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On slack 2-geodesic convex set and geodesic E-pseudoconvex function with application

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Abstract: We introduce a new class of sets named, slack 2-geodesic convex set on Riemannian manifolds and verify by a nontrivial example. We define a geodesic E-pseudoconvex function with a suitable example. Some properties of geodesic E-quasiconvex function are discussed. We establish some relationships between slack 2-geodesic convex set, geodesic E-pseudoconvex function and geodesic E-quasiconvex function. Moreover, an application of geodesic E-quasiconvex function to a nonlinear programming problem is also presented.

Key words: Slack 2-geodesic convex set, geodesic E-convex function, geodesic E-quasiconvex function, geodesic E-pseudoconvex function, Riemannian manifolds, global minima

1. Introduction

The importance of convexity can be seen in various fields of mathematics, science, and engineering. Because of its huge applications and to explore its real world problems, many generalizations have been proposed by many scientists and scholars, see [1–3, 5, 10, 15] and references therein. Youness [21] showed that many results for convex set hold for a wider range of sets called E-convex set and generalized the class of convex functions to a new class of E-convex function. Some results of E-convexity were improved in [6, 8, 18, 20]. Soleimani-damaneh [17] presented the concept of E-quasiconvex and E-pseudoconvex functions and discussed some of their properties.

The natural generalization of convexity to geodesic convexity on a complete Riemannian manifold was investigated [9, 16, 19]. Iqbal et al. [11] introduced a new class of sets and a new class of functions on Riemannian manifolds, called geodesic E-convex set and geodesic E-convex function. Later, Kumari and Jayswal [13] extended these results under geodesic E-preinvexity on Riemannian manifolds. Lupsa and Cristescu [7] and Lupsa [14] presented the concept of convexity with respect to a given set and defined strong n-convexity, slack n-convexity, strict convexity and slack convexity with respect to a given set. Motivated by the above, we introduce a new class of sets named slack 2-geodesic convex set with respect to a set on Riemannian manifolds, and its existence is shown by a proper example. The notion of geodesic E-pseudoconvex function on the Riemannian manifold is introduced and a nontrivial example for its existence is presented. Further, some relations between

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geodesic E-pseudoconvex function and geodesic E-quasiconvex function are discussed. Moreover, we study some of the properties of geodesic E-quasiconvex function and its application in a nonlinear programming problem.

2. Preliminaries

This section deals with some basic concepts of geodesic convexity on Riemannian manifolds. For the standard material on convexity and optimization on Riemannian manifolds, one can consult [19]. Throughout the work, we assume that M is an n-dimensional Riemannian manifold, $E : M \to M$ is a map, S is a nonempty subset of M (where not otherwise mentioned).

Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a map.
Youness [21] generalized the concept of convex set to E-convex set as follows:

Definition 2.1 *A set $A \subseteq \mathbb{R}^n$ is said to be E-convex with respect to a map E if* $(1-\mu)E(x) + \mu E(y) \in A$ *for all $x, y \in A$ and $\mu \in [0,1]$.*

Every convex set is E-convex set if E is an identity map. If $A \subseteq \mathbb{R}^n$ is an E-convex set, then $E(A) \subseteq A$. If $E(A)$ is a convex set and $E(A) \subseteq A$, then A is E-convex set.

Definition 2.2 *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function defined on an $E-$ convex set $A \subseteq \mathbb{R}^n$. Then the function φ is E-convex if*
$\varphi\big((1-\mu)E(x) + \mu E(y)\big) \leq (1-\mu)\varphi\big(E(x)\big) + \mu\varphi\big(E(y)\big)$ *for all $x, y \in A$ and $\mu \in [0,1]$.*

If the above inequality is strict for $E(x) \neq E(y)$, then φ is said to be strictly E-convex function. If E is an identity map, then every (strictly) convex function is (strictly) E-convex function.
Let $\gamma : [a,b] \to M$ be a piecewise C^1 curve. The length of a curve γ is defined by

$$\mathcal{L}(\gamma) = \int_a^b \|\gamma'(\mu)\|_{\gamma(t)} d(\mu).$$

For any two points $\alpha, \beta \in M$,

$$d(\alpha, \beta) := \inf\big\{\, \mathcal{L}(\gamma) \,:\, \gamma \text{ is a piecewise } C^1 \text{ curve joining } \alpha \text{ to } \beta \,\big\}.$$

Then distance d induces the original topology on a Riemannian manifold. On M, there exists a covariant derivation called Levi-Civita connection, denoted by $\nabla_X Y$, for any vector fields $X, Y \in M$.

Definition 2.3 *A geodesic is a C^∞ smooth path γ. The tangent on geodesic is parallel along the path γ, that is, γ satisfies the equation $\nabla_{d\gamma(\mu)/d\mu} d\gamma(\mu)/d\mu = 0$.*

Iqbal et al. [11] presented the following geodesic E-convex (GEC) set and geodesic E-convex (GEC) function on Riemannian manifolds.

Definition 2.4 *[11] A set S is called GEC set if there exists a unique geodesic $\gamma_{E(x),E(y)}(\mu)$ of length $d(x,y)$, which belongs to S, for all $x, y \in S$ and $\mu \in [0,1]$. Let $S \subseteq M$.*

Definition 2.5 *[11] A function $\varphi : S \to \mathbb{R}$ is said to be GEC defined on GEC set S, if*

$$\varphi\big(\gamma_{E(x),E(y)}(\mu)\big) \leq (1-\mu)\varphi\big(E(y)\big) + \mu\varphi\big(E(x)\big)$$

for all $x, y \in S$ and $\mu \in [0,1]$.

Definition 2.6 *Let $S \subseteq M$ be totally convex set. A function $\varphi : S \to \mathbb{R}$ is called quasiconvex if*

$$\varphi\big(\gamma_{x,y}(\mu)\big) \leqslant max\big\{\varphi(x), \varphi(y)\big\}$$

for all $x, y \in S$ and $\mu \in [0,1]$.

If the above inequality is strict for $x \neq y$ and $\mu \in (0,1)$, then a function φ is called to be strict quasiconvex. A function $\varphi : S \to \mathbb{R}$ is called strongly quasiconvex if

$$\varphi(\gamma_{x,y}(\mu)) < max\{\varphi(x), \varphi(y)\}$$

for all $x, y \in S$, $\mu \in (0,1)$ and $\varphi(x) \neq \varphi(y)$. Every strongly quasiconvex function is strictly quasiconvex, but the converse is not necessarily true.

Iqbal et al. [11] generalized the E-quasiconvexity [18] on Riemannian manifolds as follows:

Definition 2.7 *Let $S \subseteq M$ be a nonempty GEC set. A function $\varphi : S \to \mathbb{R}$ is said to be:*

(i) geodesic E-quasiconvex if for all $x, y \in S$ and $\mu \in [0,1]$,

$$\varphi\big(\gamma_{E(x),E(y)}(\mu)\big) \leq max\big\{\varphi\big(E(x)\big), \varphi\big(E(y)\big)\big\};$$

(ii) strictly geodesic E-quasiconvex if for all $x, y \in S$ with $\varphi\big(E(x)\big) \neq \varphi\big(E(y)\big)$ and $\mu \in (0,1)$,

$$\varphi\big(\gamma_{E(x),E(y)}(\mu)\big) < max\big\{\varphi\big(E(x)\big), \varphi\big(E(y)\big)\big\}.$$

The following important characterization of geodesic E-quasiconvex function was given by Iqbal et al. [11]

Theorem 2.8 *[11] A function $\varphi : S \to \mathbb{R}$ defined on a GEC set S is said to be geodesic E-quasiconvex function if the lower level set $L_r(\overline{\varphi})$ of its restriction $\overline{\varphi} : E(S) \to \mathbb{R}$ is convex function for each $r \in \mathbb{R}$.*

3. Slack 2-convexity

Iqbal et al. [11] defined E-epigraph of φ on $S \subseteq M$ as follows:

$$epi_E(\varphi) = \big\{\big(E(x), a\big) : x \in S, a \in \mathbb{R}, \varphi\big(E(x)\big) \leq a\big\}.$$

The α-level set corresponding to the above epigraph set is given by

$$K_{\alpha,E}(\varphi) = \big\{z \in E(S) : \varphi(z) \leq \alpha\big\}.$$

Lupsa and Cristescu [7] and Lupsa [14] defined a new class of convex sets called slack 2-convex set with respect to a given set. We generalize it on Riemannian manifolds and define slack 2-geodesic convex set.

Definition 3.1 *Suppose A and B are two subsets of M. A is said to be slack 2-geodesic convex set with respect to B if for every $x, y \in A \cap B$ and every $\mu \in [0,1]$ with the property $\gamma_{x,y}(\mu) \in B$, we have $\gamma_{x,y}(\mu) \in A$.*

In the following example (Figure 1,2), we show that a slack 2-geodesic convex set need not be geodesic convex set. It means slack 2-geodesic convex set is an independent class of sets.

Example 3.2 *Consider the sets*

$$X = \{(x,y,z) : x^2 + y^2 + z^2 = 2, z > 1\}$$

and

$$Y = \{(x-1)^2 + y^2 \leq \frac{1}{4}, z > 0\}.$$

Then the set $A = X \setminus Y$ is slack 2-geodesic convex with respect to the set

$$B = \{(x,y,z) : x^2 + y^2 + z^2 = 2, x < 0\},$$

while A is not a geodesic convex set.

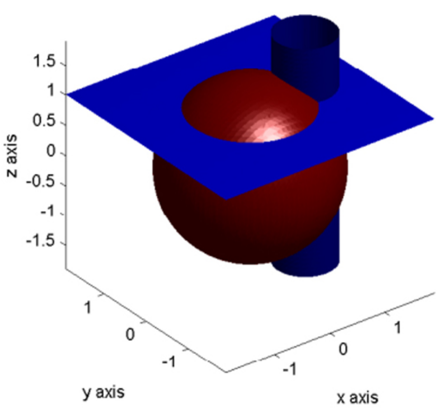

Figure 1. Set A - The cylindrical portion (set Y) removed from the set X (a better view in the adjoining figure).

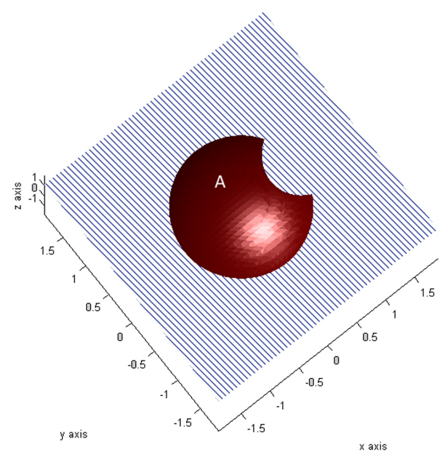

Figure 2. Set A - the top view.

Theorem 3.3 *Assume that $\varphi : S \to \mathbb{R}$ and $E : M \to M$ are two functions. Let S be a nonempty GEC subset of M. If $E(S)$ is a geodesic convex set and $epi_E(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S) \times \mathbb{R}$, then φ is a GEC function on S.*

Proof. Let $x, y \in S$ and $\mu \in [0,1]$. Since $\big(E(x), \varphi(E(x))\big) \in epi_E(\varphi)$, and $\big(E(x), \varphi(E(x))\big) \in E(S) \times \mathbb{R}$, therefore,

$$\big(E(x), \varphi\big(E(x)\big)\big), \big(E(y), \varphi\big(E(y)\big)\big) \in E(S) \times \mathbb{R} \cap epi_E(\varphi).$$

Using geodesic convexity of $E(S)$, we get

$$\gamma_{E(x),E(y)}(\mu) \in E(S),$$

hence

$$\big(\gamma_{E(x),E(y)}(\mu),(1-\mu)\varphi\big(E(x)\big)+\mu\varphi\big(E(y)\big)\big)\in E(S)\times\mathbb{R}$$

which implies that

$$\big(\gamma_{E(x),E(y)}(\mu),(1-\mu)\varphi\big(E(x)\big)+\mu\varphi\big(E(y)\big)\big)\in epi_E(\varphi) \tag{3.1}$$

as $epi_E(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S)\times\mathbb{R}$.
From (3.1), it follows that

$$\varphi\big(\gamma_{E(x),E(y)}(\mu)\big)\leq(1-\mu)\varphi\big(E(x)\big)+\mu\varphi\big(E(y)\big).$$

Hence φ is GEC function on S.

Remark 3.4 *Geodesic convexity of $E(S)$ is a must for the above theorem to hold.*

Theorem 3.5 *Assume that $\varphi:S\to\mathbb{R}$ and $E:M\to M$ are two functions. Let S be a nonempty GEC subset of M. If φ is a geodesic E convex function on S, then $epi(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S)\times\mathbb{R}$.*

Proof. Let $(x,a),(y,b)\in E(S)\times\mathbb{R}\cap epi(\varphi)$ and $\mu\in[0,1]$ such that

$$\big(\gamma_{x,y}(\mu),(1-\mu)a+\mu b\big)\in E(S)\times\mathbb{R}.$$

Then $x,y\in E(S)\subseteq S$, $\varphi(x)\leq a$, $\varphi(y)\leq b$ and $\gamma_{x,y}(\mu)\in E(S)\subseteq S.$
Since $x,y\in E(S)$, then there exist x' and y' in S such that $x=E(x'),y=E(y')$.
From the geodesic E-convexity of φ, we have

$$\begin{aligned}\varphi\big(\gamma_{x,y}(\mu)\big) &= \varphi(\gamma_{E(x'),E(y')}(\mu))\\ &\leq (1-\mu)\varphi\big(E(x')\big)+\mu\varphi\big(E(y')\big)\\ &= (1-\mu)\varphi(x)+\mu\varphi(y)\\ &\leq (1-\mu)a+\mu b.\end{aligned}$$

Therefore, $\big(\gamma_{x,y}(\mu),(1-\mu)a+\mu b\big)\in epi(\varphi)$.
Hence, $epi(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S)\times\mathbb{R}$.

The following theorem is an analogous result of Theorem 3.3, which provides another sufficient condition for φ to be a GEC function.

Theorem 3.6 *Assume that $\varphi:S\to\mathbb{R}$ and $E:M\to M$ are two functions. Let S be a nonempty GEC subset of M. If $E(S)$ is a geodesic convex set and $epi(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S)\times\mathbb{R}$, then φ is a GEC function on S.*

Proof. Let $x,y\in S$ and $\mu\in[0,1]$. Since $\big(E(x),\varphi\big(E(x)\big)\big)\in epi(\varphi)$ and $\big(E(x),\varphi\big(E(x)\big)\big)\in E(S)\times\mathbb{R}$, we have

$$\big(E(x),\varphi\big(E(x)\big)\big),\big(E(y),\varphi\big(E(y)\big)\big)\in E(S)\times\mathbb{R}\cap epi(\varphi).$$

Geodesic convexity of $E(S)$ implies that

$$\gamma_{E(x),E(y)}(\mu) \in E(S).$$

Therefore,

$$\big(\gamma_{E(x),E(y)}(\mu), (1-\mu)\varphi\big(E(x)\big) + \mu\varphi\big(E(y)\big)\big) \in E(S) \times \mathbb{R}$$

which implies that

$$\big(\gamma_{E(x),E(y)}(\mu), (1-\mu)\varphi\big(E(x)\big) + \mu\varphi\big(E(y)\big)\big) \in epi(\varphi). \tag{3.2}$$

This shows that $epi(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S) \times \mathbb{R}$.
From (3.2), we have

$$\varphi\big(\gamma_{E(x),E(y)}(\mu)\big) \le (1-\mu)\varphi\big(E(x)\big) + \mu\varphi\big(E(y)\big).$$

Hence φ is a GEC function on S.

The following theorem is a characterization of a GEC function.

Theorem 3.7 *Assume that $\varphi : S \to \mathbb{R}$ and $E : M \to M$ are two functions, where $E(S)$ is a geodesic convex set. Then, φ is a geodesic E convex function on S iff $epi(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S) \times \mathbb{R}$*

Proof. The proof can be easily deduced from Theorems 3.5 and 3.6.

The following theorem yields a necessary condition for φ to be a GEC function.

Theorem 3.8 *If $\varphi : S \to \mathbb{R}$ is a GEC function on S, then $K_\alpha(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S)$, for all $\alpha \in \mathbb{R}$.*

Proof. Let $z_1, z_2 \in K_\alpha(\varphi) \cap E(S)$ and $\mu \in [0,1]$, such that

$$\gamma_{z_1,z_2}(\mu) \in E(S) \subseteq S.$$

This implies that $z_1, z_2 \in E(S) \subseteq S$, $\varphi(z_1) \le \alpha$ and $\varphi(z_2) \le \alpha$. Also, there exist $x_1, x_2 \in S$, such that $z_1 = E(x_1)$ and $z_2 = E(x_2)$. By the geodesic E-convexity of φ, we have

$$\begin{aligned} \varphi\big(\gamma_{z_1,z_2}(\mu)\big) &= \varphi\big(\gamma_{E(x_1),E(x_2)}(\mu)\big) \\ &\le (1-\mu)\varphi\big(E(x_1)\big) + \mu\varphi\big(E(x_2)\big) \\ &= (1-\mu)\varphi(z_1) + \mu\varphi(z_2) \\ &\le \alpha. \end{aligned}$$

Hence, $\gamma_{z_1,z_2}(\mu) \in K_\alpha(\varphi)$.

In the following theorems, we give some characterizations of a geodesic E-quasiconvex function in terms of α-level set and slack 2-geodesic convex set.

Theorem 3.9 *Let $E(S)$ be a geodesic convex set and $E(S) \subseteq S$. The function $\varphi : S \to \mathbb{R}$ is a geodesic E-quasiconvex on S if and only if $K_{\alpha,E}(\varphi)$ is a geodesic convex set for each $\alpha \in \mathbb{R}$.*

Proof. Let $K_{\alpha,E}(\varphi)$ be a geodesic convex set. Then for $x_1, x_2 \in S$ and $\mu \in [0,1]$, we have

$$\gamma_{E(x_1),E(x_2)}(\mu) \in K_{\alpha,E}(\varphi),$$

for $E(x_1), E(x_2) \in K_{\alpha,E}(\varphi)$. By setting $\alpha = max\{\varphi(E(x_1)), \varphi(E(x_2))\}$, we get

$$\varphi(\gamma_{E(x_1),E(x_2)}(\mu)) \le \alpha = max\{\varphi(E(x_1)), \varphi(E(x_2))\},$$

which implies the geodesic E-quasiconvexity of φ.

Conversely, let φ be a geodesic E-quasiconvex function on S and $z_1, z_2 \in K_\alpha(\varphi)$, $\mu \in [0,1]$. Then, there exist $x_1, x_2 \in S$ such that $z_1 = E(x_1)$ and $z_2 = E(x_2)$, $\varphi(z_1) \le \alpha$ and $\varphi(z_2) \le \alpha$. Since $E(S)$ is geodesic convex set, we have $\gamma_{z_1,z_2}(\mu) \in E(S)$ and

$$\varphi\big(\gamma_{z_1,z_2}(\mu)\big) = \varphi\big(\gamma_{E(x_1),E(x_2)}(\mu)\big) \le max\big\{\varphi\big(E(x_1)\big), \varphi\big(E(x_2)\big)\big\} \le \alpha.$$

Thus, $\gamma_{z_1,z_2}(\mu) \in K_\alpha(\varphi)$.

Theorem 3.10 *Let $E(S)$ be a geodesic convex set and $E(S) \subseteq S$. The function $\varphi : S \to \mathbb{R}$ is a geodesic E-quasiconvex on S if and only if $K_\alpha(\varphi)$ is a slack 2-geodesic convex set with respect to $E(S)$ for each $\alpha \in R$.*

Proof. The proof follows on similar lines of Theorem 3.9.

4. Geodesic E-pseudoconvexity and quasiconvexity

The concept of E-pseudoconvex functions on $\mathbb{R}^n$ was introduced by Soleimani-damaneh [17] as follows:

Definition 4.1 *[17] Let S be a nonempty convex open set in $\mathbb{R}^n$. A differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be E-pseudoconvex on S, if for every $x, y \in S$ such that $\nabla\varphi(E(x))(E(y) - E(x)) \geqslant 0$, we have $\varphi(E(y)) \geqslant \varphi(E(x))$, where S is a nonempty convex open set in $\mathbb{R}^n$.*

We generalize the above concept to Riemannian manifolds and study some of their properties.

Definition 4.2 *A differentiable function $\varphi : M \to \mathbb{R}$ is said to be geodesic E-pseudoconvex on a set $S \subseteq M$ if S is GEC set and for each $x, y \in S$ with $\langle \nabla\varphi\big(E(y)\big), \gamma'_{E(x),E(y)}(0)\rangle \geqslant 0$, we have $\varphi(E(x)) \geqslant \varphi(E(y))$.*

Every differentiable GEC function is a geodesic E-pseudoconvex but the converse is not necessarily true as shown in the following example.

Example 4.3 *Let $M = \mathbb{R} \times S^1$. Define $\varphi : M \to \mathbb{R}$ and $E : M \to M$ by*

$$\varphi(x,s) = x^3 \text{ and } E(x,s) = (x^2, s^2).$$

The geodesic joining two points (x, s_1) and (y, s_2) is a part of the helix of the form $\gamma(\mu) = \big(\mu x + (1-\mu)y, e^{i[\mu\theta_1 + (1-\mu)\theta_2]}\big)$, for $\mu \in [0,1]$, where $e^{i\theta_1} = s_1$ and $e^{i\theta_2} = s_2$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. Then φ is not a GEC function. Assume that for any two points (x, s_1) and (y, s_2) in M, we have

$$\langle \nabla\varphi\big(E(y,s_2)\big), \gamma'_{E(x,s_1),E(y,s_2)}(0)\rangle \geqslant 0.$$

Then

$$\langle (6y^2,0)\big(x^2-y^2, i(\theta_1-\theta_2)e^{i\theta_2}\big)^T\rangle \geqslant 0,$$

which implies that $6x^2y^2-6y^4\geqslant 0$. *Hence* $x^2\geqslant y^2$. *So, we get*

$$\varphi\big(E(x,s_1)\big)=x^6\geqslant y^6=\varphi(E(y,s_2)).$$

This shows that φ *is a geodesic* E*-pseudoconvex function.*

Theorem 4.4 *Assume that* $\varphi : M\to\mathbb{R}$ *is a differentiable function on an open set* $S\subseteq M$ *and* $E(S)$ *is a geodesic convex set such that* $E(S)\subseteq S$. *If* φ *is a geodesic* E*-quasiconvex function on* S, *then for each* $x,y\in S$ *with* $\varphi(E(x))\leq\varphi(E(y))$, *we have* $\langle\nabla\varphi\big(E(y)\big),\gamma'_{E(x),E(y)}(0)\rangle\leq 0$.

Proof. Let $x,y\in S$ and $\varphi(E(x))\leq\varphi(E(y))$. Since S is an open set and GEC set, by geodesic E-quasiconvexity, we have

$$\varphi(\gamma_{E(x),E(y)}(\mu))-\varphi(E(y))\leq 0$$

for each $\mu\in(0,1)$. This implies that

$$\lim_{\mu\to 0}\frac{\varphi(\gamma_{E(x),E(y)}(\mu))-\varphi(E(y))}{\mu}=\langle\nabla\varphi\big(E(y)\big),\gamma'_{E(x),E(y)}(0)\rangle\leq 0.$$

which completes the proof.

Theorem 4.5 *Assume that* $\varphi : M\to R$ *is a lower semicontinuous function on an open set* $S\subseteq M$ *and* $E(S)$ *is geodesic convex set. If* φ *is a strictly geodesic E-quasiconvex function on* S *then* φ *is geodesic E-quasiconvex function.*

Proof. Let $x,y\in S$. If $\varphi(E(x))\neq\varphi(E(y))$. Then the proof is obvious. If $\varphi(E(x))=\varphi(E(y))$, we claim

$$\varphi(\gamma_{E(x),E(y)}(\mu))\leqslant max\{\varphi(E(x),E(y))\}\leqslant\varphi(E(x))$$

for each $\mu\in(0,1)$. On contrary, let

$$\varphi(\gamma_{E(x),E(y)}(\overline{\mu}))>\varphi(E(x))$$

for some $\overline{\mu}\in(0,1)$. By setting $x^0=\gamma_{E(x),E(y)}(\overline{\mu})$ and by the convexity of $E(S)$, there exists an $\overline{x}\in S$ such that

$$E(\overline{x})=x^0=\gamma_{E(x),E(y)}(\overline{\mu})\text{ and }\varphi(E(\overline{x}))>\varphi(E(x)).$$

Since φ is a lower semicontinuous function and $E(S)$ is a convex set, we have

$$\varphi\big(E(\overline{x})\big)>\varphi\big(\gamma_{E(x),E(y)}(\delta)\big)=\varphi\big(E(\hat{x})\big)>\varphi\big(E(x)\big)=\varphi\big(E(y)\big)\tag{4.1}$$

for some $\delta\in(0,1)$ and $\hat{x}\in S$. Therefore, $E(\overline{x})\in\gamma_{E(\hat{x}),E(y)}(\delta)$. Hence, by the strict geodesic E-quasiconvexity of φ, we have

$$\varphi\big(E(\overline{x})\big)<max\{\varphi\big(E(\hat{x}),E(y)\big)\}=\varphi\big(\gamma_{E(x),E(y)}(\delta)\big).\tag{4.2}$$

By (4.1) and (4.2), we obtain

$$\varphi\big(E(\overline{x})\big) > \varphi\big(\gamma_{E(x),E(y)}(\delta)\big) > \varphi\big(E(\overline{x})\big),$$

which contradicts our assumption, hence the theorem.

In the following theorems, we discuss some relationships between geodesic E-quasiconvex function and geodesic E-pseudoconvex function.

Theorem 4.6 *Assume that $\varphi : M \to R$ is a differentiable function on an open set $S \subseteq M$ and $E(S)$ is a geodesic convex set. If φ is a geodesic E-pseudoconvex function on S then φ is a strictly geodesic E-quasiconvex function on S.*

Proof. On contrary, if φ is not a strictly geodesic E-quasiconvex on S, then there exist $x, y \in M$ and $\mu \in (0,1)$, such that

$$\varphi\big(E(x)\big) < \varphi\big(E(y)\big)$$

and

$$\varphi\big(\gamma_{E(x),E(y)}(\mu)\big) \geqslant \varphi\big(E(y)\big).$$

By fixing $x^0 = \gamma_{E(x),E(y)}(\mu)$ and by geodesic convexity of $E(S)$, there exists an $\overline{x} \in S$ such that

$$E(\overline{x}) = x^0 = \gamma_{E(x),E(y)}(\mu),$$

which implies that $\varphi\big(E(\overline{x})\big) \geqslant \varphi\big(E(y)\big) > \varphi\big(E(x)\big)$. From geodesic E-pseudoconvexity of φ, we have

$$\langle \nabla\varphi\big(E(\overline{x})\big), \gamma'_{E(x),E(\overline{x})}(0)\rangle < 0,$$

which implies that

$$\langle \nabla\varphi\big(E(\overline{x})\big), \gamma'_{E(y),E(\overline{x})}(0)\rangle > 0.$$

By the geodesic E-pseudoconvexity of φ, we have

$$\varphi\big(E(\overline{x})\big) \leqslant \varphi\big(E(y)\big).$$

Therefore, $\varphi\big(E(\overline{x})\big) = \varphi\big(E(y)\big)$. On the other hand, since S is an open and GEC set, there exists an open neighbourhood around the geodesic $\gamma_{E(\overline{x}),E(y)}$, which is contained in S. Therefore,

$$\lim_{\mu\to 0} \frac{\varphi\big(\gamma_{E(y),E(\overline{x})}(\mu)\big) - \varphi\big(E(\overline{x})\big)}{\mu} = \langle \nabla\varphi\big(E(\overline{x})\big), \gamma'_{E(y),E(\overline{x})}(0)\rangle > 0.$$

This implies that there exists an $E(\hat{x}) \in S$ and a $\lambda \in (0,1)$, such that

$$E(\hat{x}) = \gamma_{E(y),E(\overline{x})}(\lambda)$$

and

$$\varphi\big(E(\hat{x})\big) > \varphi\big(E(\overline{x})\big) = \varphi\big(E(y)\big).$$

Using the geodesic E-pseudoconvexity of φ, we get

$$\langle \nabla\varphi\big(E(\hat{x})\big), \gamma'_{E(y),E(\hat{x})}(0)\rangle < 0$$

and

$$\langle \nabla\varphi\big(E(\hat{x})\big), \gamma'_{E(\overline{x}),E(\hat{x})}(0)\rangle < 0.$$

But

$$0 > \langle \nabla\varphi\big(E(\hat{x})\big), \gamma'_{E(\overline{x}),E(\hat{x})}(0)\rangle = \frac{-\lambda}{1-\lambda}\langle \nabla\varphi\big(E(\hat{x})\big), \gamma'_{E(y),E(\hat{x})}(0)\rangle > 0$$

which is a contradiction and hence the result.

Theorem 4.7 *Assume that $\varphi : M \to R$ is a differentiable function on an open set $S \subseteq M$ and E(S) is a geodesic convex set. If φ is a geodesic E-pseudoconvex function on S, then φ is a geodesic E-quasiconvex function on S.*

Proof. Since φ is a differentiable function, by Theorem 4.6, φ is a strictly geodesic E-quasiconvex function on S. Hence, by Theorem 4.5, φ is a geodesic E-quasiconvex function on S.

5. Application to a nonlinear programming problem

Consider the following nonlinear constrained problem:

$$\text{(NP)} \quad \text{Minimize } \varphi(x), \text{ such that } x \in \big\{x \in E(S) : g_i(x) \leq 0, i = 1, 2, ..., m\big\},$$

where $\varphi : S \to \mathbb{R}$ and $g_i : S \to \mathbb{R}, i = 1, 2, ..., m$ are real valued functions on S. We assume that $g_i, i = 1, 2, ..., m$ are geodesic E-quasiconvex functions and that $E(S)$ is a geodesic convex subset of M.
We denote the feasible set by Ω, where

$$\Omega = \bigcap_{i=1}^{m} \Omega_i = \bigcap_{i=1}^{m} \big\{x \in E(S) : g_i(x) \leq 0\big\}.$$

The set $\big\{x \in E(S) : g_i(x) \leq 0\big\}$ is a geodesic convex set for each $i = 1, 2, ..., m$. It is to be noted that the set Ω is the lower level set of the restriction of $g_i : S \to \mathbb{R}$ to $E(S)$ for each $i = 1, 2, ..., m$. As $g_i : S \to \mathbb{R}, i = 1, 2, ..., m,$ are geodesic E-quasiconvex functions on a geodesic convex set $E(S)$, it follows from Theorem 2.8 that $\big\{x \in E(S) : g_i(x) \leq 0\big\}$, $i = 1, 2, ..., m,$ are geodesic convex sets. It is clear that the intersection of nonempty geodesic convex sets is also geodesic convex set, we have the following obvious result:

Lemma 5.1 *Ω is geodesic convex subset of $E(S)$*

Using Lemma 5.1 and Theorem 2.8, we have the following:

Theorem 5.2 *Assume that $\varphi : S \to \mathbb{R}$ is a geodesic E-quasiconvex function, then the set of solutions of (NP) is a geodesic convex set.*

Theorem 5.3 *Assume that*

(i) $\varphi : S \to \mathbb{R}$ is a geodesic E-quasiconvex function defined on a geodesic convex set S.

(ii) x^* is a strict local optimal solution of (NP).

Then x^* is also a strict global optimal solution of (NP).

Proof. Since x^* is a strict local optimal solution of (NP), then there exists a neighbourhood, say $N_{x^*}(\epsilon)$ of x^* for which

$$\varphi\big(E(x^*)\big) < \varphi\big(E(x)\big), \text{ for each } x \in N_{x^*}(\epsilon). \tag{5.1}$$

Now, let x^* be not a strict global optimal solution of (NP). Then there exists $x' \in E(S)$ such that $\varphi\big(E(x')\big) < \varphi\big(E(x^*)\big)$.

The geodesic E-quasiconvexity of φ gives

$$\varphi\{\gamma_{E(x'),E(x^*)}(\mu)\} \leq Max\{\varphi\big(E(x')\big), \varphi\big(E(x^*)\big)\} = \varphi\big(E(x^*)\big),$$

for every $x', x^* \in E(S)$, which is a contradiction of (5.1). Hence our assumption that x^* is not a strict global optimal solution of (NP) is wrong, and hence the proof.

Theorem 5.4 *Assume that $\varphi : S \to \mathbb{R}$ is a geodesic E-quasiconvex function.*

(i) If x^* is a local optimal solution of (NP), then it is also a global optimal solution of (NP).

(ii) φ attains its minimum over Ω at no more than one point.

Proof. The proof is obvious from Lemma 5.1, Theorem 2.8 and strict geodesic quasiconvexity property.

6. Conclusion

In this paper, we have introduced the concept of slack 2-geodesic convex set on Riemannian manifolds which is the generalization of slack 2-convex set defined by Lupsa et al. [[7], [14]]. The notion of geodesic E-pseudoconvex function on Riemannian manifold is introduced and some of its properties are studied. By the suitable example, it has been shown that every differentiable GEC function is a geodesic E-pseudoconvex but the converse is not necessarily true. Application to a nonlinear programming problem with constraints has been discussed.

References

[1] Adamek M. On a problem connected with strongly convex functions. Mathematical Inequalities and Applications 2016; 16 (4): 1287-1293. doi: 10.7153/mia-19-94

[2] Adamek M. On a generalization of sandwich type theorems. Aequationes Mathematicae 2018; 92 (4): 641-647. doi: 10.1007/S00010-018-0553-8

[3] Adamek M. Remarks on $(F, t)-$ convex functions. Aequationes Mathematicae 2019; 93 (5): 851-857. doi: 10.1007/S00010-019-00646-9

[4] Bazaraa MS, Sherali HD, Shetty CM. Nonlinear Programming: Theory and Algorithms 2nd Edition. Wiley, New York, 1993.

[5] Bector CR, Singh C. B-vex Functions. Journal of Optimization Theory and Applications 1991; 71: 237-253. doi: 10.1007/BF00939919

[6] Chen X. Some properties of semi-E-convex functions. Journal of Mathematical Analysis and Applications 2002; 275 (1): 251–262. doi: 10.1016/S0022-247X(02)00325-6

[7] Cristescu G, Lupsa L. Nonconnected Convexities and Applications. Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[8] Duca DI, Lupsa L. On the E-epigraph of an E-convex function. Journal of Optimization Theory and Applications 2006; 129: 341-348. doi: 10.1007/S10957-006-9059-Y

[9] Greene RE, Shiohama G. Convex functions on complete noncompact manifolds: Topological structure. Inventiones Mathematicae 1981; 63: 129-157. doi: 10.1007/BF01389196

[10] Hanson MA, Mond B. Convex Transformable Programming Problems and Invexity. Journal of Information and Optimization Sciences 1987; 8 (2): 201-207. doi: 10.1080/02522667.1987.10698886

[11] Iqbal A, Ali S, Ahmed I. On Geodesic E-Convex Sets, Geodesic E-Convex Functions and E-Epigraphs. Journal of Optimization Theory and Applications 2012; 155 (1): 239-251. doi: 10.1007/s10957-012-0052-3

[12] Iqbal A, Ahmed I, Ali S. Some properties of geodesic semi-E-convex functions. Nonlinear Analysis: Theory, Methods & Applications 2011; 74 (17): 6805-6813. doi: 10.1016/j.na.2011.07.005

[13] Kumari B, Jayswal A. Some properties of geodesic E preinvex function and geodesic semi E preinvex function on Riemannian manifolds. Opsearch 2008; 55: 807-822. doi: 10.1007/s12597-018-0346-9

[14] Lupsa L. Slack convexity with respect to a given set. Itinerant Seminar on Functional Equations, Approximation and Convexity. Babes-Bolyai University Publishing House, Cluj-Napoca, Romania, 1985, 107-114.

[15] Merentes N, Nikadem K. Strong convexity and separation theorem. Aequationes Mathematicae 2016; 90: 47-55. doi: 10.1007/s00010-015-0360-4

[16] Rapcsak T. Smooth Nonlinear Optimization in R^n. Kluwer Academic Publishers, Dordrecht, Boston, London, 1997.

[17] Soleimani-damaneh M. E-convexity and its generalizations. International Journal of Computer Mathematics 2011; 88 (16): 3335-3349. doi: 10.1080/00207160.2011.589899

[18] Syau YR, Lee ES. Some properties of E-convex functions. Applied Mathematics Letters 2005; 18 (9): 1074-1080. doi: 10.1016/j.aml.2004.09.018

[19] Udriste C. Convex Functions and Optimization Methods on Riemannian Manifolds. Kluwer Academic, Amsterdam, 1994.

[20] Yang XM. On E-convex sets, E-convex functions, and E-convex programming. Journal of Optimization Theory and Applications 2001; 109 (3): 699-704. doi: 10.1023/A:1017532225395

[21] Youness EA. E-Convex Sets, E-convex Functions, and E-Convex Programming. Journal of Optimization Theory and Applications 1999; 102 (2): 439-450. doi: 10.1023/A:1021792726715