,\xi,Z) = 0, \quad \beta(X,Y,Z) = -\beta(Y,X,Z)\}$$

$$\mathbb{G}_4 := \{\beta \in \mathbb{G} : \beta(\xi,Y,Z) = \beta(X,\xi,Z) = 0, \quad \mathfrak{S}_{XYZ}\beta(X,Y,Z) = 0\}$$

$$\mathbb{G}_5 := \left\{\beta \in \mathbb{G} : \beta(X,Y,Z) = \frac{\theta_\beta(\xi)}{2n}\left\{\eta(Y)g(\varphi(X),\varphi(Z)) - \eta(Z)g(\varphi(X),\varphi(Y))\right\}\right\}$$

$$\mathbb{G}_6 := \left\{\beta \in \mathbb{G} : \beta(X,Y,Z) = -\frac{\theta^*_\beta(\xi)}{2n}\left\{\eta(Y)g(X,\varphi(Z)) - \eta(Z)g(X,\varphi(Y))\right\}\right\}$$

$$\begin{aligned}\mathbb{G}_7 \quad := \quad & \{\beta \in \mathbb{G} : \beta(X,Y,Z) = -\eta(Y)\beta(X,Z,\xi) + \eta(Z)\beta(X,Y,\xi) \\ & \beta(X,Y,\xi) = -\beta(Y,X,\xi) = -\beta(\varphi(X),\varphi(Y),\xi), \quad \theta^*_\beta(\xi) = 0\Big\}\end{aligned}$$

$$\begin{aligned}\mathbb{G}_8 \quad := \quad & \{\beta \in \mathbb{G} : \beta(X,Y,Z) = -\eta(Y)\beta(X,Z,\xi) + \eta(Z)\beta(X,Y,\xi) \\ & \beta(X,Y,\xi) = \beta(Y,X,\xi) = -\beta(\varphi(X),\varphi(Y),\xi), \quad \theta_\beta(\xi) = 0\}\end{aligned}$$

$$\begin{aligned}\mathbb{G}_9 \quad := \quad & \{\beta \in \mathbb{G} : \beta(X,Y,Z) = -\eta(Y)\beta(X,Z,\xi) + \eta(Z)\beta(X,Y,\xi) \\ & \beta(X,Y,\xi) = -\beta(Y,X,\xi) = \beta(\varphi(X),\varphi(Y),\xi)\}\end{aligned}$$

$$\begin{aligned}\mathbb{G}_{10} \quad := \quad & \{\beta \in \mathbb{G} : \beta(X,Y,Z) = -\eta(Y)\beta(X,Z,\xi) + \eta(Z)\beta(X,Y,\xi) \\ & \beta(X,Y,\xi) = \beta(Y,X,\xi) = \beta(\varphi(X),\varphi(Y),\xi)\}\end{aligned}$$

$$\mathbb{G}_{11} := \{\beta \in \mathbb{G} : \beta(X,Y,Z) = \eta(X)\beta(\xi,\varphi(Y),\varphi(Z))\}$$

$$\mathbb{G}_{12} := \{\beta \in \mathbb{G} : \beta(X,Y,Z) = \eta(X)\{\eta(Y)\beta(\xi,\xi,Z) - \eta(Z)\beta(\xi,\xi,Y)\}\}.$$

Such a decomposition allows any $\beta \in \mathbb{G}$ to be written as $\beta(X,Y,Z) = \beta_1(X,Y,Z) + \cdots + \beta_{12}(X,Y,Z)$, where projections $\beta_i(X,Y,Z)$, $i = 1,\cdots,12$ are as follows [10]:

$$\begin{aligned}\beta_1(X,Y,Z) \quad = \quad & \tfrac{1}{2(n-1)}\{g(X,\varphi(Y))\theta_{\beta_1}(\varphi(Z)) - g(X,\varphi(Z))\theta_{\beta_1}(\varphi(Y)) \\ & -g(\varphi(X),\varphi(Y))\theta_{\beta_1}(\varphi^2(Z)) + g(\varphi(X),\varphi(Z))\theta_{\beta_1}(\varphi^2(Y))\}\end{aligned}$$

$$\beta_2(X,Y,Z) = \frac{1}{2}\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\varphi^2(Z)\right) - \beta\left(\varphi(X),\varphi^2(Y),\varphi(Z)\right)\right\} - \beta_1(X,Y,Z)$$

$$\begin{aligned}\beta_3(X,Y,Z) \quad = \quad & \tfrac{1}{6}\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\varphi^2(Z)\right) + \beta\left(\varphi(X),\varphi^2(Y),\varphi(Z)\right)\right. \\ & +\beta\left(\varphi^2(Y),\varphi^2(Z),\varphi^2(X)\right) + \beta\left(\varphi(Y),\varphi^2(Z),\varphi(X)\right) \\ & \left.+\beta\left(\varphi^2(Z),\varphi^2(X),\varphi^2(Y)\right) + \beta\left(\varphi(Z),\varphi^2(X),\varphi(Y)\right)\right\}\end{aligned}$$

$$\beta_4(X,Y,Z) = \frac{1}{2}\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\varphi^2(Z)\right) + \beta\left(\varphi(X),\varphi^2(Y),\varphi(Z)\right)\right\} - \beta_3(X,Y,Z)$$

$$\beta_5(X,Y,Z) = \frac{\theta_{\beta_5}(\xi)}{2n}\{\eta(Y)g(\varphi(X),\varphi(Z)) - \eta(Z)g(\varphi(X),\varphi(Y))\}$$

$$\beta_6(X,Y,Z) = -\frac{\theta^*_{\beta_6}(\xi)}{2n}\{\eta(Y)g(X,\varphi(Z)) - \eta(Z)g(X,\varphi(Y))\}$$

$$\begin{aligned}\beta_7(X,Y,Z) \quad = \quad & -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right) - \beta\left(\varphi(X),\varphi(Z),\xi\right)\right. \\ & \left.-\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right) + \beta\left(\varphi(Z),\varphi(X),\xi\right)\right\} \\ & +\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right) - \beta\left(\varphi(X),\varphi(Y),\xi\right)\right. \\ & \left.-\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right) + \beta\left(\varphi(Y),\varphi(X),\xi\right)\right\} - \beta_6(X,Y,Z)\end{aligned}$$

$$\begin{aligned}\beta_8(X,Y,Z) \quad = \quad & -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right) - \beta\left(\varphi(X),\varphi(Z),\xi\right)\right. \\ & \left.+\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right) - \beta\left(\varphi(Z),\varphi(X),\xi\right)\right\} \\ & +\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right) - \beta\left(\varphi(X),\varphi(Y),\xi\right)\right. \\ & \left.+\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right) - \beta\left(\varphi(Y),\varphi(X),\xi\right)\right\} - \beta_5(X,Y,Z)\end{aligned}$$

$$\begin{aligned}\beta_9(X,Y,Z) &= -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right)+\beta\left(\varphi(X),\varphi(Z),\xi\right)\right.\\ &\quad\left.-\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right)-\beta\left(\varphi(Z),\varphi(X),\xi\right)\right\}\\ &\quad+\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right)+\beta\left(\varphi(X),\varphi(Y),\xi\right)\right.\\ &\quad\left.-\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right)-\beta\left(\varphi(Y),\varphi(X),\xi\right)\right\}\end{aligned}$$

$$\begin{aligned}\beta_{10}(X,Y,Z) &= -\tfrac{1}{4}\eta(Y)\left\{\beta\left(\varphi^2(X),\varphi^2(Z),\xi\right)+\beta\left(\varphi(X),\varphi(Z),\xi\right)\right.\\ &\quad\left.+\beta\left(\varphi^2(Z),\varphi^2(X),\xi\right)+\beta\left(\varphi(Z),\varphi(X),\xi\right)\right\}\\ &\quad+\tfrac{1}{4}\eta(Z)\left\{\beta\left(\varphi^2(X),\varphi^2(Y),\xi\right)+\beta\left(\varphi(X),\varphi(Y),\xi\right)\right.\\ &\quad\left.+\beta\left(\varphi^2(Y),\varphi^2(X),\xi\right)+\beta\left(\varphi(Y),\varphi(X),\xi\right)\right\}\end{aligned}$$

$$\beta_{11}(X,Y,Z)=\eta(X)\beta(\xi,\varphi^2(Y),\varphi^2(Z))$$

$$\beta_{12}(X,Y,Z)=\eta(X)\left\{\eta(Y)\beta(\xi,\xi,\varphi^2(Z)-\eta(Z)\beta(\xi,\xi,\varphi^2(Y)\right\}$$

If a smooth manifold N has a tensor field J (almost product structure) and a pseudo-Riemannian metric h satisfies the conditions

- $J^2(X)=X$,
- $h(J(X),J(Y))=-h(X,Y)$

for all vector fields X,Y on N, then the manifold N is called an almost para-Hermitian manifold [6]. An almost para-Hermitian manifold has even dimension ($\dim N=2n$) and the structure group of the tangent bundle reduces to the group

$$G=\left\{\begin{pmatrix} A & 0\\ 0 & (A^{-1})^t\end{pmatrix} \quad : \quad A\in GL(n,\mathbb{R})\right\}.$$

The fundamental 2-form F on N is defined by

$$F(X,Y)=h(J(X),Y)$$

for all vector fields X,Y on N. The covariant derivative of F with respect to the Levi-civita connection of h is

$$\alpha(X,Y,Z)=\left(\nabla_X F\right)(Y,Z)=h\left(\left(\nabla_X J\right)(Y),Z\right).$$

The tensor α has the following properties:

$$\alpha(X,Y,Z)=-\alpha(X,Z,Y),$$

$$\alpha\left(X,J(Y),J(Z)\right)=\alpha(X,Y,Z).$$

The tangent space T_pN at each point $p\in N$ splits as $T_pN=\mathcal{V}\oplus\mathcal{H}$ [6], where $\mathcal{V}$ and $\mathcal{H}$ are the eigenspaces of eigenvalues $+1$ and -1 of J, respectively. One can choose the basis $\{A_1,\cdots,A_n,U_1,\cdots,U_n\}$ of T_pN, where $\{A_1,\cdots,A_n\}$ and $\{U_1,\cdots,U_n\}$ are bases of $\mathcal{V}$ and $\mathcal{H}$, respectively, in which the expressions of h and J are

$$h=\begin{pmatrix} 0 & I_n\\ I_n & 0\end{pmatrix},\qquad J=\begin{pmatrix} I_n & 0\\ 0 & -I_n\end{pmatrix}.$$

The base $\{A_1,\cdots,A_n,U_1,\cdots,U_n\}$ is called a local adapted base. The subspace W of $\otimes_3^0 T_pN$ is defined as follows:

$$W:=\left\{\alpha\in\otimes^3T^* \quad | \quad \alpha(X,Y,Z)=\alpha(X,J(Y),J(Z))=-\alpha(X,Z,Y),\right\}$$

According to the symmetries of the W, this space splits into the direct sum $W = W_1 \oplus \cdots \oplus W_8$. The subspaces W_i are invariant and irreducible under $GL(n, \mathbb{R})$. The following 8 relations are used to define subspaces. An almost para-Hermitian manifold satisfying the following 8 relations, is called parakaehlerian. For $i = 1, \cdots 8$, the class W_i is characterized by all these relations except (i); the class $W_i + W_j$ is the class satisfying all relations except i and j, and so on [6]. Characteristic conditions of the different classes of almost para-Hermitian manifolds:

(i) $\mathfrak{S}_{A,B,C} (\nabla_A F)(B, C) = 0$ for all $A, B, C \in \mathcal{V}$.

(ii) $\nabla_A A \in \mathcal{V}$ for all $A \in \mathcal{V}$.

(iii) $(\nabla_A F)(U, V) = \theta(V)h(A, U) - \theta(U)h(A, V)$ for all $A \in \mathcal{V}$, $U, V \in \mathcal{H}$.

(iv) $\sum_{i=1}^{n} (\nabla_{A_i} F)(U_i, U) = 0$, where $\{A_1, \cdots, A_n, U_1, \cdots, U_n\}$ is a local adapted frame and $U \in \mathcal{H}$.

(v) $\mathfrak{S}_{U,V,W} (\nabla_U F)(V, W) = 0$ for all $U, V, W \in \mathcal{H}$.

(vi) $\nabla_U U \in \mathcal{H}$ for all $U \in \mathcal{H}$.

(vii) $(\nabla_U F)(A, B) = \theta(A)h(U, B) - \theta(B)h(U, A)$, for all $A, B \in \mathcal{V}$, $U \in \mathcal{H}$.

(viii) $\sum_{i=1}^{n} (\nabla_{U_i} F)(A_i, A) = 0$ where $\{A_1, \cdots, A_n, U_1, \cdots, U_n\}$ is a local adapted frame and $A \in \mathcal{V}$.

Here, the Lee form θ on $2n-$dimensional almost para-Hermitian manifold (M, k, J) is defined as

$$\Theta(X) = -\frac{1}{n-1}\delta F(J(X))$$

for any vector field X [6].

3. Almost para-Hermitian manifolds from almost paracontact manifold

In this section, first, we define an almost para-Hermitian structure on the product of an almost paracontact manifold with $\mathbb{R}$. Then, we give the relations between covariant derivatives.

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost paracontact metric manifold and consider the product manifold $M \times \mathbb{R}$. A vector field on the manifold $M \times \mathbb{R}$ is the form $\left(X, a\frac{d}{dt}\right)$ where t is the coordinate of $\mathbb{R}$ and a is a smooth function on $M \times \mathbb{R}$. The almost para-complex structure (or almost product structure) J on $M \times \mathbb{R}$ is defined by

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi(X) + a\xi, \eta(X)\frac{d}{dt}\right) \tag{3.1}$$

and we define a pseudo-Riemannian metric on $M \times \mathbb{R}$ with signature $(n+1, n+1)$ by

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) := g(X, Y) - ab. \tag{3.2}$$

One can easily see that

$$h\left(J\left(X,a\frac{d}{dt}\right),J\left(Y,b\frac{d}{dt}\right)\right)=-h\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right). \tag{3.3}$$

The para-Kaehler form F of the almost para-Hermitian manifold $(M\times\mathbb{R},J,h)$ is given by

$$F\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right):=h\left(J\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right). \tag{3.4}$$

Hence, one can express the form F in terms of Φ, η as

$$F\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right)=\Phi(X,Y)+a\eta(Y)-b\eta(X). \tag{3.5}$$

Let ∇ be the pseudo-Riemannian connection of (M,φ,ξ,η,g). Levi-Civita covariant derivative of the metric h on $M\times\mathbb{R}$ is obtained using the Kozsul formula as

$$\nabla_{\left(X,a\frac{d}{dt}\right)}\left(Y,b\frac{d}{dt}\right)=\left(\nabla_XY,\left(X[b]+a\frac{db}{dt}\right)\frac{d}{dt}\right). \tag{3.6}$$

Note that the covariant derivative on the product manifold $M\times\mathbb{R}$ will also be denoted with the same symbol ∇. Also, covariant derivative of the 2-form F is calculated as

$$\left(\nabla_{\left(X,a\frac{d}{dt}\right)}F\right)\left(\left(Y,b\frac{d}{dt}\right),\left(Z,c\frac{d}{dt}\right)\right)=\beta(X,Y,Z)-c\left(\nabla_X\eta\right)(Y)+b\left(\nabla_X\eta\right)(Z),$$

for any vector fields $\left(X,a\frac{d}{dt}\right)$, $\left(Y,b\frac{d}{dt}\right)$ and $\left(Z,c\frac{d}{dt}\right)$ on $M\times\mathbb{R}$. Note that the right hand side of the expression

$$\left(\nabla_{\left(X,a\frac{d}{dt}\right)}J\right)\left(Y,b\frac{d}{dt}\right)=\left(\left(\nabla_X\varphi\right)(Y)+b\nabla_X\xi,\left(\nabla_X\eta\right)(Y)\frac{d}{dt}\right)$$

does not depend on the function a similar to the almost contact case studied in [9].

Differential of the fundamental 2-form F can be evaluated as

$$\begin{aligned} dF\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right),\left(Z,c\frac{d}{dt}\right)\right) &= d\Phi(X,Y,Z)-\frac{2a}{3}d\eta(Y,Z) \\ &+\frac{2b}{3}d\eta(X,Z)-\frac{2c}{3}d\eta(X,Y). \end{aligned}$$

Let $\{e_1,\cdots,e_n,\varphi(e_1),\cdots,\varphi(e_n),\xi\}$ be a local pseudo-orthonormal $\varphi-$frame field on M. Then, one can obtain an orthonormal frame field on $M\times\mathbb{R}$ as follows:

$$\left\{(e_1,0),\cdots,(e_n,0),(\varphi(e_1),0),\cdots,(\varphi(e_n),0),(\xi,0),\left(0,\frac{d}{dt}\right)\right\}$$

Using this frame, the coderivative of F is calculated as

$$\delta F(X,a\frac{d}{dt})=-\theta(X)-a\theta^*(\xi). \tag{3.7}$$

4. Relations between almost paracontact metric manifolds and almost para-Hermitian manifolds

In this section, we examine the relations between the classes of the almost paracontact metric manifolds and almost para-Hermitian manifolds. First, it is immediate that $(M\times\mathbb{R},J,h)$ is parakahlerian if and only if almost parcontact metric manifold (M,φ,ξ,η,g) is paracosymplectic. Now, we consider the product manifold $M\times\mathbb{R}$ of type W_1.

Theorem 4.1 *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional almost paracontact metric manifold. If the product manifold $M\times\mathbb{R}$ is of class W_1, then the manifold M belongs to the class $\mathbb{G}_3$.*

Proof Let $M\times\mathbb{R}$ be of class W_1. The following equalities are satisfied:

$$h\left(\left(\nabla_{(X,a\frac{d}{dt})}J\right)(X,a\frac{d}{dt}),(Y,b\frac{d}{dt})\right)=0, \tag{4.1}$$

for all vector fields $\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)$ and

$$J\left(\nabla_U A\right)=\nabla_U A,\quad J\left(\nabla_A U\right)=-\nabla_A U,\quad J\left(\nabla_U V\right)=-\nabla_U V \tag{4.2}$$

for all $U,V\in\mathcal{H}$ and $A\in\mathcal{V}$. In equation (4.1), setting the functions $a=b=0$, we obtain

$$\left(\nabla_X\varphi\right)(X)=0, \tag{4.3}$$

$$\left(\nabla_X\eta\right)(X)=0. \tag{4.4}$$

The equality (4.3) implies that $\beta(X,X,Z)$ for all vector fields X,Z on M. Replacing X with $X+Y$ in the equality (4.3) we obtain

$$\beta(X,Y,Z)=-\beta(Y,X,Z). \tag{4.5}$$

From equations (4.4), for all vector fields X,Y on M, we get

$$g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)=0, \tag{4.6}$$

hence ξ is Killing and $\nabla_\xi\xi=0$. For any vector field X on M, we have

$$\left(\varphi(X)+X,\eta(X)\frac{d}{dt}\right)\in\mathcal{V},\quad\left(\varphi(X)-X,\eta(X)\frac{d}{dt}\right)\in\mathcal{H}.$$

It follows from the condition $\nabla_U A\in\mathcal{V}$, that

$$\nabla_{\varphi(X)}\xi-\nabla_X\xi=\varphi\left(\nabla_{\varphi(X)}\xi\right)-\varphi\left(\nabla_X\xi\right) \tag{4.7}$$

and

$$\begin{aligned}\beta\left(\varphi(X),\varphi(Y),Z\right) &+ \beta\left(\varphi(X),Y,Z\right)\\ &- \beta(X,Y,Z)-\beta\left(X,\varphi(Y),Z\right)\\ &= \eta(Y)g\left(\nabla_X\xi,Z\right)-\eta(Y)g\left(\nabla_{\varphi(X)}\xi,Z\right).\end{aligned} \tag{4.8}$$

Similarly, from equation $J\left(\nabla_A U\right)=-\nabla_A U$, we obtain

$$\nabla_{\varphi(X)}\xi+\nabla_X\xi=-\varphi\left(\nabla_{\varphi(X)}\xi\right)-\varphi\left(\nabla_X\xi\right), \tag{4.9}$$

$$\begin{aligned}-\beta\left(\varphi(X),\varphi(Y),Z\right) &+ \beta\left(\varphi(X),Y,Z\right)\\ &+ \beta(X,Y,Z)-\beta\left(X,\varphi(Y),Z\right)\\ &= \eta(Y)g\left(\nabla_X\xi,Z\right)+\eta(Y)g\left(\nabla_{\varphi(X)}\xi,Z\right).\end{aligned} \tag{4.10}$$

Substituting X in (4.8) and (4.10) with ξ we obtain

$$\beta(\xi,Y,Z)=0 \tag{4.11}$$

and

$$\beta(Y,\xi,Z)=0 \tag{4.12}$$

is obtained from the equation (4.5). From the equations (4.5), (4.11), and (4.12), it follows that the almost paracontact metric manifold M is of the class $\mathbb{G}_3$. □

Similarly, it can be shown that if the product manifold $M\times\mathbb{R}$ is of class W_5, then the manifold M also belongs to the class $\mathbb{G}_3$. Then we can say that if the $M\times\mathbb{R}$ is of class $W_1\oplus W_5$, then the manifold M also belongs to the class $\mathbb{G}_3$.

The reverse of this statement is also true:

Theorem 4.2 *If the almost paracontact metric manifold (M,φ,ξ,η,g) is of class $\mathbb{G}_3$, the manifold $M\times\mathbb{R}$ belongs to the class $W_1\oplus W_5$.*

Proof Let (M,φ,η,ξ,g) be an almost paracontact metric manifold of class $\mathbb{G}_3$. Defining relations of the class $\mathbb{G}_3$ are

$$\beta(X,Y,Z)=\beta(\varphi(X),\varphi(Y),Z) \tag{4.13}$$

and

$$\beta(X,Y,Z)=\frac{1}{3}\left\{\beta(X,Y,Z)+\beta(Y,Z,X)+\beta(Z,X,Y)\right\}. \tag{4.14}$$

From equation (4.13), we obtain

$$\beta(\xi,Y,Z)=\beta(X,\xi,Z)=0 \tag{4.15}$$

for all vector fiels X,Y,Z on M. From equation (4.14) we obtain

$$\beta(X,Y,Z)=-\beta(Y,X,Z). \tag{4.16}$$

Since $\beta(X,\xi,\varphi(Z))=0$, ξ is parallel. Then for all $(X,a\frac{d}{dt})$ vector field on $M\times\mathbb{R}$ one can obtain

$$\left(\nabla_{(X,a\frac{d}{dt})}J\right)(X,a\frac{d}{dt}) = \left(\left(\nabla_X\varphi\right)(X)+a\nabla_X\xi,\left(\nabla_X\eta\right)(X)\frac{d}{dt}\right) \tag{4.17}$$

$$= (0,0). \tag{4.18}$$

Let $U \in \mathcal{H}$, $A \in \mathcal{V}$. Then we can take $U = \left(X, -\eta(X)\frac{d}{dt}\right)$ and $A = \left(Y, \eta(Y)\frac{d}{dt}\right)$ such that $\varphi(X) = -\varphi^2(X)$ and $\varphi(Y) = \varphi^2(Y)$. Since $\varphi(X) = -\varphi^2(X)$ and $\varphi(Y) = \varphi^2(Y)$, we have

$$\beta(X, Y, Z) = \beta(\varphi(X), \varphi(Y), Z) = -\beta(\varphi^2(X), \varphi^2(Y), Z) = -\beta(X, Y, Z).$$

Hence $(\nabla_X \varphi)(Y) = 0$. On employing the expression $\varphi(Y) = Y - \eta(Y)\xi$ when used in the equation $(\nabla_X \varphi)(Y) = 0$, one obtains

$$\nabla_X Y - \eta(\nabla_X Y)\xi = \varphi(\nabla_X Y).$$

Hence, we have

$$\begin{aligned} J(\nabla_U A) &= J\left(\nabla_X Y,\ \eta(\nabla_X Y)\frac{d}{dt}\right) \\ &= \left(\varphi(\nabla_X Y) + \eta(\nabla_X Y)\xi,\ \eta(\nabla_X Y)\frac{d}{dt}\right) \\ &= \left(\nabla_X Y, \eta(\nabla_X Y)\frac{d}{dt}\right) \\ &= \nabla_U A. \end{aligned}$$

Similarly, it is proved that $J(\nabla_A U) = -\nabla_A U$. We know that defining relations of the class $W_1 \oplus W_5$ [6] are

$$\left(\nabla_{\left(X, a\frac{d}{dt}\right)} F\right)\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = 0,$$

$$\nabla_U A \in \mathcal{V},\ \nabla_A U \in \mathcal{H},\ \text{for all } A \in \mathcal{V},\ U \in \mathcal{U}.$$

Thus, it has been shown that the almost para-Hermitian manifold $M \times \mathbb{R}$ belongs to the class $W_1 \oplus W_5$. □

As a result, the manifold M is of the class $\mathbb{G}_3$ if and only if the product manifold $M \times \mathbb{R}$ belongs to the class $W_1 \oplus W_5$.

At this step, we examine the class W_2. First we show that if the manifold $M \times \mathbb{R}$ is of class W_2, then M belongs to the class $\mathbb{G}_4 \oplus \mathbb{G}_{10}$.

Theorem 4.3 *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost paracontact metric manifold. If the $M \times \mathbb{R}$ is of class W_2, then the manifold M belongs to the class $\mathbb{G}_4 \oplus \mathbb{G}_{10}$.*

Proof Let $M \times \mathbb{R}$ be of class W_2. Then the following equalities are satisfied:

$$dF = 0, \tag{4.19}$$

$$\nabla_U A \in \mathcal{V}, \quad \nabla_A U \in \mathcal{H}, \quad \nabla_U V \in \mathcal{H}. \tag{4.20}$$

Since

$$\begin{aligned} 0 &= dF\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) \\ &= d\Phi(X, Y, Z) - \frac{2c}{3} d\eta(X, Y) - \frac{2a}{3} d\eta(Y, Z) + \frac{2b}{3} d\eta(X, Z) \end{aligned}$$

for all vector fields $\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)$, setting $a = b = c = 0$, one can obtain

$$d\Phi(X, Y, Z) = 0. \tag{4.21}$$

If we take $a = b = 0$ and $c = 1$,

$$d\eta(X, Y) = 0 \tag{4.22}$$

is obtained. In addition, one can obtain

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \quad \nabla_\xi \xi = 0$$

for all vector fields X, Y on M from equation (4.22). Since $\nabla_U A \in \mathcal{V}$, we have (4.7), (4.8). In equation (4.8), if $X = \xi$ is taken,

$$\beta(\xi, Y, Z) = -\beta\left(\xi, \varphi(Y), Z\right) \tag{4.23}$$

is obtained. Since $\nabla_A U \in \mathcal{H}$ is satisfied in the class W_2, we have equations (4.9) and (4.10). Setting $X = \xi$ in (4.10),

$$\beta(\xi, Y, Z) - \beta(\xi, \varphi(Y), Z) = 0 \tag{4.24}$$

is obtained. Hence, from (4.23) and (4.24) we get

$$\beta(\xi, Y, Z) = 0. \tag{4.25}$$

Moreover, from the equation

$$0 = d\Phi(\xi, Y, Z) = \beta(\xi, Y, Z) + \beta(Y, Z, \xi) + \beta(Z, \xi, Y),$$

one obtains the equation

$$\beta(Y, Z, \xi) = \beta(Z, Y, \xi).$$

In (4.8) and (4.10), setting $Y = \varphi(Y)$ we gives

$$\beta\left(\varphi(X), \varphi(Y), Z\right) = \beta\left(X, \varphi^2(Y), Z\right), \tag{4.26}$$

$$\beta\left(\varphi(X), \varphi^2(Y), Z\right) = \beta\left(X, \varphi(Y), Z\right). \tag{4.27}$$

From equations (4.26) and (4.27),

$$\begin{aligned}
\beta\left(\varphi^2(X), \varphi^2(Y), \varphi^2(Z)\right) &= \beta\left(\varphi(X), \varphi(Y), \varphi^2(Z)\right) \\
&= -\beta\left(\varphi(X), \varphi^2(Z), \varphi(Y)\right) \\
&= -\beta\left(\varphi^2(X), \varphi(Z), \varphi(Y)\right) \\
&= \beta\left(\varphi^2(X), \varphi(Y), \varphi(Z)\right) \\
&= \beta\left(\varphi(X), \varphi^2(Y), \varphi(Z)\right)
\end{aligned} \tag{4.28}$$

is obtained. We then have $\beta_2(X, Y, Z) = -\beta_1(X, Y, Z)$ and $\beta_1(X, Y, Z) = \beta_2(X, Y, Z) = 0$. From equation (4.28), we get $\beta_3(X, Y, Z) = 0$. In (4.26), setting $X = \varphi(X)$, $Y = \varphi(Y)$ and $Z = \xi$, $\beta_7(X, Y, Z) = -\beta_6(X, Y, Z)$ is obtained. Hence, we get $\beta_7(X, Y, Z) = \beta_6(X, Y, Z) = 0$. Similarly $\beta_8(X, Y, Z) = -\beta_5(X, Y, Z)$ and $\beta_8(X, Y, Z) = \beta_5(X, Y, Z) = 0$ is obtained. For all vector fields X, Y, since

$$\beta(X, Y, \xi) = \beta(Y, X, \xi),$$

we get $\beta(\varphi^2(X), \varphi^2(Y), \xi) = \beta(\varphi^2(Y), \varphi^2(X), \xi)$, from which follows $\beta_9(X, Y, Z) = 0$. For all vector fields X, Y, having $\beta(\xi, Y, Z) = 0$, one obtains $\beta_{11}(X, Y, Z) = \beta_{12}(X, Y, Z) = 0$. We thus arrive at

$$\beta(X, Y, Z) = \beta_4(X, Y, Z) + \beta_{10}(X, Y, Z)$$

Hence, the almost paracontact metric manifold M is in the class of $\mathbb{G}_4 \oplus \mathbb{G}_{10}$. □

It may similarly be shown that if the product manifold $M \times \mathbb{R}$ is of class W_6, the manifold M also belongs to the class $\mathbb{G}_4 \oplus \mathbb{G}_{10}$. We can then say that if the $M \times \mathbb{R}$ is of class $W_2 \oplus W_6$, then the manifold M also belongs to the class $\mathbb{G}_4 \oplus \mathbb{G}_{10}$.

At this step, we investigate the reverse of this theorem. First, we consider a manifold of the class $\mathbb{G}_4$.

Theorem 4.4 *If the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is of the class $\mathbb{G}_4$, the manifold $M \times \mathbb{R}$ belongs to the class $W_2 \oplus W_6$.*

Proof Let $(M, \varphi, \eta, \xi, g)$ be an almost paracontact metric manifold of class $\mathbb{G}_4$. We know that the defining relations of the class $\mathbb{G}_4$ are

$$\beta(X,Y,Z) + \beta(Y,Z,X) + \beta(Z,X,Y) = 0 \tag{4.29}$$

and

$$\beta(\xi, Y, Z) = \beta(X, \xi, Z) = 0. \tag{4.30}$$

From equation (4.30), we obtain ξ is parallel, which together with equation (4.29) being satisfying, implies that

$$dF = 0.$$

Let $U \in \mathcal{H}$ and $A \in \mathcal{V}$. Then, we take $U = \left(X, -\eta(X)\frac{d}{dt}\right)$ and $A = \left(Y, \eta(Y)\frac{d}{dt}\right)$ such that $\varphi^2(X) = -\varphi(X)$, $\varphi^2(Y) = \varphi(Y)$. In the class $\mathbb{G}_4$, since $\beta(\varphi(X), \varphi(Y), Z) = \beta(X, Y, Z)$ is satisfied, it follows that

$$\beta(X,Y,Z) = \beta(\varphi(X), \varphi(Y), Z)) = -\beta(\varphi^2(X), \varphi^2(Y), Z)) = -\beta(X,Y,Z).$$

Hence, $(\nabla_X \varphi)(Y) = 0$ when $\varphi^2(X) = -\varphi(X)$ and $\varphi^2(Y) = \varphi(Y)$. Since

$$\begin{aligned} 0 = (\nabla_X \varphi)(Y) &= \nabla_X \varphi(Y) - \varphi(\nabla_X Y) \\ &= \nabla_X \varphi^2(Y) - \varphi(\nabla_X Y) \\ &= \nabla_X Y - X[\eta(Y)]\xi - \varphi(\nabla_X Y) \\ &= \nabla_X Y - \eta(\nabla_X Y)\xi - \varphi(\nabla_X Y), \end{aligned}$$

one can obtain

$$\begin{aligned} J(\nabla_U A) &= J\left(\nabla_X Y, \eta(\nabla_X Y)\tfrac{d}{dt}\right) \\ &= \left(\varphi(\nabla_X Y) + \eta(\nabla_X Y)\xi, \eta(\nabla_X Y)\tfrac{d}{dt}\right) \\ &= \left(\nabla_X Y, \eta(\nabla_X Y)\tfrac{d}{dt}\right) = \nabla_U A. \end{aligned}$$

Similarly, it can be shown that $\nabla_A U \in \mathcal{H}$ where $A \in \mathcal{V}$, $U \in \mathcal{H}$. Hence, we get the manifold $M \times \mathbb{R}$ in the class $W_2 \oplus W_6$. □

Using similar methods, it is proven that the product manifold $M \times \mathbb{R}$ is of the class $W_2 \oplus W_6$ if the almost paracontact metric manifold M is of the class $\mathbb{G}_{10}$. Hence we get the following theorem:

Theorem 4.5 *The manifold $M \times \mathbb{R}$ is of class $W_2 \oplus W_6$ if and only if M belongs to class $\mathbb{G}_4 \oplus \mathbb{G}_{10}$.*

Now the para-Hermitian semi-parakahlerian classes W_3 and W_7 will be analyzed.

First we prove that if the manifold $M \times \mathbb{R}$ is of class W_3, then M belongs to the class $\mathbb{G}_2 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8$. In addition, it can similarly be proved that if the manifold $M \times \mathbb{R}$ is of class W_7, then M also belongs to the class $\mathbb{G}_2 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8$.

Theorem 4.6 *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional almost paracontact metric manifold. If the $M\times\mathbb{R}$ is of class W_3, then the manifold M belongs to the class $\mathbb{G}_2\oplus\mathbb{G}_7\oplus\mathbb{G}_8$.*

Proof Since the product manifold $M\times\mathbb{R}$ is of the class W_3, for all vector fields $\left(X, a\frac{d}{dt}\right)$,

$$0=\delta F\left(X, a\frac{d}{dt}\right)=-\theta_\beta(X)-a\theta^*_\beta(\xi).$$

Setting $a=0$, one obtains

$$\theta_\beta(X)=0$$

and setting $X=0$ and $a=1$, $\theta^*(\xi)=0$ is obtained. In the class W_3, since $\nabla_A B\in\mathcal{V}$, where $A,B\in\mathcal{V}$, we get

$$\nabla_{\varphi(X)}\xi+\nabla_X\xi=\varphi\left(\nabla_{\varphi(X)}\right)+\varphi\left(\nabla_X\xi\right) \tag{4.31}$$

and

$$\begin{aligned}\beta(\varphi(X),\varphi(Y),Z) &+ \beta(X,\varphi(Y),Z)+\beta(\varphi(X),Y,Z)+\beta(X,Y,Z)\\ &= -\eta(Y)g\left(\nabla_{\varphi(X)}\xi,Z\right)-\eta(Y)g\left(\nabla_X\xi,Z\right).\end{aligned} \tag{4.32}$$

Since $\nabla_U V\in\mathcal{H}$, where $U,V\in\mathcal{H}$, we have equations

$$\nabla_{\varphi(X)}\xi-\nabla_X\xi=-\varphi\left(\nabla_{\varphi(X)}\xi\right)+\varphi\left(\nabla_X\xi\right), \tag{4.33}$$

$$\begin{aligned}-\beta\left(\varphi(X),\varphi(Y),Z\right) &+ \beta\left(\varphi(X),Y,Z\right)\\ &- \beta(X,Y,Z)+\beta\left(X,\varphi(Y),Z\right)\\ &= -\eta(Y)g\left(\nabla_X\xi,Z\right)+\eta(Y)g\left(\nabla_{\varphi(X)}\xi,Z\right).\end{aligned} \tag{4.34}$$

From equations (4.32) and (4.34) , one can obtain

$$\beta(\varphi(X),Y,Z)+\beta(X,\varphi(Y),Z)=-\eta(Y)g\left(\nabla_X\xi,Z\right) \tag{4.35}$$

and

$$\beta(\varphi(X),\varphi(Y),Z)+\beta(X,Y,Z)=-\eta(Y)g\left(\nabla_{\varphi(X)}\xi,Z\right). \tag{4.36}$$

In equation (4.36), taking $X=\xi$,

$$\beta(\xi,Y,Z)=0 \tag{4.37}$$

is obtained. In equation (4.36), setting $Z=\xi$ we obtain

$$\beta(\varphi(X),\varphi(Y),\xi)=-\beta(X,Y,\xi). \tag{4.38}$$

Equation (4.37) implies that $\beta_{11}(X,Y,Z)=\beta_{12}(X,Y,Z)=0$. The identity $\beta_9(X,Y,Z)=\beta_{10}(X,Y,Z)=0$ follows from equation (4.38). Setting $X=\varphi(X)$ and $Y=\varphi(Y)$ in equation (4.36) results in

$$\beta(\varphi^2(X),\varphi^2(Y),Z)=-\beta(\varphi(X),\varphi(Y),Z).$$

Consequently, we get

$$\beta\left(\varphi^2(X), \varphi^2(Y), \varphi^2(Z)\right) + \beta\left(\varphi(X), \varphi^2(Y), \varphi(Z)\right) = 0,$$

which yields $\beta_3(X,Y,Z) = \beta_4(X,Y,Z) = 0$. Since $\delta\Phi = 0$, $\theta_{\beta_1}(X) = \theta_{\beta_5}(X) = 0$ and

$$\beta_1(X,Y,Z) = \beta_5(X,Y,Z) = 0$$

is obtained. Moreover, since $\theta^*_\beta(\xi) = 0$, $\theta^*_{\beta_6}(\xi) = 0$ and $\beta_6(X,Y,Z) = 0$ are obtained. Thus the manifold M is of the class $\mathbb{G}_2 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8$. □

At this step, we consider the almost paracontact metric manifold M of the class $\mathbb{G}_2$.

Theorem 4.7 *If the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is of the class $\mathbb{G}_2$, the manifold $M \times \mathbb{R}$ belongs to the class $W_3 \oplus W_7$.*

Proof Let the manifold M be of class $\mathbb{G}_2$. From the defining relation of this class, since

$$\beta(X, \xi, \varphi(Y)) = 0,$$

for any vector field X, Y, we get $\nabla_X \xi = 0$. Let $\{e_1, e_2, \cdots, e_n, \varphi(e_1), \cdots, \varphi(e_n), \xi\}$ be a frame field on M. In this class, since

$$0 = \beta(e_i, \xi, \varphi(e_i)) = -\beta(e_i, \varphi(e_i), \xi)) = \beta(\varphi(e_i), e_i, \xi),$$

we have $\theta^*(\xi) = 0$. In addition, we know that $\theta(X) = 0$ in the class $\mathbb{G}_2$ for all X. Hence we get

$$\delta F\left(X, a\frac{d}{dt}\right) = -\theta_\beta(X) - \theta^*_\beta(\xi) = 0 - a.0 = 0$$

for any vector field $\left(X, a\frac{d}{dt}\right)$.

Let $A, B \in \mathcal{V}$. We can take $A = \left(X, \eta(X)\frac{d}{dt}\right)$ and $B = \left(Y, \eta(Y)\frac{d}{dt}\right)$ such that $\varphi(X) = \varphi^2(X), \varphi(Y) = \varphi^2(Y)$. Then

$$\beta(X,Y,Z) = -\beta(\varphi(X), \varphi(Y), Z) = -\beta(\varphi^2(X), \varphi^2(Y), Z) = -\beta(X,Y,Z)$$

and we have $(\nabla_X \varphi)(Y) = 0$ since $\varphi(X) = \varphi^2(X), \varphi(Y) = \varphi^2(Y)$. Moreover

$$\begin{aligned} 0 = (\nabla_X \varphi)(Y) &= \nabla_X \varphi(Y) - \varphi(\nabla_X Y) \\ &= \nabla_X \varphi^2(Y) - \varphi(\nabla_X Y) \\ &= \nabla_X Y - \eta(\nabla_X Y)\xi - \varphi(\nabla_X Y) \end{aligned}$$

can be obtained. Then we have

$$\begin{aligned} \nabla_A B &= \left(\nabla_X Y, \eta(\nabla_X Y)\tfrac{d}{dt}\right) \\ &= \left(\eta(\nabla_X Y)\xi + \varphi(\nabla_X Y), \eta(\nabla_X Y)\tfrac{d}{dt}\right) \\ &= J(\nabla_A B), \end{aligned}$$

that is $\nabla_A B \in \mathcal{V}$.

Let $U, V \in \mathcal{H}$. We take $U = \left(X, -\eta(X)\frac{d}{dt}\right), V = \left(Y, -\eta(Y)\frac{d}{dt}\right)$, such that $\varphi(X) = -\varphi^2(X), \quad \varphi(Y) = -\varphi^2(Y)$. Then

$$\beta(X,Y,Z) = -\beta(\varphi(X), \varphi(Y), Z) = -\beta(\varphi^2(X), \varphi^2(Y), Z) = -\beta(X,Y,Z)$$

and we have $(\nabla_X\varphi)(Y) = 0$ since $\varphi(X) = -\varphi^2(X)$, $\varphi(Y) = -\varphi^2(Y)$. Moreover

$$\begin{aligned} 0 = (\nabla_X\varphi)(Y) &= \nabla_X\varphi(Y) - \varphi(\nabla_X Y) \\ &= -\nabla_X\varphi^2(Y) - \varphi(\nabla_X Y) \\ &= -\nabla_X Y + \eta(\nabla_X Y)\xi - \varphi(\nabla_X Y) \end{aligned}$$

can be obtained. Thus we have

$$\begin{aligned} \nabla_U V &= \left(\nabla_X Y, -\eta(\nabla_X Y)\tfrac{d}{dt}\right) \\ &= \left(\eta(\nabla_X Y)\xi - \varphi(\nabla_X Y), \eta(\nabla_X Y)\tfrac{d}{dt}\right) \\ &= -J(\nabla_U V), \end{aligned}$$

that is $\nabla_U V \in \mathcal{H}$. As a result, we get that if the manifold M is of the class $\mathbb{G}_2$, then the product manifold $M \times \mathbb{R}$ is of the class $W_3 \oplus W_7$. □

Similarly, if the manifold M is of class $\mathbb{G}_7$ (or $\mathbb{G}_8$), it can be shown that the product manifold $M \times \mathbb{R}$ is of class $W_3 \oplus W_7$. Hence we get the following theorem:

Theorem 4.8 *The manifold $M \times \mathbb{R}$ is of class $W_3 \oplus W_7$ if and only if M belongs to class $\mathbb{G}_2 \oplus \mathbb{G}_7 \oplus \mathbb{G}_8$.*

Finally, we study the primitive classes W_4 and W_8.

Theorem 4.9 *If the manifold $M \times \mathbb{R}$ is from class W_4, which is one of the primitive classes, the manifold M is from the $\mathbb{G}_5 \oplus \mathbb{G}_6$ class.*

Proof Let the product manifold $(M \times \mathbb{R}, h, J)$ be of class W_4. Then the following equality satisfied [6]:

$$\begin{aligned} \frac{1}{2n}\Bigg\{ &\delta F\left(Y, b\frac{d}{dt}\right) h\left(\left(X, a\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) \\ &-\delta F\left(Z, c\frac{d}{dt}\right) h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) \\ &+\delta F\left(J\left(Y, b\frac{d}{dt}\right)\right) h\left(\left(X, a\frac{d}{dt}\right), J\left(Z, c\frac{d}{dt}\right)\right) \\ &-\delta F\left(J\left(Z, c\frac{d}{dt}\right)\right) h\left(\left(X, a\frac{d}{dt}\right), J\left(Y, b\frac{d}{dt}\right)\right)\Bigg\} \\ &= \left(\nabla_{(X, a\frac{d}{dt})}F\right)\left(\left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) \end{aligned} \tag{4.39}$$

for all vector fields $\left(X, a\frac{d}{dt}\right)$, $\left(Y, b\frac{d}{dt}\right)$, $\left(Z, c\frac{d}{dt}\right)$ and

$$\nabla_A B \in \mathcal{V}, \quad \nabla_U A \in \mathcal{V}, \quad \nabla_U V \in \mathcal{H}$$

for all vector field $A, B \in \mathcal{V}$, $U, V \in \mathcal{H}$. Equation (4.39) can be written as

$$\begin{aligned}
&\beta(X,Y,Z) + b\left(\nabla_X \eta\right)(Z) - c\left(\nabla_X \eta\right)(Y) \\
&= \frac{1}{2n}\left\{-\theta_\beta(Y) g(X,Z) + \theta_\beta(Z) g(X,Y)\right. \\
&-\theta_\beta(\varphi(Y)) g(X,\varphi(Z)) + \theta_\beta(\varphi(Z)) g(X,\varphi(Y)) \\
&\left.-\eta(Y)\theta^*_\beta(\xi) g(X,\varphi(Z)) + \eta(Z)\theta^*_\beta(\xi) g(X,\varphi(Y))\right\} \\
&+\frac{1}{2n} ac\left\{\theta_\beta(Y) - \eta(Y)\theta_\beta(\xi)\right\} - \frac{1}{2n} ab\left\{\theta_\beta(Z) - \eta(Z)\theta_\beta(\xi)\right\} \\
&+\frac{1}{2n} b\left\{\theta^*_\beta(\xi) g(\varphi(X),\varphi(Z)) - \theta_\beta(\xi) g(X,\varphi(Z))\right\} \\
&-\frac{1}{2n} c\left\{\theta^*_\beta(\xi) g(\varphi(X),\varphi(Y)) - \theta_\beta(\xi) g(X,\varphi(Y))\right\} \\
&+\frac{1}{2n}\left\{b\eta(X)\theta_\beta\left(\varphi(Z)\right) - c\eta(X)\theta_\beta\left(\varphi(Y)\right)\right\} \\
&+\frac{1}{2n} a\left\{-\eta(Y)\theta_\beta\left(\varphi(Z)\right) + \eta(Z)\theta_\beta\left(\varphi(Y)\right)\right\}
\end{aligned} \tag{4.40}$$

In equation (4.39), taking $X = 0$, $Y = 0$, $a = b = 1$ and $c = 0$ gives

$$\delta\Phi(Z) = \eta(Z)\delta\Phi(\xi).$$

In the last equation, setting $Z = \varphi(Z)$ we get $\delta\Phi(\varphi(Z)) = 0$. In equation (4.39), setting $a = b = c = 0$, we get

$$\begin{aligned}
\beta(X,Y,Z) \quad = \quad & \frac{1}{2n}\left\{\theta(Z) g(X,Y) - \theta(Y) g(X,Z)\right. \\
& \left.-\eta(Y)\theta^*(\xi) g(X,\varphi(Z)) + \eta(Z)\theta^*(\xi) g(X,\varphi(Y))\right\}.
\end{aligned} \tag{4.41}$$

From the last equation (4.41), we get $\beta(\xi,\xi,\varphi^2(Z)) = 0$ and $\beta(X,\varphi(Y),\varphi(Z)) = 0$. Hence

$$\beta_{11}(X,Y,Z) = \beta_{12}(X,Y,Z) = 0.$$

Since $\beta(X,\varphi(Y),\varphi(Z)) = 0$, we get $\beta_1(X,Y,Z) = 0 - \beta_2(X,Y,Z)$ and $\beta_3(X,Y,Z) = 0 - \beta_4(X,Y,Z)$, then

$$\beta_1(X,Y,Z) = \beta_2(X,Y,Z) = \beta_3(X,Y,Z) = \beta_4(X,Y,Z) = 0$$

From equation (4.41), one can obtain

$$\beta(\varphi(X),\varphi(Y),\xi) = -\beta(X,Y,\xi). \tag{4.42}$$

then we get $\beta_9(X,Y,Z) = \beta_{10}(X,Y,Z) = 0$. From the defining relation of the class $\mathbb{G}_7$, we have $\theta^*_{\beta_7}(\xi) = 0$. In addition, since equation (4.42) is satisfied, we get $\theta^*_{\beta_8}(\xi) = 0$, also one can obtain easily $\theta^*_{\beta_5}(\xi) = 0$. Then $\theta^*(\xi) = \theta_{\beta_6}(\xi) = 0$. When similar calculations are carried out, $\theta(\xi) = \theta_{\beta_5}(\xi) = 0$ is obtained.

Since $\nabla_U A \in \mathcal{V}$, $\nabla_U V \in \mathcal{H}$, $\nabla_A B \in \mathcal{V}$, equations (4.8), (4.34), and (4.32) are satisfied. Then from these equations we have

$$\beta(\varphi(X),\varphi(Y),\xi) = -\beta(\varphi(X),Y,\xi) = \beta(X,\varphi(Y),\xi) = -\beta(X,Y,\xi).$$

From these equations we have $\theta(\xi) = -\theta^*(\xi)$. However, we obtained no further contribution from the equations above to eliminate either the class $\mathbb{G}_5$ or $\mathbb{G}_6$. When all the obtained results are used, we arrive at

$$\begin{aligned} \beta(X, Y, Z) &= \frac{\theta(\xi)}{2n}\{-\eta(Y)g(X, Z) + \eta(Z)g(X, Y)\} \\ &\quad + \frac{\theta^*(\xi)}{2n}\{-\eta(Y)g(X, \varphi(Z)) + \eta(Z)g(X, \varphi(Y))\}, \end{aligned} \tag{4.43}$$

and as a result, we have that the almost paracontact metric manifold M is of the class $\mathbb{G}_5 \oplus \mathbb{G}_6$. □

In a similar manner, it can be proved that if the manifold $M \times \mathbb{R}$ is from class W_8, which is another primitive class, the manifold M is from the $\mathbb{G}_5 \oplus \mathbb{G}_6$ class too. We can express the following theorem:

Theorem 4.10 *If the manifold $M \times \mathbb{R}$ is from class $W_4 \oplus W_8$, then the manifold M is from the $\mathbb{G}_5 \oplus \mathbb{G}_6$ class.*

In addition, we can get the inverse of this theorem:

Theorem 4.11 *If the manifold M belongs to the class $\mathbb{G}_5$, then $M \times \mathbb{R}$ is from class $W_4 \oplus W_8$.*

Proof Let the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ be of class $\mathbb{G}_5$. Then $\beta(X, Y, Z) = \frac{\theta(\xi)}{2n}\{\eta(Y)g(\varphi(X), \varphi(Z)) - \eta(Z)g(\varphi(X), \varphi(Y))\}$. Since

$$\beta(e_i, \varphi(e_i), X) = \beta(\varphi(e_i), e_i, X) = 0$$

we obtain $\theta^*(X) = 0$. In addition, in the class $\mathbb{G}_5$ we have $\theta(X) = \eta(X)\theta(\xi)$. Then we get $\theta(\varphi(X)) = 0$. The right hand side of equation (4.40) reduces as

$$\begin{aligned} &\frac{1}{2n}\theta(\xi)\{-\eta(Y)g(X, Z) + \eta(Z)g(X, Y)\} \\ &-\frac{b}{2n}\theta(\xi)g(X, \varphi(Z)) + \frac{c}{2n}\theta(\xi)g(X, \varphi(Y)) \end{aligned} \tag{4.44}$$

On the other hand, since equation

$$(\nabla_X \eta)(y) = \beta(X, \xi, \varphi(Y)) = \beta_5(X, \xi, \varphi(Y)) = \frac{\theta(\xi)}{2n}g(\varphi(X), y)$$

is satisfied, equation (4.40) is obtained.

Let $A, B \in \mathcal{V}$. Then we can take $A = \left(X, \eta(X)\frac{d}{dt}\right)$, $B = \left(Y, \eta(Y)\frac{d}{dt}\right)$ such that $\varphi^2(X) = \varphi(X)$, $\varphi^2(Y) = \varphi(Y)$. We note that $\nabla_\xi \xi = 0$ in the class $\mathbb{G}_5$. Moreover, since $\varphi^2(X) = \varphi(X)$, $\varphi^2(Y) = \varphi(Y)$, we have $g(\nabla_X \xi, Y) = 0$. Then we have

$$\nabla_A B = \left(\nabla_X Y, \eta(\nabla_X Y)\frac{d}{dt}\right)$$

and

$$J(\nabla_A B) = \left(\varphi(\nabla_X Y) + \eta(\nabla_X Y)\xi, \eta(\nabla_X Y) \cdot \frac{d}{dt}\right)$$

In the class $\mathbb{G}_5$ we have

$$\left(\nabla_X\varphi\right)(Y)=\frac{\theta(\xi)}{2n}\left\{-\eta(Y)\varphi^2(X)-g\left(\varphi(X),\varphi(Y)\right)\xi\right\}. \tag{4.45}$$

Since $\varphi^2(X)=\varphi(X)$ and $\varphi^2(Y)=\varphi(Y)$, left hand side of equation (4.45) is

$$\begin{aligned}\left(\nabla_X\varphi\right)(Y) &= \nabla_X\varphi(Y)-\varphi\left(\nabla_XY\right)\\ &= \nabla_X\varphi^2(Y)-\varphi\left(\nabla_XY\right)\\ &= \nabla_XY-X\left[\eta(Y)\right]\xi-\eta(Y)\nabla_X\xi-\varphi\left(\nabla_XY\right),\end{aligned}$$

and right-hand side of equation (4.45) is

$$-\frac{\theta_{\beta_5}(\xi)}{2n}\left\{\eta(Y)\varphi^2(X)+g\left(\varphi(X),\varphi(Y)\right)\xi\right\}=-\eta(Y)\nabla_X\xi.$$

Then, we get

$$\nabla_XY-\eta\left(\nabla_XY\right)\xi=\varphi\left(\nabla_XY\right).$$

Hence we obtain that

$$\begin{aligned}J\left(\nabla_AB\right) &= \left(\varphi\left(\nabla_XY\right)+X[\eta(Y)]\xi,\eta\left(\nabla_XY\right)\tfrac{d}{dt}\right)\\ &= \left(\nabla_XY,\eta\left(\nabla_XY\right)\tfrac{d}{dt}\right)\\ &= \nabla_AB.\end{aligned}$$

Let $U,V\in\mathcal{H}$. Then we can take $U=\left(X,-\eta(X)\frac{d}{dt}\right)$, $V=\left(Y,-\eta(Y)\frac{d}{dt}\right)$ such that $\varphi^2(X)=-\varphi(X)$, $\varphi^2(Y)=-\varphi(Y)$. Since $\varphi^2(X)=-\varphi(X)$, $\varphi^2(Y)=-\varphi(Y)$, we have $\beta(X,\xi,\varphi(Y))=0$ and hence we get $g\left(\nabla_X\xi,Y\right)=0$ and $X\left[\eta(Y)\right]=\eta\left(\nabla_XY\right)$. In addition, we have

$$\nabla_UV=\left(\nabla_XY,-\eta\left(\nabla_XY\right)\frac{d}{dt}\right)$$

and

$$J\left(\nabla_UV\right)=\left(\varphi\left(\nabla_XY\right)-\eta\left(\nabla_XY\right)\xi,\eta\left(\nabla_XY\right)\frac{d}{dt}\right).$$

In the class $\mathbb{G}_5$, we have equation (4.45) and since $\varphi^2(X)=-\varphi(X)$ and $\varphi^2(Y)=-\varphi(Y)$, left hand side of equation (4.45) is

$$\begin{aligned}\left(\nabla_X\varphi\right)(Y) &= \nabla_X\varphi(Y)-\varphi\left(\nabla_XY\right)\\ &= -\nabla_X\varphi^2(Y)-\varphi\left(\nabla_XY\right)\\ &= -\nabla_XY+X\left[\eta(Y)\right]\xi+\eta(Y)\nabla_X\xi-\varphi\left(\nabla_XY\right),\end{aligned}$$

and right-hand side of equation (4.45) is

$$-\frac{\theta_{\beta_5}(\xi)}{2n}\left\{\eta(Y)\varphi^2(X)+g\left(\varphi(X),\varphi(Y)\right)\xi\right\}=\eta(Y)\nabla_X\xi.$$

It follows that

$$-\nabla_XY+\eta\left(\nabla_XY\right)\xi=\varphi\left(\nabla_XY\right).$$

Hence, we obtain that

$$\begin{aligned} J\left(\nabla_U V\right) &= \left(\varphi\left(\nabla_X Y\right) - X[\eta(Y)]\xi, \eta\left(\nabla_X Y\right)\tfrac{d}{dt}\right) \\ &= \left(-\nabla_X Y, \eta\left(\nabla_X Y\right)\tfrac{d}{dt}\right) \\ &= -\nabla_U V. \end{aligned}$$

Since all conditions are satisfied, the product manifold $M \times \mathbb{R}$ is of the class $W_4 \oplus W_8$. □

Similarly, it can be shown that if the almost paracontact metric manifold is of the class $\mathbb{G}_6$, then the manifold $M \times \mathbb{R}$ is of the class $W_4 \oplus W_8$. Thus, we can express the following theorem:

Theorem 4.12 *The almost contact metric manifold M is of the class $\mathbb{G}_5 \oplus \mathbb{G}_6$ iff the almost para-Hermitian manifold $M \times \mathbb{R}$ is of the class $W_4 \oplus W_8$.*

Finally, we give some examples:

Let G be the $7-$dimensional Lie group whose Lie algebra $\mathfrak{g}$ with basis $\{E_0, E_1, \cdots, E_6\}$ of left invariant fields and non-zero brackets of this Lie algebra are:

$$[E_0, E_1] = E_4, \quad [E_0, E_2] = E_5, \quad [E_0, E_3] = E_6,$$

$$[E_0, E_4] = E_1, \quad [E_0, E_5] = E_2, \quad [E_0, E_6] = E_3.$$

We define a left-invariant almost paracontact metric structure on G as:

$$\varphi(E_1) = E_4, \quad \varphi(E_2) = E_5, \quad \varphi(E_3) = E_6,$$

$$\varphi(E_4) = E_1, \quad \varphi(E_5) = E_2, \quad \varphi(E_6) = E_3, \quad \varphi(E_0) = 0,$$

$$\xi = E_0, \quad \eta(E_0) = 1$$

$$g(E_0, E_0) = g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

$$g(E_4, E_4) = g(E_5, E_5) = g(E_6, E_6) = -1, \; g(E_i, E_j) = 0, \; i \neq j.$$

One can easily check that $(L, \varphi, \xi, \eta, g)$ is a 7-dimensional almost paracontact metric manifold with parallel structure, i.e. $\nabla\varphi = 0$. If the product of M with $\mathbb{R}$ is taken, we obtain the para-Hermitian structure on $M \times \mathbb{R}$. This example can also be generalized to higher dimensions.

Consider the following vector fields on $\mathbb{R}^3$

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial y}$$

and pseudo-Riemannian metric

$$g = dx \otimes dx + dy \otimes dy + (x^2 - 1)dz \otimes dz - x\left(dy \otimes dz + dz \otimes dy\right).$$

Lie brackets of vector fields are

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

From Koszul's formula, we have

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = \frac{1}{2}E_3, \quad \nabla_{E_1} E_3 = \frac{1}{2}E_2,$$

$$\nabla_{E_2}E_1 = -\frac{1}{2}E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_2}E_3 = \frac{1}{2}E_1,$$

$$\nabla_{E_3}E_1 = \frac{1}{2}E_2, \quad \nabla_{E_3}E_2 = \frac{1}{2}E_1, \quad \nabla_{E_3}E_3 = 0.$$

If we take $\xi = E_3$, $\varphi(E_1) = E_2$, $\varphi(E_2) = E_1$, we have almost paracontact metric structure of the class $\mathbb{G}_5$. Then we obtain an almost para-Hermitian structure of the class $W_4 \oplus W_8$ on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ by using Theorem 4.11.

Hyperboloids $\mathbb{H}^{2n+1}_{n+1}(1)$ and the hyperbolic Heisenberg group $\mathcal{H}^{2n+1}$ are paraSasakian manifolds [7]. Then we can obtain almost para-Hermitian manifolds $\mathbb{H}^{2n+1}_{n+1}(1) \times \mathbb{R}$ and $\mathcal{H}^{2n+1} \times \mathbb{R}$ which are of the class $W_4 \oplus W_8$ from the Theorem 4.11.

A manifold from the class $\mathbb{G}_6$ is called $\alpha-$paraKenmotsu manifold. Then if a paraKenmotsu manifold product with $\mathbb{R}$, the almost para-Hermitian manifold $M \times \mathbb{R}$ is of the class $W_4 \oplus W_8$ from the Theorem 4.11. Examples of $\alpha-$paraKenmotsu manifolds can be found in [10, 11].

We know that $\dim \mathbb{G}_3 = \frac{(n-2)(n-1)n}{3}$ [10]. Then an almost paracontact metric manifold of class $\mathbb{G}_3$ will have a dimension of at least 7. We could not find an example of class $\mathbb{G}_3$ in the literature. We have investigated at least 7 dimensional Lie algebras. However, we have not achieved results so far. Our calculations on this subject continue.

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