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Rough approximations based on different topologies via ideals

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Abstract: In this paper, we generalize the notations of rough sets based on the topological space. Firstly, we produce various topologies by using the concept of ideal, C_j-neighbourhoods and P_j-neighbourhoods. When we compare these topologies with previous topologies, we see that these topologies are more general. Then we introduce new methods to find the approximations by using these generated topologies. When we compare these methods with the previous methods, we see that these methods are more accurate.

Key words: Topological spaces, ideals, rough sets, $\mathfrak{I}_{C_j}$-approximations, $\mathfrak{I}_{P_j}$-approximations

1. Introduction

The problem of managing and perceiving knowledge is a crucial issue in the area of information systems. There are many new ways how to manage and perceive knowledge. One of them is the rough set theory. Rough set theory was investigated by Pawlak [12, 13] as a mathematical approach that deals with uncertainty and vagueness of imprecise data. It has a wide variety of executions in modern life fields such as biology, chemistry, engineering, etc. The central idea in this theory is approximation operators, which are characterized by equivalence classes. They have properties of closure, interior and boundary operators in topology. Thus, approximation for qualitative concepts is obtained without assumption using these topological concepts. However, the equivalence relations are limiter for theoretical and practical viewpoints. Therefore, many researchers introduced several types of generalization of Pawlak's rough set theory using topological concepts and they replaced equivalence relations with binary relations (see [10, 14, 15]).

Lin [11] and Yao[16] examined rough sets concerning neighbourhood systems for the interpretation of granules. Abd El-Monsef et al. [1] defined different neighbourhood systems to approximate rough sets and introduced new neighbourhood systems which appear as a generalized type of neighbourhood spaces. Amer et al. [4] obtained new j-nearly approximations as mathematical instruments modifying and generalizing the j-approximations in the j-neighbourhood space. Atef et al. [5] generalized three types of rough set models hold on j-neighbourhood space. They also introduced the notions of P_j-neighbourhood using j-neighbourhoods and investigated their properties. Abd El-Monsef et al. [3] presented the covering rough sets based on j-neighbourhoods by approximation operations. Al-Shami et al.[6] defined different topologies by using concepts of P_j-neighbourhoods and examined some basic properties. Al-Shami [7] established the notions

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of C_j-neighbourhoods using j-neighbourhoods and studied their characteristics. Hosny [8] generated different topologies using ideal $\mathfrak{I}$ and constructed new approximations namely $\mathfrak{I}$-j-approximations via these topologies. Also, he proved that these approximations are extended the notation of j-approximations. Hosny [9] presented the concepts of j-nearly open sets with respect to ideals and j-nearly approximations in terms of ideals as generalizations of j-nearly approximations.

The layout of our work is as follows. In Section 2, we have included the definitions and theorems that we will use throughout the paper. In Section 3, we generate eight different topologies by using ideal and C_j-neighbourhoods and study the relationships between them. In Section 4, we introduce eight methods to find the new approximations by using topologies described in Section 3. Then we examine the fundamental properties of these approximation operators and obtain the best two approximations comparing the exactness of these types of approximations. Besides, we compare them with those given by Hosny [8] and obtain that they are more accurate when the relation is reflexive. In Section 5, we generate eight different topologies by using ideal and P_j-neighbourhoods and study the relationships between them. Besides, we compare them with the topologies defined by Al-Shami et al.[6] and show that they are more general. In Section 6, we introduce eight methods to find the new approximations by using topologies described in Section 5. Besides, we examined relations between the accuracy of our approximations. We summarize all comparisons we made throughout the paper with tables and we give counterexamples to support the study. Finally, we apply these topological results to model for a real-life problem.

2. Preliminaries

Definition 2.1 *[2] Let R be an arbitrary binary relation on a nonempty finite set X. The j-neighbourhood of $x \in X$ (briefly, $N_j(x)$) for each $j \in J = \{r, l, <r>, <l>, u, i, <u>, <i>\}$ is defined as:*

(a) r-neighbourhood: $N_r(x) = \{y \in X : xRy\}$;

(b) l-neighbourhood: $N_l(x) = \{y \in X : yRx\}$;

(c) $<r>$-neighbourhood: $N_{<r>}(x) = \bigcap_{x \in N_r(y)} N_r(y)$;

(d) $<l>$-neighbourhood: $N_{<l>}(x) = \bigcap_{x \in N_l(y)} N_l(y)$;

(e) u-neighbourhood: $N_u(x) = N_r(x) \cup N_l(x)$;

(f) i-neighbourhood: $N_i(x) = N_r(x) \cap N_l(x)$;

(g) $<u>$-neighbourhood: $N_{<u>}(x) = N_{<r>}(x) \cup N_{<l>}(x)$;

(h) $<i>$-neighbourhood: $N_{<i>}(x) = N_{<r>}(x) \cap N_{<l>}(x)$.

Definition 2.2 *[2] Let $\zeta_j : X \to \mathcal{P}(X)$ be a mapping which assigns for each $x \in X$ its j-neighbourhood in $\mathcal{P}(X)$. The triple (X, R, ζ_j) is called j-neighbourhood space (j-NS).*

Theorem 2.3 *[8] Let (X, R, ζ_j) be a j-NS, $A \subseteq X$ and $\mathfrak{I}$ be an ideal on X. Then, for every $j \in J$, the collection*

$$\tau_j^{\mathfrak{I}} = \{A \subseteq X : N_j(x) \cap A' \in \mathfrak{I} \text{ for every } x \in A\}$$

is a topology on X.

Definition 2.4 *[8] Let (X, R, ζ_j) be a j-NS, $A \subseteq X$ and $j \in J$. The $\Im_j$-lower approximations, $\Im_j$-upper approximations, $\Im_j$-boundary regions and $\Im_j$-accuracy measures of A are described respectively by:*

(a) $\underline{R}_j^{\Im}(A) = int_j^{\Im}(A)$, *where* $int_j^{\Im}(A)$ *represents the interior of* A *in* $(X, \tau_j^{\Im})$;

(b) $\overline{R}_j^{\Im}(A) = cl_j^{\Im}(A)$, *where* $cl_j^{\Im}(A)$ *represents the closure of* A *in* $(X, \tau_j^{\Im})$;

(c) $B_j^{\Im}(A) = \overline{R}_j^{\Im}(A) - \underline{R}_j^{\Im}(A)$;

(d) $\sigma_j^{\Im}(A) = \frac{|\underline{R}_j^{\Im}(A)|}{|\overline{R}_j^{\Im}(A)|}$, *where* $|\overline{R}_j^{\Im}(A)| \neq 0$.

Definition 2.5 *[5] The P_j-neighbourhoods of $x \in X$ for each $j \in J$ is described as:*

(a) $P_r(x) = \{y \in X : N_r(y) = N_r(x)\}$;

(b) $P_l(x) = \{y \in X : N_l(y) = N_l(x)\}$;

(c) $P_u(x) = P_r(x) \cup P_l(x)$;

(d) $P_i(x) = P_r(x) \cap P_l(x)$;

(e) $P_{<r>}(x) = \{y \in X : N_{<r>}(y) = N_{<r>}(x)\}$;

(f) $P_{<l>}(x) = \{y \in X : N_{<l>}(y) = N_{<l>}(x)\}$;

(g) $P_{<u>}(x) = P_{<r>}(x) \cup P_{<l>}(x)$;

(h) $P_{<i>}(x) = P_{<r>}(x) \cap P_{<l>}(x)$.

Definition 2.6 *[7] The C_j-neighbourhoods of $x \in X$ for each $j \in J$ is described as:*

(a) $C_r(x) = \{y \in X : N_r(y) \subseteq N_r(x)\}$;

(b) $C_l(x) = \{y \in X : N_l(y) \subseteq N_l(x)\}$;

(c) $C_u(x) = C_r(x) \cup C_l(x)$;

(d) $C_i(x) = C_r(x) \cap C_l(x)$;

(e) $C_{<r>}(x) = \{y \in X : N_{<r>}(y) \subseteq N_{<r>}(x)\}$;

(f) $C_{<l>}(x) = \{y \in X : N_{<l>}(y) \subseteq N_{<l>}(x)\}$;

(g) $C_{<u>}(x) = C_{<r>}(x) \cup C_{<l>}(x)$;

(h) $C_{<i>}(x) = C_{<r>}(x) \cap C_{<l>}(x)$.

Theorem 2.7 *[7] C_j-neighbourhoods have the following properties for each $x \in X$:*

(a) $C_i(x) \subseteq C_r(x) \subseteq C_u(x)$ *and* $C_i(x) \subseteq C_l(x) \subseteq C_u(x)$;

(b) $C_{<i>}(x) \subseteq C_{<r>}(x) \subseteq C_{<u>}(x)$ *and* $C_{<i>}(x) \subseteq C_{<l>}(x) \subseteq C_{<u>}(x)$;

(c) *If R is reflexive, then $C_j(x) \subseteq N_j(x)$ for each $j \in J$;*

(d) $P_j(x) \subseteq C_j(x)$ *for each* $j \in J$.

Theorem 2.8 *[6] Let (X, R, ζ_j) be a j-NS and $A \subseteq X$. Then, for every $j \in J$, the collection*

$$\tau_{P,j} = \{A \subseteq X : P_j(x) \subseteq A \text{ for every } x \in A\}$$

is topology on X.

3. Topologies based on C_j-neighbourhoods by using ideals

In this section, we generated different topologies by using ideals and C_j-neighbourhoods. Then, we investigated the relationships between them.

Theorem 3.1 *Let (X, R, ζ_j) be a j-NS, $A \subseteq X$ and $\mathfrak{I}$ be an ideal on X. Then, for every $j \in J$, the collection*

$$\tau_{C_j}^{\mathfrak{I}} = \{A \subseteq X : C_j(x) \cap A' \in \mathfrak{I} \text{ for every } x \in A\}$$

is a topology on X.

Proof

(t_1) It is obvious.

(t_2) Let $A_\delta \in \tau_{C_j}^{\mathfrak{I}}$ for each $\delta \in \Delta$ and $x \in \bigcup_{\delta\in\Delta} A_\delta$. Then, there exists a $\delta_0 \in \Delta$ such that $x \in A_{\delta_0}$. Thus, we get $C_j(x) \cap A'_{\delta_0} \in \mathfrak{I}$. This implies that $C_j(x) \cap (\bigcup_{\delta\in\Delta} A_\delta)' \in \mathfrak{I}$. Hence, $\bigcup_{\delta\in\Delta} A_\delta \in \tau_{C_j}^{\mathfrak{I}}$.

(t_3) Let $A, B \in \tau_{C_j}^{\mathfrak{I}}$ and $x \in A \cap B$. Then, we have $C_j(x) \cap A' \in \mathfrak{I}$ and $C_j(x) \cap B' \in \mathfrak{I}$. This implies that $C_j(x) \cap (A \cap B)' \in \mathfrak{I}$. Hence, $A \cap B \in \tau_{C_j}^{\mathfrak{I}}$.

□

Theorem 3.2 *Let (X, R, ζ_j) be a j-NS, $A \subseteq X$ and $\mathfrak{I}$ be an ideal on X. Then, $\tau_{C_j}^{\mathfrak{I}}$ is finer than the topology $\tau_{C,j} = \{A \subseteq X : C_j(x) \subseteq A \text{ for every } x \in A\}$ generated by C_j-neighbourhood for every $j \in J$.*

Proof Let $A \in \tau_{C,j}$. Then, $C_j(x) \subseteq A$ for every $x \in A$. Hence, $C_j(x) \cap A' = \emptyset \in \mathfrak{I}$ for every $x \in A$. Therefore, $A \in \tau_{C_j}^{\mathfrak{I}}$. □

Example 3.3 *Let $X = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$, $R = \{(\varrho_1, \varrho_2), (\varrho_1, \varrho_3), (\varrho_2, \varrho_2), (\varrho_2, \varrho_3), (\varrho_3, \varrho_2)\}$ and $\mathfrak{I} = \{\emptyset, \{\varrho_2\}, \{\varrho_4\}, \{\varrho_2, \varrho_4\}\}$.*

(a) $\tau_{C,r} = \{X, \emptyset, \{\varrho_4\}, \{\varrho_3, \varrho_4\}\}$ and $\tau_{C_r}^{\mathfrak{I}} = \{X, \emptyset, \{\varrho_3\}, \{\varrho_1, \varrho_3\}, \{\varrho_4\}, \{\varrho_3, \varrho_4\}, \{\varrho_1, \varrho_2, \varrho_3\}, \{\varrho_1, \varrho_3, \varrho_4\}\}$. Thus, $\tau_{C,r} \subsetneq \tau_{C_r}^{\mathfrak{I}}$.

(b) $\tau_{C,l} = \{X, \emptyset, \{\varrho_1, \varrho_4\}, \{\varrho_1, \varrho_3, \varrho_4\}\}$ and $\tau_{C_l}^{\mathfrak{I}} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_1, \varrho_3\}, \{\varrho_1, \varrho_4\}, \{\varrho_1, \varrho_2, \varrho_3\}, \{\varrho_1, \varrho_3, \varrho_4\}\}$ Thus, $\tau_{C,l} \subsetneq \tau_{C_l}^{\mathfrak{I}}$.

Besides, $\tau_{C,j} \subseteq \tau_{C_j}^{\mathfrak{I}}$ for every $j \in \{<r>, <l>, i, u, <i>, <u>\}$.

Remark 3.4 *If $\mathfrak{I} = \{\emptyset\}$ in Theorem 3.2, then $\tau_{C,j}$ coincides with $\tau_{C_j}^{\mathfrak{I}}$ for each $j \in J$.*

Proposition 3.5 *The followings are hold:*

(a) $\tau_{C_u}^{\mathfrak{I}} \subseteq \tau_{C_r}^{\mathfrak{I}} \subseteq \tau_{C_i}^{\mathfrak{I}}$;

(b) $\tau_{C_u}^{\mathfrak{I}} \subseteq \tau_{C_l}^{\mathfrak{I}} \subseteq \tau_{C_i}^{\mathfrak{I}}$;

(c) $\tau_{C_{<u>}}^{\mathfrak{I}} \subseteq \tau_{C_{<r>}}^{\mathfrak{I}} \subseteq \tau_{C_{<i>}}^{\mathfrak{I}}$;

(d) $\tau_{C_{<u>}}^{\mathfrak{I}} \subseteq \tau_{C_{<l>}}^{\mathfrak{I}} \subseteq \tau_{C_{<i>}}^{\mathfrak{I}}$.

Proof (a) Let $A \in \tau^{\Im}_{C_u}$. Then, $C_u(x) \cap A' \in \Im$ for every $x \in A$. By definition of $C_u(x)$, we have $C_r(x) \cap A' \in \Im$. This implies $A \in \tau^{\Im}_{C_r}$. Also, by definition of $C_i(x)$, we have $C_i(x) \cap A' \subseteq C_r(x) \cap A'$ and $C_i(x) \cap A' \in \Im$. This implies that $A \in \tau^{\Im}_{C_i}$. Thus, $\tau^{\Im}_{C_u} \subseteq \tau^{\Im}_{C_r} \subseteq \tau^{\Im}_{C_i}$.

(b) Let $A \in \tau^{\Im}_{C_u}$. Then, $C_u(x) \cap A' \in \Im$ for every $x \in A$. By definition of $C_u(x)$, we have $C_l(x) \cap A' \in \Im$. This implies $A \in \tau^{\Im}_{C_l}$. Besides, by definition of $C_i(x)$, we have $C_i(x) \cap A' \subseteq C_l(x) \cap A'$ and $C_i(x) \cap A' \in \Im$. This implies that $A \in \tau^{\Im}_{C_i}$. Thus, $\tau^{\Im}_{C_u} \subseteq \tau^{\Im}_{C_l} \subseteq \tau^{\Im}_{C_i}$.

(c) and (d) can be proved in a similar way. □

Example 3.6 *Let* $X = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}, R = \{(\varrho_1, \varrho_1), (\varrho_3, \varrho_3), (\varrho_1, \varrho_3), (\varrho_2, \varrho_4), (\varrho_4, \varrho_2), (\varrho_3, \varrho_2)\}$ *and* $\Im = \{\emptyset, \{\varrho_2\}\}$. *Then*

(a) $\tau^{\Im}_{C_u} \subsetneq \tau^{\Im}_{C_r} \subsetneq \tau^{\Im}_{C_i}$;

(b) $\tau^{\Im}_{C_u} \subsetneq \tau^{\Im}_{C_l} \subsetneq \tau^{\Im}_{C_i}$;

Also, we can see that $\tau^{\Im}_{C_r}$ *and* $\tau^{\Im}_{C_{<r>}}$, $\tau^{\Im}_{C_l}$ *and* $\tau^{\Im}_{C_{<l>}}$, $\tau^{\Im}_{C_i}$ *and* $\tau^{\Im}_{C_{<i>}}$,*and* $\tau^{\Im}_{C_u}$ *and* $\tau^{\Im}_{C_{<u>}}$ *may not be comparable.*

Theorem 3.7 *Let* (X, R, ζ_j) *be a* j*-NS and* $\Im$ *be an ideal on* X. *If* R *is reflexive, then* $\tau^{\Im}_{C_j}$ *is finer than* $\tau^{\Im}_j$ *for every* $j \in J$.

Proof Let $A \in \tau^{\Im}_j$. Then, $N_j(x) \cap A' \in \Im$ for every $x \in A$. Since R is reflexive, $C_j(x) \cap A' \in \Im$ for every $x \in A$. Therefore, $A \in \tau^{\Im}_{C_j}$. □

Example 3.8 *Let* $X = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$, $R = \{(\varrho_1, \varrho_1), (\varrho_2, \varrho_2), (\varrho_3, \varrho_3), (\varrho_4, \varrho_4), (\varrho_1, \varrho_2), (\varrho_3, \varrho_1)\}$ *and* $\Im = \{\emptyset, \{\varrho_4\}\}$. *For* $j = r$, *we get* $\tau^{\Im}_r = \{X, \emptyset, \{\varrho_2\}, \{\varrho_1, \varrho_2\}, \{\varrho_4\}, \{\varrho_2, \varrho_4\}, \{\varrho_1, \varrho_2, \varrho_3\}, \{\varrho_1, \varrho_2, \varrho_4\}\}$ *and* $\tau^{\Im}_{C_r} = \{X, \emptyset, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_1, \varrho_2\}, \{\varrho_4\}, \{\varrho_2, \varrho_3\}, \{\varrho_2, \varrho_4\}, \{\varrho_3, \varrho_4\}, \{\varrho_1, \varrho_2, \varrho_3\}, \{\varrho_1, \varrho_2, \varrho_4\}, \{\varrho_2, \varrho_3, \varrho_4\}\}$. *Thus,* $\tau^{\Im}_r \neq \tau^{\Im}_{C_r}$.

4. Rough approximations induced by $\tau^{\Im}_{C_j}$-topologies

We defined new approximations according to $\tau^{\Im}_{C_j}$ and obtained the best approximation. Then we investigated the properties of these approximations and compared them with those given by Hosny [8].

Definition 4.1 *Let* (X, R, ζ_j) *be a* j*-NS,* $A \subseteq X$ *and* $\Im$ *be an ideal on* X. *Then, for every* $j \in J$, *a subset* A *is called* $\Im_{C_j}$*-open set if* $A \in \tau^{\Im}_{C_j}$ *and the complement of* $\Im_{C_j}$*-open set is called* $\Im_{C_j}$*-closed set. The family* $\mathcal{K}^{\Im}_{C_j}$ *of all* $\Im_{C_j}$*-closed sets is described by* $\mathcal{K}^{\Im}_{C_j} = \{F \subseteq X : F' \in \tau^{\Im}_{C_j}\}$.

Definition 4.2 *For every* $j \in J$, *the* $\Im_{C_j}$*-lower and* $\Im_{C_j}$*-upper approximations of* A *are described respectively by*

$$\underline{R}^{\Im}_{C_j}(A) = \cup\{G \in \tau^{\Im}_{C_j} : G \subseteq A\} = int^{\Im}_{C_j}(A)$$

$$\overline{R}^{\mathfrak{I}}_{C_j}(A) = \cap\{F \in \mathcal{K}^{\mathfrak{I}}_{C_j} : A \subseteq F\} = cl^{\mathfrak{I}}_{C_j}(A),$$

where $int^{\mathfrak{I}}_{C_j}(A)$ *(*$cl^{\mathfrak{I}}_{C_j}(A)$*) represents* $\mathfrak{I}_{C_j}$*-interior (*$\mathfrak{I}_{C_j}$*-closure) of* A*.*

Proposition 4.3 *For every* $j \in J$*, the followings are hold:*

(a) $\underline{R}^{\mathfrak{I}}_{C_j}(A) \subseteq A \subseteq \overline{R}_{C_j}(A)$;

(b) $A \subseteq B$ *implies* $\overline{R}^{\mathfrak{I}}_{C_j}(A) \subseteq \overline{R}^{\mathfrak{I}}_{C_j}(B)$;

(c) $A \subseteq B$ *implies* $\underline{R}^{\mathfrak{I}}_{C_j}(A) \subseteq \underline{R}^{\mathfrak{I}}_{C_j}(B)$;

(d) $\underline{R}^{\mathfrak{I}}_{C_j}(\emptyset) = \overline{R}^{\mathfrak{I}}_{C_j}(\emptyset) = \emptyset$

$\underline{R}^{\mathfrak{I}}_{C_j}(X) = \overline{R}^{\mathfrak{I}}_{C_j}(X) = X$;

(e) $\underline{R}^{\mathfrak{I}}_{C_j}(A \cup B) \supseteq \underline{R}^{\mathfrak{I}}_{C_j}(A) \cup \underline{R}^{\mathfrak{I}}_{C_j}(B)$

$\overline{R}^{\mathfrak{I}}_{C_j}(A \cap B) \subseteq \overline{R}^{\mathfrak{I}}_{C_j}(A) \cap \overline{R}^{\mathfrak{I}}_{C_j}(B)$;

(f) $\overline{R}^{\mathfrak{I}}_{C_j}(A \cup B) = \overline{R}^{\mathfrak{I}}_{C_j}(A) \cup \overline{R}^{\mathfrak{I}}_{C_j}(B)$

$\underline{R}^{\mathfrak{I}}_{C_j}(A \cap B) = \underline{R}^{\mathfrak{I}}_{C_j}(A) \cap \underline{R}^{\mathfrak{I}}_{C_j}(B)$;

(g) $(\underline{R}^{\mathfrak{I}}_{C_j}(A))' = \overline{R}^{\mathfrak{I}}_{C_j}(A')$

$((\overline{R}^{\mathfrak{I}}_{C_j}(A))' = \underline{R}^{\mathfrak{I}}_{C_j}(A')$;

(h) $\overline{R}^{\mathfrak{I}}_{C_j}(\overline{R}^{\mathfrak{I}}_{C_j}(A)) = \overline{R}^{\mathfrak{I}}_{C_j}(A)$

$\underline{R}^{\mathfrak{I}}_{C_j}(\underline{R}^{\mathfrak{I}}_{C_j}(A)) = \underline{R}^{\mathfrak{I}}_{C_j}(A)$.

Proof The proofs are clear from the definitions of $\mathfrak{I}_{C_j}$-interior and $\mathfrak{I}_{C_j}$-closure. □

Example 4.4 *Consider Example 3.3.*

(a) Let $A = \{\varrho_3, \varrho_4\}$ *and* $B = \{\varrho_1, \varrho_2, \varrho_4\}$*. Then, we have* $\overline{R}^{\mathfrak{I}}_{C_r}(A) = X$ *and* $\underline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_4\}$*. Thus,* $A \subsetneq \overline{R}^{\mathfrak{I}}_{C_r}(A)$ *and* $\underline{R}^{\mathfrak{I}}_{C_r}(B) \subsetneq B$*.*

(b) Let $A = \{\varrho_1, \varrho_2, \varrho_4\}$ *and* $B = \{\varrho_2, \varrho_3\}$*. Then, we get* $\underline{R}^{\mathfrak{I}}_{C_r}(A) = \{\varrho_4\}$*,* $\underline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_3\}$ *and* $\underline{R}^{\mathfrak{I}}_{C_r}(A \cup B) = X$*. Hence,* $\underline{R}^{\mathfrak{I}}_{C_r}(A \cup B) = X \neq \underline{R}^{\mathfrak{I}}_{C_r}(A) \cup \underline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_3, \varrho_4\}$*.*

(c) Let $A = \{\varrho_1, \varrho_2\}$ *and* $B = \{\varrho_2, \varrho_3\}$*. Then, we get* $\overline{R}^{\mathfrak{I}}_{C_r}(A) = \{\varrho_1, \varrho_2\}$*,* $\overline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_1, \varrho_2, \varrho_3\}$ *and* $\overline{R}^{\mathfrak{I}}_{C_r}(A \cap B) = \{\varrho_2\}$*. From here,* $\overline{R}^{\mathfrak{I}}_{C_r}(A \cap B) = \{\varrho_2\} \neq \overline{R}^{\mathfrak{I}}_{C_r}(A) \cap \overline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_1, \varrho_2\}$*.*

(d) Let $A = \{\varrho_1, \varrho_2\}$ *and* $B = \{\varrho_2, \varrho_3\}$*. Then, we get* $\overline{R}^{\mathfrak{I}}_{C_r}(A) = \{\varrho_1, \varrho_2\}$ *and* $\overline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_1, \varrho_2, \varrho_3\}$*. Thus,* $\overline{R}^{\mathfrak{I}}_{C_r}(A) \subset \overline{R}^{\mathfrak{I}}_{C_r}(B)$ *but* $A \nsubseteq B$*.*

(e) Let $A = \{\varrho_1, \varrho_2\}$ *and* $B = \{\varrho_1, \varrho_3\}$*. Then, we obtain* $\underline{R}^{\mathfrak{I}}_{C_r}(A) = \emptyset \subset \underline{R}^{\mathfrak{I}}_{C_r}(B) = \{\varrho_1, \varrho_3\}$*. Therefore,* $\underline{R}^{\mathfrak{I}}_{C_r}(A) \subset \underline{R}^{\mathfrak{I}}_{C_r}(B)$ *but* $A \nsubseteq B$*.*

Proposition 4.5 *The followings are hold:*

(a) $\underline{R}^{\Im}_{C_u}(A) \subseteq \underline{R}^{\Im}_{C_r}(A) \subseteq \underline{R}^{\Im}_{C_i}(A)$;

(b) $\underline{R}^{\Im}_{C_u}(A) \subseteq \underline{R}^{\Im}_{C_l}(A) \subseteq \underline{R}^{\Im}_{C_i}(A)$;

(c) $\underline{R}^{\Im}_{C_{<u>}}(A) \subseteq \underline{R}^{\Im}_{C_{<r>}}(A) \subseteq \underline{R}^{\Im}_{C_{<i>}}(A)$;

(d) $\underline{R}^{\Im}_{C_{<u>}}(A) \subseteq \underline{R}^{\Im}_{C_{<l>}}(A) \subseteq \underline{R}^{\Im}_{C_{<i>}}(A)$.

Proof The proofs are clear from Proposition 3.5. □

Proposition 4.6 *The followings are hold:*

(a) $\overline{R}^{\Im}_{C_i}(A) \subseteq \overline{R}^{\Im}_{C_r}(A) \subseteq \overline{R}^{\Im}_{C_u}(A)$;

(b) $\overline{R}^{\Im}_{C_i}(A) \subseteq \overline{R}^{\Im}_{C_l}(A) \subseteq \overline{R}^{\Im}_{C_u}(A)$;

(c) $\overline{R}^{\Im}_{C_{<i>}}(A) \subseteq \overline{R}^{\Im}_{C_{<r>}}(A) \subseteq \overline{R}^{\Im}_{C_{<u>}}(A)$;

(d) $\overline{R}^{\Im}_{C_{<i>}}(A) \subseteq \overline{R}^{\Im}_{C_{<l>}}(A) \subseteq \overline{R}^{\Im}_{C_{<u>}}(A)$.

Proof The proofs are clear from Proposition 3.5. □

Theorem 4.7 *If R is reflexive, then the followings are hold:*

(a) $\underline{R}^{\Im}_{j}(A) \subseteq \underline{R}^{\Im}_{C_j}(A)$;

(b) $\overline{R}^{\Im}_{C_j}(A) \subseteq \overline{R}^{\Im}_{j}(A)$.

Proof The proofs are obvious from Theorem 3.7. □

Definition 4.8 *For every $j \in J$, a subset A is called $\Im_{C_j}$-exact set if $\overline{R}^{\Im}_{C_j}(A) = \underline{R}^{\Im}_{C_j}(A)$. Otherwise, A is called $\Im_{C_j}$-rough set.*

Proposition 4.9 *If R is reflexive then, for every $j \in J$, every $\Im_j$-exact set in X is $\Im_{C_j}$-exact (equivalently, every $\Im_{C_j}$-rough set in X is $\Im_j$-rough).*

Proof The proof is obvious from Proposition 4.3 and Theorem 4.7. □

Example 4.10 *Consider Example 3.8. Thus, $A = \{\varrho_3\}$ is $\Im_{C_r}$-exact but it is not $\Im_r$-exact.*

Definition 4.11 *For every $j \in J$, the $\Im_{C_j}$-boundary regions and $\Im_{C_j}$-accuracy of approximations of A are defined respectively by*

$$B^{\Im}_{C_j}(A) = \overline{R}^{\Im}_{C_j}(A) - \underline{R}^{\Im}_{C_j}(A)$$

$$\sigma^{\Im}_{C_j}(A) = \frac{|\underline{R}^{\Im}_{C_j}(A)|}{|\overline{R}^{\Im}_{C_j}(A)|}, \text{ where } |\overline{R}^{\Im}_{C_j}(A)| \neq 0.$$

Corollary 4.12 *The followings are hold:*

(a) $B^{\Im}_{C_i}(A) \subseteq B^{\Im}_{C_r}(A) \subseteq B^{\Im}_{C_u}(A)$;

(b) $B^{\mathfrak{J}}_{C_i}(A) \subseteq B^{\mathfrak{J}}_{C_l}(A) \subseteq B^{\mathfrak{J}}_{C_u}(A)$;

(c) $B^{\mathfrak{J}}_{C_{<i>}}(A) \subseteq B^{\mathfrak{J}}_{C_{<r>}}(A) \subseteq B^{\mathfrak{J}}_{C_{<u>}}(A)$;

(d) $B^{\mathfrak{J}}_{C_{<i>}}(A) \subseteq B^{\mathfrak{J}}_{C_{<l>}}(A) \subseteq B^{\mathfrak{J}}_{C_{<u>}}(A)$.

Corollary 4.13 *The followings are hold:*

(a) $\sigma^{\mathfrak{J}}_{C_u}(A) \leq \sigma^{\mathfrak{J}}_{C_r}(A) \leq \sigma^{\mathfrak{J}}_{C_i}(A)$;

(b) $\sigma^{\mathfrak{J}}_{C_u}(A) \leq \sigma^{\mathfrak{J}}_{C_l}(A) \leq \sigma^{\mathfrak{J}}_{C_i}(A)$;

(c) $\sigma^{\mathfrak{J}}_{C_{<u>}}(A) \leq \sigma^{\mathfrak{J}}_{C_{<r>}}(A) \leq \sigma^{\mathfrak{J}}_{C_{<i>}}(A)$;

(d) $\sigma^{\mathfrak{J}}_{C_{<u>}}(A) \leq \sigma^{\mathfrak{J}}_{C_{<l>}}(A) \leq \sigma^{\mathfrak{J}}_{C_{<i>}}(A)$.

Remark 4.14 *(a) In Table 1, the boundary regions and accuracy measures are calculated by using our approach according to the Example 3.6. Thus, it is seen that $\mathfrak{J}_{C_i}$-accuracy is more precise than $\mathfrak{J}_{C_j}$-accuracy for $j = \{r, l, u\}$.*

(b) In Table 2, the lower approximations, upper approximations, boundary regions and accuracy measures are calculated by using Hosny's approach [8] and our approach according to Example 3.3 taking the ideal $\mathcal{I} = \{\emptyset, \{\varrho_4\}\}$. Thus, it is seen that the accuracy measure of our approach and Hosny's approach are incomparable.

Table 1. Comparison between the boundary regions and accuracy measures by using our approach for $j = \{r, l, u, i\}$.

A	for $j = r$		for j=l		for j=u		for j=i	
	$B^{\mathfrak{J}}_{C_r}(A)$	$\sigma^{\mathfrak{J}}_{C_r}(A)$	$B^{\mathfrak{J}}_{C_l}(A)$	$\sigma^{\mathfrak{J}}_{C_l}(A)$	$B^{\mathfrak{J}}_{C_u}(A)$	$\sigma^{\mathfrak{J}}_{C_u}(A)$	$B^{\mathfrak{J}}_{C_i}(A)$	$\sigma^{\mathfrak{J}}_{C_i}(A)$
$\{\varrho_1\}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{1}{2}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_2\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\varrho_3\}$	$\{\varrho_3\}$	0	$\{\varrho_3\}$	0	$\{\varrho_3\}$	0	$\emptyset$	1
$\{\varrho_4\}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_1, \varrho_2\}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{2}{3}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\varrho_1, \varrho_3\}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_1, \varrho_4\}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\varrho_2, \varrho_3\}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_2, \varrho_4\}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\varrho_3, \varrho_4\}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{1}{2}$	$\{\varrho_3\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_3\}$	$\frac{3}{4}$	$\{\varrho_3\}$	$\frac{3}{4}$	$\{\varrho_3\}$	$\frac{3}{4}$	$\emptyset$	1
$\{\varrho_1, \varrho_3, \varrho_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\varrho_2, \varrho_3, \varrho_4\}$	$\emptyset$	1	$\{\varrho_3\}$	$\frac{2}{3}$	$\{\varrho_3\}$	$\frac{2}{3}$	$\emptyset$	1
X	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1

Table 2. Comparison between the lower approximations, upper approximations, boundary regions and accuracy measures by using Hosny's approach [8] and our approach for $j = r$.

A	Hosny's approach [8]				Our approach			
	$\underline{R}^{\Im}_{r}(A)$	$\overline{R}^{\Im}_{r}(A)$	$B^{\Im}_{r}(A)$	$\sigma^{\Im}_{r}(A)$	$\underline{R}^{\Im}_{C_r}(A)$	$\overline{R}^{\Im}_{C_r}(A)$	$B^{\Im}_{C_r}(A)$	$\sigma^{\Im}_{C_r}(A)$
$\{\varrho_1\}$	$\emptyset$	$\{\varrho_1\}$	$\{\varrho_1\}$	0	$\emptyset$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_1,\varrho_2\}$	0
$\{\varrho_2\}$	$\emptyset$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	0	$\emptyset$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_1,\varrho_2\}$	0
$\{\varrho_3\}$	$\emptyset$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	0	$\{\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2\}$	$\frac{1}{3}$
$\{\varrho_4\}$	$\{\varrho_4\}$	$\{\varrho_4\}$	$\emptyset$	1	$\{\varrho_4\}$	$\{\varrho_4\}$	$\emptyset$	1
$\{\varrho_1,\varrho_2\}$	$\emptyset$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	0	$\emptyset$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_1,\varrho_2\}$	0
$\{\varrho_1,\varrho_3\}$	$\emptyset$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	0	$\{\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2\}$	$\frac{1}{3}$
$\{\varrho_1,\varrho_4\}$	$\{\varrho_4\}$	$\{\varrho_1,\varrho_4\}$	$\{\varrho_1\}$	$\frac{1}{2}$	$\{\varrho_4\}$	$\{\varrho_1,\varrho_2,\varrho_4\}$	$\{\varrho_1,\varrho_2\}$	$\frac{1}{3}$
$\{\varrho_2,\varrho_3\}$	$\{\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1\}$	$\frac{2}{3}$	$\{\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2\}$	$\frac{1}{3}$
$\{\varrho_2,\varrho_4\}$	$\{\varrho_4\}$	X	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\frac{1}{4}$	$\{\varrho_4\}$	$\{\varrho_1,\varrho_2,\varrho_4\}$	$\{\varrho_1,\varrho_2\}$	$\frac{1}{3}$
$\{\varrho_3,\varrho_4\}$	$\{\varrho_4\}$	X	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\frac{1}{4}$	$\{\varrho_3,\varrho_4\}$	X	$\{\varrho_1,\varrho_2\}$	$\frac{1}{2}$
$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\emptyset$	1	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\emptyset$	1
$\{\varrho_1,\varrho_2,\varrho_4\}$	$\{\varrho_4\}$	X	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\frac{1}{4}$	$\{\varrho_4\}$	$\{\varrho_1,\varrho_2,\varrho_4\}$	$\{\varrho_1,\varrho_2\}$	$\frac{1}{3}$
$\{\varrho_1,\varrho_3,\varrho_4\}$	$\{\varrho_4\}$	X	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\frac{1}{4}$	$\{\varrho_3,\varrho_4\}$	X	$\{\varrho_1,\varrho_2\}$	$\frac{1}{2}$
$\{\varrho_2,\varrho_3,\varrho_4\}$	$\{\varrho_2,\varrho_3,\varrho_4\}$	X	$\{\varrho_1\}$	$\frac{3}{4}$	$\{\varrho_3,\varrho_4\}$	X	$\{\varrho_1,\varrho_2\}$	$\frac{1}{2}$
X	X	X	$\emptyset$	1	X	X	$\emptyset$	1

5. Topologies based on P_j-neighbourhoods by using ideals

In this section, we generated eight different topologies by using ideal and P_j-neighbourhoods. Then, we studied the relationships between them.

Theorem 5.1 *Let (X, R, ζ_j) be a j-NS, $A \subseteq X$ and $\Im$ be an ideal on X. Then, for every $j \in J$, the collection*

$$\tau^{\Im}_{P_j} = \{A \subseteq X : P_j(x) \cap A' \in \Im \text{ for every } x \in A\}$$

is a topology on X.

Proof

The proof is similar to that of Theorem 3.1.

□

Theorem 5.2 $\tau_{P,j} \subseteq \tau^{\Im}_{P_j}$ *for every $j \in J$.*

Proof Let $A \in \tau_{P,j}$. Then, $P_j(x) \subseteq A$ for every $x \in A$. Hence, $P_j(x) \cap A' = \emptyset \in \Im$ for every $x \in A$. Therefore, $A \in \tau^{\Im}_{P_j}$. □

Example 5.3 *Consider Example 3.3*

(a) $\tau_{P,r} = \{X, \emptyset, \{\varrho_3\}, \{\varrho_4\}, \{\varrho_1,\varrho_2\}, \{\varrho_3,\varrho_4\}, \{\varrho_1,\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}\}$ *and* $\tau^{\Im}_{P_r} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_3\}, \{\varrho_4\}, \{\varrho_1,\varrho_3\}, \{\varrho_1,\varrho_4\}, \{\varrho_1,\varrho_2\}, \{\varrho_3,\varrho_4\}, \{\varrho_1,\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_3,\varrho_4\}, \{\varrho_1,\varrho_2,\varrho_4\}\}$ *Thus,* $\tau_{P,r} \subsetneq \tau^{\Im}_{P_r}$.

(b) $\tau_{P,l} = \{X, \emptyset, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_1,\varrho_4\}, \{\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_3,\varrho_4\}\}$ *and* $\tau^{\Im}_{P_l} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_1,\varrho_2\}, \{\varrho_1,\varrho_4\}, \{\varrho_1,\varrho_3\}, \{\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_3,\varrho_4\}\}$. *Thus,* $\tau_{P,l} \subsetneq \tau^{\Im}_{P_l}$.

(c) $\tau_{P,u} = \{X, \emptyset, \{\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}\}$ *and* $\tau^{\Im}_{P_u} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_3\}, \{\varrho_1,\varrho_2\}, \{\varrho_1,\varrho_3\}, \{\varrho_1,\varrho_4\}, \{\varrho_1,\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_3,\varrho_4\}\}$. *Thus,* $\tau_{P,u} \subsetneq \tau^{\Im}_{P_u}$.

(d) $\tau_{P,<r>} = \{X, \emptyset, \{\varrho_2\}, \{\varrho_1,\varrho_4\}, \{\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}\}$ *and* $\tau^{\Im}_{P_{<r>}} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_1,\varrho_2\}, \{\varrho_1,\varrho_3\}, \{\varrho_1,\varrho_4\}, \{\varrho_1,\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_3,\varrho_4\}\}$. *Thus,* $\tau_{P,<r>} \subsetneq \tau^{\Im}_{P_{<r>}}$.

(e) $\tau_{P,<u>} = \{X, \emptyset, \{\varrho_1,\varrho_2,\varrho_4\}\}$ *and* $\tau^{\Im}_{P_{<u>}} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_3\}, \{\varrho_1,\varrho_2\}, \{\varrho_1,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_3\}, \{\varrho_1,\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_3,\varrho_4\}\}$. *Thus,* $\tau_{P,<u>} \subsetneq \tau^{\Im}_{P_{<u>}}$.

Besides, $\tau_{P,j} \subseteq \tau^{\Im}_{P_j}$ *for every* $j \in \{ i, <l>, <i> \}$.

Remark 5.4 *If* $\Im = \{\emptyset\}$ *in Theorem 5.2, then* $\tau_{P,j}$ *coincides with* $\tau^{\Im}_{P_j}$ *for each* $j \in J$.

Proposition 5.5 *The followings are hold:*

(a) $\tau^{\Im}_{P_u} \subseteq \tau^{\Im}_{P_r} \subseteq \tau^{\Im}_{P_i}$;

(b) $\tau^{\Im}_{P_u} \subseteq \tau^{\Im}_{P_l} \subseteq \tau^{\Im}_{P_i}$;

(c) $\tau^{\Im}_{P_{<u>}} \subseteq \tau^{\Im}_{P_{<r>}} \subseteq \tau^{\Im}_{P_{<i>}}$;

(d) $\tau^{\Im}_{P_{<u>}} \subseteq \tau^{\Im}_{P_{<l>}} \subseteq \tau^{\Im}_{P_{<i>}}$.

Proof The proof is similar to that of Proposition 3.5. □

Also, we can see that $\tau^{\Im}_{P_r}$ and $\tau^{\Im}_{P_{<r>}}$, $\tau^{\Im}_{P_l}$ and $\tau^{\Im}_{P_{<l>}}$, $\tau^{\Im}_{P_i}$ and $\tau^{\Im}_{P_{<i>}}$, and $\tau^{\Im}_{P_u}$ and $\tau^{\Im}_{P_{<u>}}$ may not be comparable.

Theorem 5.6 $\tau^{\Im}_{P_j}$ *is finer than* $\tau^{\Im}_{C_j}$ *for every* $j \in J$.

Proof Let $A \in \tau^{\Im}_{C_j}$. Then, $C_j(x) \cap A' \in \Im$ for every $x \in A$. From Theorem 2.7, $P_j(x) \cap A' \in \Im$ for every $x \in A$. Therefore, $A \in \tau^{\Im}_{P_j}$. □

Example 5.7 *Consider Example 3.6. Then we get* $\tau^{\Im}_{P_r} = P(X)$ *and* $\tau^{\Im}_{C_r} = \{X, \emptyset, \{\varrho_1\}, \{\varrho_2\}, \{\varrho_4\}, \{\varrho_1,\varrho_2\}, \{\varrho_1,\varrho_4\}, \{\varrho_3,\varrho_4\}, \{\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_2,\varrho_4\}, \{\varrho_1,\varrho_3,\varrho_4\}, \{\varrho_2,\varrho_3,\varrho_4\}\}$. *Thus,* $\tau^{\Im}_{P_r} \neq \tau^{\Im}_{C_r}$.

6. Rough approximations induced by $\tau^{\Im}_{P_j}$-topologies

We introduced various methods to find the approximations by using generated topologies described in previous section. Then we studied properties of the new approximations and examined relations between the accuracies of our approximations.

Definition 6.1 *Let* (X, R, ζ_j) *be a* j*-NS,* $A \subseteq X$ *and* $\Im$ *be an ideal on* X*. Then, for every* $j \in J$*, a subset* A *is called* $\Im_{P_j}$*-open set if* $A \in \tau^{\Im}_{P_j}$ *and the complement of* $\Im_{P_j}$*-open set is called* $\Im_{P_j}$*-closed set. The family* $\mathcal{K}^{\Im}_{P_j}$ *of all* $\Im_{P_j}$*-closed sets is defined by* $\mathcal{K}^{\Im}_{P_j} = \{F \subseteq X : F' \in \tau^{\Im}_{P_j}\}$.

Definition 6.2 *For every $j \in J$, the $\Im_{P_j}$-lower and $\Im_{P_j}$-upper approximations of A are defined respectively by*

$$\underline{R}^{\Im}_{P_j}(A) = \cup\{G \in \tau^{\Im}_{P_j} : G \subseteq A\} = int^{\Im}_{P_j}(A)$$

$$\overline{R}^{\Im}_{P_j}(A) = \cap\{F \in \mathcal{K}^{\Im}_{P_j} : A \subseteq F\} = cl^{\Im}_{P_j}(A),$$

where $int^{\Im}_{P_j}(A)$ ($cl^{\Im}_{P_j}(A)$) represents $\Im_{P_j}$-interior ($\Im_{P_j}$-closure) of A.

Proposition 6.3 *For every $j \in J$, the followings are hold:*

(a) $\underline{R}^{\Im}_{P_j}(A) \subseteq A \subseteq \overline{R}_{P_j}(A)$;

(b) $A \subseteq B$ implies $\overline{R}^{\Im}_{P_j}(A) \subseteq \overline{R}^{\Im}_{P_j}(B)$;

(c) $A \subseteq B$ implies $\underline{R}^{\Im}_{P_j}(A) \subseteq \underline{R}^{\Im}_{P_j}(B)$;

(d) $\underline{R}^{\Im}_{P_j}(\emptyset) = \overline{R}^{\Im}_{P_j}(\emptyset) = \emptyset$

$\underline{R}^{\Im}_{P_j}(X) = \overline{R}^{\Im}_{P_j}(X) = X$;

(e) $\underline{R}^{\Im}_{P_j}(A \cup B) \supseteq \underline{R}^{\Im}_{P_j}(A) \cup \underline{R}^{\Im}_{P_j}(B)$

$\overline{R}^{\Im}_{P_j}(A \cap B) \subseteq \overline{R}^{\Im}_{P_j}(A) \cap \overline{R}^{\Im}_{P_j}(B)$;

(f) $\overline{R}^{\Im}_{P_j}(A \cup B) = \overline{R}^{\Im}_{P_j}(A) \cup \overline{R}^{\Im}_{P_j}(B)$

$\underline{R}^{\Im}_{P_j}(A \cap B) = \underline{R}^{\Im}_{P_j}(A) \cap \underline{R}^{\Im}_{P_j}(B)$;

(g) $(\underline{R}^{\Im}_{P_j}(A))' = \overline{R}^{\Im}_{P_j}(A')$

$((\overline{R}^{\Im}_{P_j}(A))' = \underline{R}^{\Im}_{P_j}(A')$;

(h) $\overline{R}^{\Im}_{P_j}(\overline{R}^{\Im}_{P_j}(A)) = \overline{R}^{\Im}_{P_j}(A)$

$\underline{R}^{\Im}_{P_j}(\underline{R}^{\Im}_{P_j}(A)) = \underline{R}^{\Im}_{P_j}(A)$.

Proof The proofs are clear from the definitions of $\Im_{P_j}$-interior and $\Im_{P_j}$-closure. □

Example 6.4 *Consider Example 5.3.*

(a) Let $A = \{\varrho_1\}$ and $B = \{\varrho_2, \varrho_3\}$. Then, we have $\overline{R}^{\Im}_{P_r}(A) = \{\varrho_1, \varrho_2\}$. and $\underline{R}^{\Im}_{P_r}(B) = \{\varrho_3\}$. Thus, $A \subsetneq \overline{R}^{\Im}_{P_r}(A)$ and $\underline{R}^{\Im}_{P_r}(B) \subsetneq B$.

(b) Let $A = \{\varrho_2, \varrho_4\}$ and $B = \{\varrho_1, \varrho_4\}$. Then, we get $\underline{R}^{\Im}_{P_u}(A) = \emptyset$, $\underline{R}^{\Im}_{P_u}(B) = \{\varrho_1, \varrho_4\}$ and $\underline{R}^{\Im}_{P_u}(A \cup B) = \{\varrho_1, \varrho_2, \varrho_4\}$. Hence, $\underline{R}^{\Im}_{P_u}(A \cup B) = \{\varrho_1, \varrho_2, \varrho_4\} \neq \underline{R}^{\Im}_{P_u}(A) \cup \underline{R}^{\Im}_{P_u}(B) = \{\varrho_1, \varrho_4\}$.

(c) Let $A = \{\varrho_1, \varrho_4\}$ and $B = \{\varrho_2, \varrho_4\}$. Then, we get $\overline{R}^{\Im}_{P_{<u>}}(A) = \{\varrho_1, \varrho_2, \varrho_4\}$, $\overline{R}^{\Im}_{P_{<u>}}(B) = \{\varrho_2, \varrho_4\}$ and $\overline{R}^{\Im}_{P_{<u>}}(A \cap B) = \{\varrho_4\}$. From here, $\overline{R}^{\Im}_{P_{<u>}}(A \cap B) = \{\varrho_4\} \neq \overline{R}^{\Im}_{P_{<u>}}(A) \cap \overline{R}^{\Im}_{P_{<u>}}(B) = \{\varrho_2, \varrho_4\}$.

(d) Let $A = \{\varrho_2, \varrho_4\}$ and $B = \{\varrho_1, \varrho_2, \varrho_3\}$. Then, we get $\overline{R}^{\Im}_{P_{<r>}}(A) = \{\varrho_2, \varrho_4\}$ and $\overline{R}^{\Im}_{P_{<r>}}(B) = X$. Thus, $\overline{R}^{\Im}_{P_{<l>}}(A) \subset \overline{R}^{\Im}_{P_{<r>}}(B)$ but $A \not\subseteq B$.

(e) Let $A = \{\varrho_2, \varrho_3\}$ and $B = \{\varrho_1, \varrho_3\}$. Then, we obtain $\underline{R}^{\Im}_{P_u}(A) = \{\varrho_3\}$ and $\underline{R}^{\Im}_{P_u}(B) = \{\varrho_1, \varrho_3\}$. Therefore, $\underline{R}^{\Im}_{P_u}(A) \subset \underline{R}^{\Im}_{P_u}(B)$ but $A \not\subseteq B$.

Theorem 6.5 *The followings are hold:*

(a) $\underline{R}^{\Im}_{C_j}(A) \subseteq \underline{R}^{\Im}_{P_j}(A)$;

(b) $\overline{R}^{\Im}_{P_j}(A) \subseteq \overline{R}^{\Im}_{C_j}(A)$.

Proof The proofs are obvious. □

Proposition 6.6 *The followings are hold:*

(a) $\underline{R}^{\Im}_{P_u}(A) \subseteq \underline{R}^{\Im}_{P_r}(A) \subseteq \underline{R}^{\Im}_{P_i}(A)$;

(b) $\underline{R}^{\Im}_{P_u}(A) \subseteq \underline{R}^{\Im}_{P_l}(A) \subseteq \underline{R}^{\Im}_{P_i}(A)$;

(c) $\underline{R}^{\Im}_{P_{<u>}}(A) \subseteq \underline{R}^{\Im}_{P_{<r>}}(A) \subseteq \underline{R}^{\Im}_{P_{<i>}}(A)$;

(d) $\underline{R}^{\Im}_{P_{<u>}}(A) \subseteq \underline{R}^{\Im}_{P_{<l>}}(A) \subseteq \underline{R}^{\Im}_{P_{<i>}}(A)$.

Proof The proofs are clear from Proposition 5.5. □

Proposition 6.7 *The followings are hold:*

(a) $\overline{R}^{\Im}_{P_i}(A) \subseteq \overline{R}^{\Im}_{P_r}(A) \subseteq \overline{R}^{\Im}_{P_u}(A)$;

(b) $\overline{R}^{\Im}_{P_i}(A) \subseteq \overline{R}^{\Im}_{P_l}(A) \subseteq \overline{R}^{\Im}_{P_u}(A)$;

(c) $\overline{R}^{\Im}_{P_{<i>}}(A) \subseteq \overline{R}^{\Im}_{P_{<r>}}(A) \subseteq \overline{R}^{\Im}_{P_{<u>}}(A)$;

(d) $\overline{R}^{\Im}_{P_{<i>}}(A) \subseteq \overline{R}^{\Im}_{P_{<l>}}(A) \subseteq \overline{R}^{\Im}_{P_{<u>}}(A)$.

Proof The proofs are clear from Proposition 5.5. □

Definition 6.8 *For every $j \in J$, a subset A is called $\Im_{P_j}$-exact set if $\overline{R}^{\Im}_{P_j}(A) = \underline{R}^{\Im}_{P_j}(A)$. Otherwise, A is called $\Im_{P_j}$-rough set.*

Proposition 6.9 *Every $\Im_{C_j}$-exact set in X is $\Im_{P_j}$-exact (equivalently, every $\Im_{P_j}$-rough set in X is $\Im_{C_j}$-rough) for every $j \in J$.*

Proof The proof is obvious from Proposition 6.3 and Theorem 6.5. □

Example 6.10 *Consider Example 5.7. Then, $A = \{\varrho_1, \varrho_2, \varrho_4\}$ is $\Im_{P_r}$-exact but it is not $\Im_{C_r}$-exact.*

Definition 6.11 *For every $j \in J$, the $\Im_{P_j}$-boundary regions and $\Im_{P_j}$-accuracy of approximations of A are defined respectively by*

$$B^{\Im}_{P_j}(A) = \overline{R}^{\Im}_{P_j}(A) - \underline{R}^{\Im}_{P_j}(A).$$

$$\sigma^{\Im}_{P_j}(A) = \frac{|\underline{R}^{\Im}_{P_j}(A)|}{|\overline{R}^{\Im}_{P_j}(A)|}, \text{ where } |\overline{R}^{\Im}_{P_j}(A)| \neq 0.$$

Corollary 6.12 *For every $j \in J$, then the followings are hold:*

(a) $B^{\Im}_{P_j}(A) \subseteq B^{\Im}_{C_j}(A)$;

(b) $\sigma^{\Im}_{C_j}(A) \leq \sigma^{\Im}_{P_j}(A)$.

Corollary 6.13 *The followings are hold:*

(a) $B^{\Im}_{P_i}(A) \subseteq B^{\Im}_{P_r}(A) \subseteq B^{\Im}_{P_u}(A)$;

(b) $B^{\Im}_{P_i}(A) \subseteq B^{\Im}_{P_l}(A) \subseteq B^{\Im}_{P_u}(A)$;

(c) $B^{\Im}_{P_{<i>}}(A) \subseteq B^{\Im}_{P_{<r>}}(A) \subseteq B^{\Im}_{P_{<u>}}(A)$;

(d) $B^{\Im}_{P_{<i>}}(A) \subseteq B^{\Im}_{P_{<l>}}(A) \subseteq B^{\Im}_{P_{<u>}}(A)$.

Corollary 6.14 *The followings are hold:*

(a) $\sigma^{\Im}_{P_u}(A) \leq \sigma^{\Im}_{P_r}(A) \leq \sigma^{\Im}_{P_i}(A)$;

(b) $\sigma^{\Im}_{P_u}(A) \leq \sigma^{\Im}_{P_l}(A) \leq \sigma^{\Im}_{P_i}(A)$;

(c) $\sigma^{\Im}_{P_{<u>}}(A) \leq \sigma^{\Im}_{P_{<r>}}(A) \leq \sigma^{\Im}_{P_{<i>}}(A)$;

(d) $\sigma^{\Im}_{P_{<u>}}(A) \leq \sigma^{\Im}_{P_{<l>}}(A) \leq \sigma^{\Im}_{P_{<i>}}(A)$.

Remark 6.15

(a) In Table 3, the boundary regions and accuracy measures are calculated by using our approach according to Example 5.3. Thus, it is seen that $\Im_{P_i}$*-accuracy is more precise than* $\Im_{P_j}$*-accuracy for* $j = \{r, l, u\}$.

(b) In Table 4, the lower approximations, upper approximations, boundary regions and accuracy measures are calculated by using our approaches according to Example 3.3. Thus, it is seen that $\Im_{P_j}$*-accuracy is more precise than* $\Im_{C_j}$*-accuracy.*

7. Application

The main purpose of this section is to give simple practice example in order to compare Hosny's approach [8] and one of our approaches. We used a reflexive relation generated from a multivalued information system.

Example 7.1 *Consider the following multivalued information system as in Table 5 containing data about 5 persons.*
Let R_k *be the binary relation such that*

$$xR_k\, y \text{ if and only if } v_k(x) \subseteq v_k(y) \text{ for each } k = 1, 2, 3$$

where
$R_1 = Languages = \{English, German, Italy, France\}$
$R_2 = Sports = \{Handball, Basketball, Voleyball, Tennis\}$
$R_3 = Skills = \{Swimming, Climbing, Running, Fishing\}$.
In order to represent to the set of all condition attributes, we generated the following relation from all above relation as follows: $xR = \bigcap xR_k$. *So we have* $1R = \{1, 3\}$, $2R = \{2\}$, $3R = \{3\}$, $4R = \{4\}$, $5R = \{5\}$. *Then* $P_R(1) = \{1\}$, $P_R(2) = \{2\}$, $P_R(3) = \{3\}$, $P_R(4) = \{4\}$ *and* $P_R(5) = \{5\}$. *Suppose that* $A = \{2, 3, 5\}$ *(Decision: accept) and* $B = \{1, 4\}$ *(Decision: reject). Then, taking the ideal* $\mathcal{I} = \{\emptyset, \{2\}, \{4\}, \{2, 4\}\}$, *we get Table 6.*

From this practical example, accuracy measures of our approach are higher than Hosny's approach [8]. Hence, these approaches are very useful in rough set theory.

Table 3. Comparison between the boundary regions and accuracy measures by using our approach for $j = \{r, l, u, i\}$.

A	for $j = r$		for j=l		for j=u		for j=i	
	$B^{\Im}_{P_r}(A)$	$\sigma^{\Im}_{P_r}(A)$	$B^{\Im}_{P_l}(A)$	$\sigma^{\Im}_{P_l}(A)$	$B^{\Im}_{P_u}(A)$	$\sigma^{\Im}_{P_u}(A)$	$B^{\Im}_{P_i}(A)$	$\sigma^{\Im}_{P_i}(A)$
$\{\varrho_1\}$	$\{\varrho_2\}$	$\frac{1}{2}$	$\{\varrho_4\}$	$\frac{1}{2}$	$\{\varrho_2, \varrho_4\}$	$\frac{1}{3}$	$\emptyset$	1
$\{\varrho_2\}$	$\{\varrho_2\}$	0	$\emptyset$	1	$\{\varrho_2\}$	0	$\emptyset$	1
$\{\varrho_3\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\varrho_4\}$	$\emptyset$	1	$\{\varrho_4\}$	0	$\{\varrho_4\}$	0	$\emptyset$	1
$\{\varrho_1, \varrho_2\}$	$\emptyset$	1	$\{\varrho_4\}$	$\frac{2}{3}$	$\{\varrho_4\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\varrho_1, \varrho_3\}$	$\{\varrho_2\}$	$\frac{2}{3}$	$\{\varrho_4\}$	$\frac{2}{3}$	$\{\varrho_2, \varrho_4\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_1, \varrho_4\}$	$\{\varrho_2\}$	$\frac{2}{3}$	$\emptyset$	1	$\{\varrho_2\}$	$\frac{2}{3}$	$\emptyset$	1
$\{\varrho_2, \varrho_3\}$	$\{\varrho_2\}$	$\frac{1}{2}$	$\emptyset$	1	$\{\varrho_2\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_2, \varrho_4\}$	$\{\varrho_2\}$	$\frac{1}{2}$	$\{\varrho_4\}$	$\frac{1}{2}$	$\{\varrho_2, \varrho_4\}$	0	$\emptyset$	1
$\{\varrho_3, \varrho_4\}$	$\emptyset$	1	$\{\varrho_4\}$	$\frac{1}{2}$	$\{\varrho_4\}$	$\frac{1}{2}$	$\emptyset$	1
$\{\varrho_1, \varrho_2, \varrho_3\}$	$\emptyset$	1	$\{\varrho_4\}$	$\frac{3}{4}$	$\{\varrho_4\}$	$\frac{3}{4}$	$\emptyset$	1
$\{\varrho_1, \varrho_2, \varrho_4\}$	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1
$\{\varrho_1, \varrho_3, \varrho_4\}$	$\{\varrho_2\}$	$\frac{3}{4}$	$\emptyset$	1	$\{\varrho_2\}$	$\frac{3}{4}$	$\emptyset$	1
$\{\varrho_2, \varrho_3, \varrho_4\}$	$\{\varrho_2\}$	$\frac{2}{3}$	$\{\varrho_4\}$	$\frac{2}{3}$	$\{\varrho_2, \varrho_4\}$	$\frac{1}{3}$	$\emptyset$	1
X	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1	$\emptyset$	1

Table 4. Comparison between the lower approximations, upper approximations, boundary regions and accuracy measures by using our approaches for $j = r$.

A	Our approach according to P_j-neighbourhoods				Our approach according to C_j-neighbourhoods			
	$\underline{R}^{\Im}_{P_r}(A)$	$\overline{R}^{\Im}_{P_r}(A)$	$B^{\Im}_{P_r}(A)$	$\sigma^{\Im}_{P_r}(A)$	$\underline{R}^{\Im}_{C_r}(A)$	$\overline{R}^{\Im}_{C_r}(A)$	$B^{\Im}_{C_r}(A)$	$\sigma^{\Im}_{C_r}(A)$
$\{\varrho_1\}$	$\{\varrho_1\}$	$\{\varrho_1, \varrho_2\}$	$\{\varrho_2\}$	$\frac{1}{2}$	$\emptyset$	$\{\varrho_1, \varrho_2\}$	$\{\varrho_1, \varrho_2\}$	0
$\{\varrho_2\}$	$\emptyset$	$\{\varrho_2\}$	$\{\varrho_2\}$	0	$\emptyset$	$\{\varrho_2\}$	$\{\varrho_2\}$	0
$\{\varrho_3\}$	$\{\varrho_3\}$	$\{\varrho_3\}$	$\emptyset$	1	$\{\varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_1, \varrho_2\}$	$\frac{1}{3}$
$\{\varrho_4\}$	$\{\varrho_4\}$	$\{\varrho_4\}$	$\emptyset$	1	$\{\varrho_4\}$	$\{\varrho_4\}$	$\emptyset$	1
$\{\varrho_1, \varrho_2\}$	$\{\varrho_1, \varrho_2\}$	$\{\varrho_1, \varrho_2\}$	$\emptyset$	1	$\emptyset$	$\{\varrho_1, \varrho_2\}$	$\{\varrho_1, \varrho_2\}$	0
$\{\varrho_1, \varrho_3\}$	$\{\varrho_1, \varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_2\}$	$\frac{2}{3}$	$\{\varrho_1, \varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_2\}$	$\frac{2}{3}$
$\{\varrho_1, \varrho_4\}$	$\{\varrho_1, \varrho_4\}$	$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_2\}$	$\frac{2}{3}$	$\{\varrho_4\}$	$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_1, \varrho_2\}$	$\frac{1}{3}$
$\{\varrho_2, \varrho_3\}$	$\{\varrho_3\}$	$\{\varrho_2, \varrho_3\}$	$\{\varrho_2\}$	$\frac{1}{2}$	$\{\varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_1, \varrho_2\}$	$\frac{1}{3}$
$\{\varrho_2, \varrho_4\}$	$\{\varrho_4\}$	$\{\varrho_2, \varrho_4\}$	$\{\varrho_2\}$	$\frac{1}{2}$	$\{\varrho_4\}$	$\{\varrho_2, \varrho_4\}$	$\{\varrho_2\}$	$\frac{1}{2}$
$\{\varrho_3, \varrho_4\}$	$\{\varrho_3, \varrho_4\}$	$\{\varrho_3, \varrho_4\}$	$\emptyset$	1	$\{\varrho_3, \varrho_4\}$	X	$\{\varrho_1, \varrho_2\}$	$\frac{1}{2}$
$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\emptyset$	1	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\emptyset$	1
$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_1, \varrho_2, \varrho_4\}$	$\emptyset$	1	$\{\varrho_4\}$	$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_1, \varrho_2\}$	$\frac{1}{3}$
$\{\varrho_1, \varrho_3, \varrho_4\}$	$\{\varrho_1, \varrho_3, \varrho_4\}$	X	$\{\varrho_2\}$	$\frac{3}{4}$	$\{\varrho_1, \varrho_3, \varrho_4\}$	X	$\{\varrho_2\}$	$\frac{3}{4}$
$\{\varrho_2, \varrho_3, \varrho_4\}$	$\{\varrho_3, \varrho_4\}$	$\{\varrho_2, \varrho_3, \varrho_4\}$	$\{\varrho_2\}$	$\frac{2}{3}$	$\{\varrho_3, \varrho_4\}$	X	$\{\varrho_1, \varrho_2\}$	$\frac{1}{2}$
X	X	X	$\emptyset$	1	X	X	$\emptyset$	1

Table 5. Multivalued information system (MVIS).

Person	Languages	Sports	Skills	Decision
1	{I}	{H}	{S}	Reject
2	{E,F,G}	{B,T}	{S,F}	Accept
3	{E,I}	{H,T}	{S,R}	Accept
4	{G}	{V}	{F}	Reject
5	{E,G}	{V,B,T}	{C,R}	Accept

Table 6. Comparisons between Hosny's approach [8] and our approach.

	Hosny's approach [8]			Our approach		
	Lower	Upper	Accuracy	Lower	Upper	Accuracy
$\{2,3,5\}$	$\{2,3,5\}$	$\{1,2,3,5\}$	$\frac{3}{4}$	$\{2,3,5\}$	$\{2,3,5\}$	1
$\{1,4\}$	$\{4\}$	$\{1,4\}$	$\frac{1}{2}$	$\{1,4\}$	$\{1,4\}$	1

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