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A series evaluation technique based on a modified Abel lemma

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Abstract: We introduce a technique for determining infinite series identities through something of a combination of the modified Abel lemma on summation by parts and a method of undetermined coefficients. We succeed in applying our technique in our proving a nontrivial variant of Gauss' hypergeometric identity, giving us an evaluation for a family of ${}_3F_2(1)$-series with three free parameters, and to establish a ${}_3F_2(-1)$-variant of Kummer's hypergeometric identity. Also, we apply the technique upon which this article is based to formulate a new and simplified proof of a remarkable series evaluation recently derived by Cantarini via the generalized Clebsch–Gordan integral.

Key words: Abel lemma, Kummer's identity, closed form, generalized hypergeometric functions

1. Introduction

Niels Henrik Abel's original 1826 formulation [1] of what is now known as *Abel's lemma on summation by parts* has an important place in the history of mathematical analysis. What is referred to as the *modified Abel lemma on summation by parts*, as in the summation identity in (1.1) below, has generated much in the way of progress in the field of special functions theory, when it comes to the study of hypergeometric functions [14–16, 20, 21, 32, 33, 37] and harmonic sums [12, 17, 34, 35]. If we let ∇ and Δ be such that $\nabla\tau_n = \tau_n - \tau_{n-1}$ and $\Delta\tau_n = \tau_n - \tau_{n+1}$ for a sequence $(\tau_n : n \in \mathbb{N}_0)$, then

$$\sum_{n=1}^{\infty} B_n \nabla A_n = \left(\lim_{m\to\infty} A_m B_{m+1}\right) - A_0 B_1 + \sum_{n=1}^{\infty} A_n \Delta B_n \tag{1.1}$$

if this limit exists and one of the two infinite sums given above converges. In this article, we introduce a series evaluation technique that may be thought of as being given by a combination of the modified Abel lemma in (1.1) and a clever use of a method of undetermined coefficients.

As in [10], our article is inspired by much about the history of the application of hypergeometric transforms in relation to symbolic/closed-form evaluations, making reference, as in [10], to Borwein and Crandall's article [4] on the importance of and definitions associated with the phrase *closed-form evaluation*. Our explorations of the summation lemma in (1.1), in conjunction with our recent work on WZ theory [6, 7] and on lemniscate-like constants [7, 9], have led us to construct a new and simplified proof of a series due to Cantarini inspired by the work of Ramanujan. Generalizing this alternate proof led us to the technique that is outlined in Section 1.2 and

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that is the basis of our article. In Section 2, we apply this technique to determine new and nontrivial variants of classical hypergeometric identities. In Section 3, we briefly review Cantarini's Fourier–Legendre-based proof of the remarkable hypergeometric formula

$$\sum_{n=0}^{\infty}\left(-\frac{1}{64}\right)^{n}\binom{2n}{n}^{3}\frac{(4n+1)^2}{(4n-1)(4n+3)}=-\frac{32\left(2+\sqrt{2}\right)\Gamma^{2}\left(\frac{1}{4}\right)}{\Gamma^{4}\left(\frac{1}{8}\right)}, \tag{1.2}$$

which Maple, Mathematica, MATLAB, etc., cannot obtain, letting Γ denote the Γ-function as defined by Euler integral

$$\Gamma(x)=\int_{0}^{\infty}u^{x-1}e^{-u}\,du.$$

In Section 3.2, we apply our main technique, as formulated in Section 1.2 below, to obtain a simplified and dramatically different proof of Cantarini's formula shown in (1.2).

In Section 1.1 below, we briefly review some previous mathematical literature that is relevant for the purposes of this article. For the time being, we briefly review some preliminary notation.

The shifted factorial function, which is also referred to as the *Pochhammer symbol*, is such that

$$(x)_0=1 \quad \text{and} \quad (x)_n=x(x+1)\cdots(x+n-1)$$

for $n\in\mathbb{N}$ (see, e.g., [26], section $5.2(iii)$). Generalized hypergeometric series are defined and often denoted as follows:

$${}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p;\\ \beta_1,\dots,\beta_q;\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}$$

(see, e.g., [26], section $16.2(i)$). It is often convenient to write:

$$\begin{bmatrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{bmatrix}_n=\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}.$$

We later refer to the family of *Legendre polynomials*, which may be defined according to the Rodrigues formula:

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

(see, e.g., [26], section $14.7(ii)$, equation $14.7.13$).

1.1. Some relevant literature

In 2015, Zhang [37] employed the modified Abel lemma on summation by parts to extend both the Watson and Whipple sums, and to determine Ramanujan-like series for $\frac{1}{\pi}$. This serves as a source of inspiration for much of our work. However, the methods used in [37] are inequivalent to our summation technique, as formulated in Section 1.2. As in [37], writing

$$\Omega(a,b,c,d)=\sum_{n=0}^{\infty}\begin{bmatrix}a,c,\frac{b+d+1}{2}\\ b+1,d+1,\frac{a+c+1}{2}\end{bmatrix}_n$$

to denote a common extension of the Watson and Whipple sums, linear relations that allow us to write $\Omega(a,b,c,d)$ in terms of $\Omega(a+p,b+q,c+r,d+s)$ are obtained via the modified Abel lemma shown in (1.1), and such recurrences for Ω are utilized in [37] to derive fast convergent hypergeometric series for $\frac{1}{\pi}$. In contrast, our summation technique, as described below in Section 1.2, is formulated in a much more general way, without relying on Watson/Whipple-type sums, and does not rely on "iteration patterns" as in [37], but instead is based on the idea of choosing a sequence $(A_n : n \in \mathbb{N}_0)$, given a hypergeometric sequence $(B_n : n \in \mathbb{N}_0)$, such that products of the form

$$A_n \left(1 - \frac{B_{n+1}}{B_n}\right)$$

simplify in a certain way, as explained below.

In reference to the recursive techniques from [37], the modified Abel lemma, as displayed in (1.1), was used previously and in something of a similar fashion in [22] (cf. [15]), in order to determine recursions involving the partial sums of series of the following form:

$$\sum_{n=0}^{\infty}(a+2n)\begin{bmatrix} b,c,d,e, \\ 1+a-b,1-a-c,1-a-d,1+a-e \end{bmatrix}_n .$$

Again, this recursive approach stands in constrast to our methodologies that are described in Section 1.2. With regard to [15, 22, 37], the modified Abel lemma is used in a similarly recursive fashion in [16] to determine ${}_4F_3\left(\frac{3}{4}\right)$-identities. For some further research on hypergeometric functions and generalizations relevant to our work, see [13, 24, 27, 30].

1.2. Main technique

Our main technique may be broadly summarized with the six steps listed below. Successfully going through with the set-up suggested as follows, within the context of a given application, can be quite intricate, as in with our proof of the Kummer-type ${}_3F_2(-1)$-identity introduced in Section 2.1.

Step 1: Set the sequence $(B_n : n \in \mathbb{N})$ to be hypergeometric.

Step 2: Simplify the rational expression $r_1(n) = \frac{B_{n+1}}{B_n}$.

Step 3: Rewrite $B_n - B_{n+1} = B_n\,(1 - r_1(n))$ in the latter series in (1.1) accordingly.

Step 4: Set A_n to be an expression involving a rational function $r_2(n)$ as a factor. We set this rational function $r_2(n)$ to be of the form $\frac{1}{a_1 n + a_2}$ or $\frac{a_3 n + a_4}{a_1 n + a_2}$, where each expression of the form a_i is a scalar determined as follows, writing $A_n = C_n r_2(n)$, for a sequence $(C_n : n \in \mathbb{N}_0)$.

Step 5: According to our construction, the summand of the latter series in (1.1) is of the form

$$B_n C_n r_2(n)\,(1 - r_1(n)) .$$

Apply partial fraction decomposition to $r_2(n)\,(1 - r_1(n))$.

Step 6: The above partial fraction decomposition is to involve a term given by a scalar multiple of $\frac{1}{a_1 n + a_2}$. Solve for the unknown coefficients of the form a_i, if possible, so that this term vanishes.

After the application of the above steps, we would want, ideally, the modified Abel lemma, as in (1.1), to provide us with new series identities: intuitively, and in general, finding closed forms for binomial sums involving summand factors of the form $\frac{1}{mx+b}$ is much more manageable for $m \in \{1, 2\}$, and often becomes very difficult even for $m = 3$ or $m = 4$ (see relevant material from [8, 9]), and, in this regard, the above steps are "designed" to try to eliminate undesirable factors of the form $\frac{1}{mx+b}$ for higher-order values m. This is clarified in our applications, as below, of the above procedure. We let the terms A*-sequence* and B*-sequence* refer, respectively, to the sequences $(A_n : n \in \mathbb{N})$ and $(B_n : n \in \mathbb{N})$ involved in the above algorithm.

2. Variants of classical hypergeometric identities

We consider using our Abel summation-based method so as to be applicable to hypergeometric sums involving multiple parameters. In this direction, a natural place to start off would be with our making use of classical hypergeometric series. We recall Gauss' famous hypergeometric identity whereby

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} \; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.1}$$

for $\Re(c-a-b) > 0$. As it turns out, we may apply the technique in Section 1.2 using Gauss' identity, so as to obtain a proof of a nontrivial ${}_3F_2$-variant of (2.1) involving three free parameters. It appears that this ${}_3F_2(1)$-identity, as given in Theorem 2.1, is new.

In view of the summand of the hypergeometric expression in (2.1), we set

$$B_n = \frac{(a)_n (b)_n}{(c)_n n!}, \tag{2.2}$$

and we proceed to follow the steps listed in Section 1.2. The second series in (1.1) may then be written as

$$\sum_{n=1}^{\infty} B_n A_n \left(1 - \frac{(n+a)(n+b)}{(n+1)(n+c)}\right), \tag{2.3}$$

so we need, according to the steps in Section 1.2, to use a method of undetermined coefficients to find a suitable rational function A_n such that the product of the last two summand factors in (2.3) involves a vanishing term as described in Section 1.2. This leads us to the partial fraction decomposition

$$\frac{1}{n + \frac{ab-c}{a+b-c-1}} \left(1 - \frac{(n+a)(n+b)}{(n+1)(n+c)}\right) = \frac{1-a-b+c}{(c-1)(n+1)} - \frac{1-a-b+c}{(c-1)(n+c)}, \tag{2.4}$$

which, in turn, leads us to the ${}_3F_2(1)$-identity given in Theorem 2.1. Maple, Mathematica, MATLAB, etc., cannot evaluate the below ${}_3F_2(1)$-expression for free parameters a, b, and γ.

Theorem 2.1 *For free parameters a, b, and γ, the identity*

$$ {}_3F_2\left[\begin{array}{r} a, b, \gamma; \\ a+b+\frac{a+b-ab-1}{\gamma}-1, \gamma+2; \end{array} 1\right] = \frac{(\gamma+1)\Gamma\left(a+b+\frac{a+b-ab-1}{\gamma}-1\right)\Gamma\left(\frac{a+b-ab+\gamma-1}{\gamma}\right)}{\Gamma\left(a+\frac{a+b-ab-1}{\gamma}\right)\Gamma\left(b+\frac{a+b-ab-1}{\gamma}\right)} $$

holds if the left-hand side is convergent and the right-hand side is defined for given values of a, b, and γ.

Proof In the modified Abel lemma, we set the B-sequence to be as in (2.2). As suggested in (2.4), we are to set the A-sequence in (1.1) to be as follows:

$$ A_n = \frac{1}{n - \frac{c-ab}{a+b-c-1}}, $$

letting a, b, and c be suitably bounded parameters. So, making use of the partial fraction decomposition in (2.4), a direct application of the modified Abel lemma then gives us the equality of

$$ \sum_{n=0}^{\infty} \frac{1}{(1-a-b+ab(a+b-c-1)n)(ab-c+(a+b-c-1)n)} \frac{(a)_n(b)_n}{n!(c)_n} $$

and

$$ -\frac{1}{(1-a)(1-b)(1-a-b+c)} - \frac{1}{1-a-b+c} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} \frac{1}{(n+1)(n+c)}. $$

This is easily seen to be equivalent to the equality

$$ {}_3F_2\left[\begin{array}{r} a, b, \frac{(a-1)(b-1)}{a+b-c-1}; \\ c, \frac{a+b+ab-2c-1}{a+b-c-1}; \end{array} 1\right] = \frac{(c-ab)\Gamma(c)\Gamma(1-a-b+c)}{\Gamma(1-a+c)\Gamma(1-b+c)}, $$

and the desired result then follows by making the substituion $\gamma = \frac{(a-1)(b-1)}{a+b-c-1}$. □

It is unclear as to how the telescoping series-based techniques applied in [18] to prove ${}_3F_2(1)$-identities could be used to prove Theorem 2.1. We encourage the use of the telescoping methods from [18] in conjunction with our results. Karlsson, in [23], introduced techniques for reducing ${}_3F_2(1)$-series into linear combinations of ${}_2F_1(1)$-series (cf. [29]), and this serves as something of a starting point for generalizing our proof of Theorem 2.1 using linear combinations of ${}_2F_1(1)$-expressions. For the sake of brevity, we leave this to a future project.

2.1. A variant of Kummer's identity

Since we have successfully applied the technique in Section 1.2 to Gauss' hypergeometric identity, this raises the question as to how a similar kind of approach could be applied to other classical ${}_2F_1$-identities. So, we are led to mimic our proof of Theorem 2.1 using Kummer's identity

$$ {}_2F_1\left[\begin{array}{r} a, b; \\ 1+a-b; \end{array} -1\right] = \frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma(a-b+1)}{\Gamma(a+1)\Gamma\left(\frac{a}{2}-b+1\right)} \tag{2.5} $$

for suitably bounded a and b. See, for example, [26], section 15.4(iii), formula 15.4.26. By setting, in the summation identity in (1.1), the B-sequence to be the summand of the left-hand side of (2.5), this gives us, following the procedure in Section 1.2, that the series on the right-hand side of (1.1) may be rewritten as:

$$\sum_{n=1}^{\infty} B_n A_n \left(1 + \frac{(n+a)(n+b)}{(n+1)(n+a-b+1)}\right).$$

Again, following through with the steps in Section 1.2, writing c_i in place of a_i to avoid confusion with the parameters in (2.5), we apply partial fraction decomposition to the product

$$\frac{1}{c_1 n + c_2}\left(1 + \frac{(n+a)(n+b)}{(n+1)(n+a-b+1)}\right),$$

to obtain:

$$\frac{-ab + a + 2b^2 - 3b + 1}{(a-b)(ac_1 - bc_1 + c_1 - c_2)(n+a-b+1)} + \frac{ab - a - b + 1}{(a-b)(c_2 - c_1)(n+1)} + \\ \frac{abc_1^2 + ac_1^2 - 2ac_1c_2 - bc_1^2 + c_1^2 - 2c_1c_2 + 2c_2^2}{(c_2 - c_1)(-ac_1 + bc_1 - c_1 + c_2)(c_1 n + c_2)}.$$

Setting the scalar coefficient of $\frac{1}{c_1 n + c_2}$ to 0, and solving for c_1 and c_2, we obtain that

$$c_2 = \frac{1}{2}\left(c_1 + ac_1 \pm \sqrt{-c_1^2 + a^2c_1^2 + 2bc_1^2 - 2abc_1^2}\right),$$

and this leads us to the following hypergeometric identity.

Theorem 2.2 *For suitably bounded parameters a and b, the hypergeometric series*

$${}_3F_2\left[\begin{matrix} a, b, c; \\ c+2, 1+a-b; \end{matrix} \; -1\right]$$

evaluates as

$$\frac{\sqrt{\pi}2^{-a}c(c+1)\Gamma(a-b+1)\left(\frac{2(a-2c-1)}{(a-1)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(a-2b+3)\right)} + \frac{2}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}\right)}{b-1},$$

writing $c = -\frac{1}{2} + \frac{a}{2} - \frac{1}{2}\sqrt{(a-1)(1+a-2b)}$.

Proof In the modified Abel lemma, as in (1.1), we set

$$A_n = \frac{1}{n + \frac{1}{2}\left(1 + a - \sqrt{(-1+a)(1+a-2b)}\right)}$$

and

$$B_n = \frac{(-1)^n (a)_n (b)_n}{n!(1+a-b)_n}.$$

This gives us that

$$-\frac{4\,{}_3F_2\left[\begin{matrix} a, \frac{a}{2}-\frac{1}{2}\sqrt{(a-1)(a-2b+1)}-\frac{1}{2}, b; \\ \frac{a}{2}-\frac{1}{2}\sqrt{(a-1)(a-2b+1)}+\frac{3}{2}, a-b+1; \end{matrix} \ -1\right]}{\left(\sqrt{(a-1)(a-2b+1)}-a-1\right)\left(\sqrt{(a-1)(a-2b+1)}-a+1\right)}$$

which equals the following:

$$-\frac{2}{-1+a^2+b-a\left(\sqrt{(a-1)(1+a-2b)}+b\right)}+$$

$$\left(\lim_{m\to\infty} A_m B_{m+1}\right) - A_0 B_1 +$$

$$\sum_{n=1}^{\infty} B_n \left(\frac{1}{1+n} + \frac{\sqrt{(a-1)(1+a-2b)}}{(1+n)(1+a-b+n)} + \frac{b+n}{(1+n)(1+a-b+n)}\right).$$

The right-hand side may be written as a combination of closed-form expressions, values of the Γ-function, and expressions of the following form:

$$ {}_2F_1\left[\begin{matrix} a, b; \\ 2+a-b; \end{matrix} \ -1\right]. \tag{2.6}$$

Vidunas' work [31] on generalizing Kummer's identity is not applied to series of the above form, but by setting

$$A_n = \frac{(n+1)\Gamma(n+a+1)}{a\Gamma(a)\Gamma(n+2)}$$

and

$$B_n = \frac{(-1)^n (b)_n}{(1+a-b)_n}$$

in the modified Abel lemma, we can show that (2.6) evaluates as

$$\frac{\sqrt{\pi}2^{-a}\left(\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(a-2b+3)\right)} - \frac{1}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}\right)\Gamma(a-b+2)}{b-1},$$

and this ${}_2F_1(-1)$-identity is a special case of Theorem 3 from [28]. The desired result then follows by setting

$$c = -\frac{1}{2} + \frac{a}{2} - \frac{1}{2}\sqrt{(a-1)(1+a-2b)},$$

as above. $\square$

Example 2.3 *We have that*

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{6}, \frac{1}{2}, \frac{4}{3}; \\ \frac{11}{6}, \frac{11}{6}; \end{matrix} \ -1\right] = \frac{25}{24\sqrt{3}} + \frac{5\ 2^{2/3}\sqrt{\pi}\Gamma\left(\frac{11}{6}\right)}{\Gamma^2\left(\frac{1}{6}\right)},$$

as a direct consequence of Theorem 2.2. Maple, Mathematica, MATLAB, etc., are not able to evaluate the above series.

Example 2.4 *The equality*

$$ {}_3F_2\left[\begin{matrix} -\frac{2}{15}, \frac{29}{45}, \frac{7}{5}; \\ \frac{79}{45}, \frac{28}{15}; \end{matrix} \ -1\right] = \frac{39\sqrt{\pi}\Gamma\left(\frac{79}{45}\right)}{40\ 2^{2/5}\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{7}{10}\right)} + \frac{13\sqrt{\pi}\Gamma\left(\frac{79}{45}\right)}{40\ 2^{2/5}\Gamma\left(\frac{19}{18}\right)\Gamma\left(\frac{6}{5}\right)} $$

follows directly from our variant of Kummer's theorem. Maple, Mathematica, MATLAB, etc., cannot evaluate the above series.

Our recent work [19] concerned applications of an extension with three integer parameters of Kummer's summation theorem. We encourage the generalization of Theorem 2.2 using the generalization of Kummer's theorem from [19].

The applications in [21] of the modified Abel lemma in the determination of terminating hypergeometric series identities motivate our applying our summation technique from Section 1.2 to evaluate finite hypergeometric sums. For the sake of brevity, we leave this for a future research project.

2.2. A curious "nonpower" series identity

What is typically meant by a power series refers to an expression of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$ where a_n is a fixed scalar for each member n of $\mathbb{N}_0$, i.e. so that the value of a_n does not depend on x. So, an expression such as

$$ \sum_{n=0}^{\infty} b_n(x)(x-c)^n \tag{2.7} $$

where $b_n(x)$ depends on x nontrivially and does not contain a factor that cancels with $(x-c)^n$, would not typically be considered as a "power series" per se. Our experimentation with the technique in Section 1.2 has led us to discover an interesting result concerning series of this latter form, i.e. "nonpower" series as in (2.7).

If we set $B_n = \frac{x^n}{\binom{2n}{n}}$ in the modified Abel lemma, we would need to find an expression $r_2(n) = \frac{1}{a_1 n + a_2}$, presumably dependent on the variable x, such that: in the partial fraction decomposition of

$$ \frac{1}{a_1 n + a_2}\left(1 - \frac{x(n+1)}{2(2n+1)}\right), $$

the coefficient of $\frac{1}{a_1 n + a_2}$ is zero. Following through with the method of undetermined coefficients according to Section 1.2, we are led to find that:

$$ \frac{1}{n + \frac{x-2}{x-4}}\left(1 - \frac{x(n+1)}{2(2n+1)}\right) = \frac{4-x}{2(2n+1)}. \tag{2.8} $$

This leads us to the following result, which we later apply to prove new ${}_3F_2$-values.

Theorem 2.5 *The identity*

$$ \sum_{n=0}^{\infty} \frac{x^n}{\binom{2n}{n}(nx-4n+2)(nx+x-4n-2)} = \frac{\frac{x}{2} - \frac{2\sqrt{(4-x)x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)}{x} - 2}{(x-4)^2} $$

holds for suitably bounded x.

Proof As above, we set B_n to be equal to $\frac{x^n}{\binom{2n}{n}}$, as in (1.1). In (1.1), we also let, in view of (2.8), the sequence $(A_n : n \in \mathbb{N}_0)$ be such that

$$A_n = \frac{1}{n + \frac{x-2}{x-4}}$$

for all $n \in \mathbb{N}_0$. So, since we may rewrite the infinite series in

$$\left(\lim_{m\to\infty} A_m B_{m+1}\right) - A_0 B_1 + \sum_{n=1}^{\infty} A_n \Delta B_n$$

as

$$\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}} \frac{1}{n + \frac{x-2}{x-4}} \left(1 - \frac{x(n+1)}{2(2n+1)}\right),$$

the desired result then easily follows, in view of the partial fraction decomposition in (2.8), from the generating function evaluation for the sequence $n \mapsto \frac{1}{(2n+1)\binom{2n}{n}}$, and according to the modified Abel lemma in (1.1). $\square$

Assigning special values to the variable x in Theorem 2.5, we obtain new closed forms for ${}_3F_2$-series.

Example 2.6 *Letting φ denote the famous golden ratio constant, the closed-form evaluation*

$${}_3F_2\left[\begin{matrix} -\frac{2}{5}, 1, 1; \\ \frac{1}{2}, \frac{8}{5}; \end{matrix} \; -\frac{1}{4}\right] = \frac{12\ln(\varphi)}{5\sqrt{5}} + \frac{3}{5}.$$

follows in a direct way from Theorem 2.5. Maple, Mathematica, MATLAB, etc., cannot evaluate this series.

Example 2.7 *We can also prove that*

$${}_3F_2\left[\begin{matrix} -\frac{2}{3}, 1, 1; \\ \frac{1}{2}, \frac{4}{3}; \end{matrix} \; \frac{1}{4}\right] = \frac{1}{3} + \frac{2\pi}{9\sqrt{3}}$$

in a direct way via Theorem 2.5. Maple, Mathematica, MATLAB, etc., cannot evaluate this series.

Example 2.8 *The hypergeometric formula*

$${}_3F_2\left[\begin{matrix} -\frac{1}{4}, 1, 1; \\ \frac{1}{2}, \frac{7}{4}; \end{matrix} \; -1\right] = \frac{3}{4} + \frac{3\ln\left(1+\sqrt{2}\right)}{4\sqrt{2}}$$

follows directly from Theorem 2.5. Maple, Mathematica, MATLAB, etc., cannot evaluate the above series.

3. Cantarini's Ramanujan-like series

As in [10] (cf. [38]), the generalized Clebsch–Gordan integral refers to

$$\int_{-1}^{1} P_\mu(x) P_\nu(x) P_\nu(-x)\, dx \tag{3.1}$$

for complex parameters ν and μ, letting $\{P_n(x) : n \in \mathbb{N}_0\}$ denote the family of Legendre polynomials. By rewriting

$$ {}_2F_1\left[\begin{matrix} -v, v+1; \\ 1; \end{matrix} \; x \right] {}_2F_1\left[\begin{matrix} -v, v+1; \\ 1; \end{matrix} \; 1-x \right] $$

as

$$ -\frac{\sin(\pi v)}{2} \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{\Gamma(n-v)\Gamma(n+v+1)(4n+1)}{\Gamma\left(n-v+\frac{1}{2}\right)\Gamma\left(n+v+\frac{3}{2}\right)} P_{2n}(2x-1) $$

for a complex parameter v and for $x \in (0,1)$, as in [10], using the fact that

$$ {}_2F_1\left[\begin{matrix} -v, v+1; \\ 1; \end{matrix} \; x \right] = P_v(1-2x), $$

it is shown, in [10], using integrals as in (3.1), that

$$ \sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 \frac{\Gamma(n-v)\Gamma(n+v+1)(4n+1)}{\Gamma\left(n-v+\frac{1}{2}\right)\Gamma\left(n+v+\frac{3}{2}\right)} $$

admits an explicit evaluation as $-\frac{\cot\left(\frac{\pi v}{2}\right)\Gamma^2\left(\frac{v+1}{2}\right)}{\pi\Gamma^2\left(\frac{v+2}{2}\right)}$ for a complex value v outside of $(-2\mathbb{N}+1) \cup 2\mathbb{N}$, with (1.2) as a corollary. This relies on Zhou's identity [38] whereby

$$ \begin{aligned} &\int_{-1}^{1} P_v(x)P_v(-x)P_{2m}(x)\,dx \\ &= -\frac{\sin(\pi v)}{2} \left(\frac{1}{4^n}\binom{2m}{m}\right)^2 \frac{\Gamma(m-v)\Gamma(m+v+1)}{\Gamma\left(m-v+\frac{1}{2}\right)\Gamma\left(m+v+\frac{3}{2}\right)} \end{aligned} \tag{3.2} $$

under certain conditions. Zhou's work in [38] on the evaluation of integrals as in (3.2) concerns relatively sophisticated concepts from what is referred to as Clebsch–Gordan theory. Our new proof of the Cantarini's formula in (1.2) is dramatically simpler and does not involve Legendre polynomials.

3.1. Preliminaries on complete elliptic integrals

The complete elliptic integrals of the first and second kinds may, respectively, be defined as follows:

$$ \mathbf{K}(k) = \frac{\pi}{2} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}; \\ 1; \end{matrix} \; k^2 \right], \quad \mathbf{E}(k) = \frac{\pi}{2} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2}, -\frac{1}{2}; \\ 1; \end{matrix} \; k^2 \right]. $$

As in [6], we record the following generating function (g.f.) identity:

$$ \sum_{n=0}^{\infty} \binom{2n}{n}^3 x^n = \frac{4\mathbf{K}^2\left(\frac{\sqrt{1-\sqrt{1-64x}}}{\sqrt{2}}\right)}{\pi^2}. \tag{3.3} $$

As in [2], we recall that this power series identity may be proved using Clausen's hypergeometric product. We later make implicit use of the following identity for $\mathbf{K}$ evaluated at a complex argument, as in [6]:

$$ \mathbf{K}\left(ik/k'\right) = k'\mathbf{K}(k), \tag{3.4} $$

where $k' = \sqrt{1-k^2}$. The identity

$$\mathbf{E} = \frac{\pi}{4\sqrt{r}\mathbf{K}} + \left(1 - \frac{\alpha(r)}{\sqrt{r}}\right)\mathbf{K}, \tag{3.5}$$

is used in a crucial way in [6], where α denotes the elliptic alpha function (see [3, §5]). This elliptic alpha function identity is of such importance in [6] because it shows how the evaluation of an equivalent form of the Ramanujan-like series

$$\sum_{k=0}^{\infty} \left(-\frac{1}{64}\right)^k \frac{\binom{2k}{k}^3}{k+1} \left(\frac{4k+3}{(k+1)^2}\left(4k^2+6k+3\right)O_k + 4\right) = 8 - \frac{16}{\pi}$$

introduced in [6] via the WZ method boils down to the evaluation of the following $\mathbf{K}$- and $\mathbf{E}$-values, which cannot be evaluated by Maple, Mathematica, MATLAB, etc., letting $O_k = 1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}$ denote the k^{th} odd harmonic number. We had obtained these values using a known elliptic singular value, together with (3.4) and (3.5):

$$\mathbf{K}\left(i\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}\right) = \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{2^{11/4}\sqrt{\pi}}. \tag{3.6}$$

and

$$\mathbf{E}\left(i\sqrt{\frac{1}{\sqrt{2}} - \frac{1}{2}}\right) = \frac{\sqrt[4]{2}\pi^{3/2}}{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} + \frac{\left(2+\sqrt{2}\right)\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{2^{19/4}\sqrt{\pi}}. \tag{3.7}$$

3.2. A new and simplified proof

Following the procedure outlined in Section 1.2, we begin by setting

$$B_n = \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3. \tag{3.8}$$

According to the aforementioned procedure, we should find a rational function $\frac{1}{a_1 n + a_2}$ such that

$$\frac{1}{a_1 n + a_2}\left(1 + \frac{(2n+1)^3}{8(n+1)^3}\right) \tag{3.9}$$

simplifies in such a way so that $\frac{1}{a_1 n + a_2}$ vanishes in the partial fraction decomposition of (3.9). Solving for a_1 and a_2, accordingly, we find that:

$$\frac{1}{4n+3}\left(1 + \frac{(2n+1)^3}{8(n+1)^3}\right) = \frac{1}{8(n+1)^3} - \frac{1}{4(n+1)^2} + \frac{1}{2(n+1)}. \tag{3.10}$$

This leads us toward our alternate proof of (1.2).

We apply, as below, our evaluation method from Section 1.2 to construct a completely different proof (cf. [10]) of what we refer to as Cantarini's formula, which may be formulated in an equivalent way as follows [10]:

$$\sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 \frac{1}{(4n-1)(4n+3)} = \frac{(\sqrt{2}-2)\Gamma^4\left(\frac{1}{8}\right)}{64\pi^2\Gamma^2\left(\frac{1}{4}\right)} + \frac{16\Gamma^2\left(\frac{1}{4}\right)}{\left(\sqrt{2}-2\right)\Gamma^4\left(\frac{1}{8}\right)}, \tag{3.11}$$

We let B_n be as in (3.8). As suggested in (3.10), we are to set $A_n = \frac{1}{4n+3}$ in the modified Abel lemma. From the g.f. identity in (3.3), it is easily verifiable that the g.f.'s for each of the following sequences is explicitly evaluable in terms of the **K**- and **E**-functions:

$$\left(\binom{2n}{n}^3 \frac{1}{n+1} : n \in \mathbb{N}_0\right), \tag{3.12}$$

$$\left(\binom{2n}{n}^3 \frac{1}{(n+1)^2} : n \in \mathbb{N}_0\right), \tag{3.13}$$

$$\left(\binom{2n}{n}^3 \frac{1}{(n+1)^3} : n \in \mathbb{N}_0\right). \tag{3.14}$$

Indeed, these g.f.'s evaluated at $-\frac{1}{64}$ are all expressible in terms of combinations of closed-form expressions and combinations of the special elliptic values highlighted in (3.6) and (3.7). Leaving the computational verification of this as an exercise, this effectively completes the proof, since, according to the modified Abel lemma, we have shown how the series

$$\sum_{n=1}^{\infty} B_n \nabla A_n = \sum_{n=1}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 \left(\frac{1}{4n+3} - \frac{1}{4n-1}\right)$$

may be symbolically evaluated.

Using variants/generalizations of the above proof, we may obtain new Ramanujan-like series for $\frac{1}{\pi}$, such as

$$\sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 \frac{64n^5 + 112n^4 + 80n^3 + 12n^2 - 16n - 3}{(4n^2 - 2n + 1)(4n^2 + 6n + 3)(n+1)} = -\frac{4}{\pi},$$

noting that Maple, Mathematica, MATLAB, etc., are unable to evaluate the above series, i.e. not even in terms of **K**- or **E**-expressions. We may also mimic the above proof so as to obtain many families of variants of Ramanujan's series for $\frac{1}{\pi}$. For example, an application of the technique in Section 1.2, not unlike our proof of (3.11), may be used to prove that

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{12n} \binom{2n}{n}^3 \frac{81648n^4 - 2160n^3 - 141840n^2 - 34500n - 2267}{(6n+1)(6n+7)} = -\frac{1024}{\pi}. \tag{3.15}$$

It seems that there had not previously been much known about Ramanujan-like series for $\frac{1}{\pi}$ with summands that contain expressions as in $\frac{1}{6n+1}$, apart from the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 (4n+1)}{\left(\frac{1}{6}\right)_n^2 (6n+1)^2 9^n} = \frac{\sqrt{3}\sqrt[3]{2}\Gamma^3\left(\frac{1}{3}\right)}{12\pi}$$

recently given in [25] using summation techniques in the vein of [36], which relied on classical hypergeometric methodologies.

3.3. Applications to generalized binomial coefficients

In this subsection, we further demonstrate how the modified Abel lemma is very versatile as a tool, according to our technique from Section 1.2. We begin by observing that by taking $(-1)^n A_n$ in place of A_n in formula (1.1), we get

$$\sum_{n=1}^{\infty} B_n (-1)^n (A_n + A_{n-1}) = \left(\lim_{m\to+\infty} (-1)^m A_m B_{m+1}\right) - A_0 B_1 + \sum_{n=1}^{\infty} (-1)^n A_n \Delta B_n. \tag{3.16}$$

Now, consider the identity

$$\sum_{n=0}^{\infty} \binom{x}{n}^2 \frac{(-1)^n y^n}{n+1} = {}_2F_1\left[\begin{matrix} -x, -x; \\ 2; \end{matrix} \; -y\right] \quad |y| < 1, \quad x \in \mathbb{R}, \tag{3.17}$$

which can be easily proved using the series representation of the ${}_2F_1$-function. Then, recalling that

$$\int {}_2F_1\left[\begin{matrix} \alpha, \beta; \\ \gamma; \end{matrix} \; y\right] dy = \frac{(\gamma - 1)}{(\alpha - 1)(\beta - 1)} \, {}_2F_1\left[\begin{matrix} \alpha - 1, \beta - 1; \\ \gamma - 1; \end{matrix} \; y\right]$$

(see [5], equation 5, section 1.15.1) and Kummer's identity, as given in (2.5), we get that

$$\sum_{n=1}^{\infty} \binom{x}{n}^2 \frac{(-1)^n}{(n+1)^2} = \frac{1}{(x+1)^2} - \frac{\sqrt{\pi}\, 2^{2+x}}{(x+1)^3 \Gamma\left(\frac{1+x}{2}\right) \Gamma\left(-\frac{x}{2}\right)} - 1 \tag{3.18}$$

where the indeterminate form must be interpreted as limits. Now, assume that x is not an odd integer. Taking $A_n := \frac{(-1)^n}{n + \frac{1-x}{2}}$ and $B_n := \binom{x}{n}^2$ and using (3.16) we obtain

$$\sum_{n=1}^{\infty} \binom{x}{n}^2 \frac{(-1)^n (2n - x)}{(2n - x - 1)(2n - x + 1)} = -\frac{x^2}{2(1 - x)} + \frac{x+1}{2} \sum_{n \geq 1} \binom{x}{n}^2 \frac{(-1)^n}{(n+1)^2}.$$

Finally, inserting (3.18), we obtain

$$\sum_{n=0}^{\infty} \binom{x}{n}^2 \frac{(-1)^n (2n - x)}{(2n - x - 1)(2n - x + 1)} = -\frac{\sqrt{\pi} 2^{2+x}}{2(x+1)^2 \Gamma\left(\frac{1+x}{2}\right) \Gamma\left(-\frac{x}{2}\right)}. \tag{3.19}$$

From equation (3.19) we can deduce some closed forms that Mathematica, Maple, MATLAB, etc., fail to calculate.

Example 3.1 *We may prove the following equations, via a straightforward application of the formula shown in*

(3.19).

$$\sum_{n=0}^{\infty}\binom{-1/2}{n}^{2}\frac{(-1)^{n}(4n+1)}{(4n-1)(4n+3)}=-\frac{2\sqrt{2\pi}}{\Gamma^{2}\left(\frac{1}{4}\right)},$$
$$\sum_{n=0}^{\infty}\binom{-1/3}{n}^{2}\frac{(-1)^{n}(6n+1)}{(6n-2)(6n+4)}=-\frac{\sqrt{3}\pi}{2\Gamma^{2}\left(\frac{1}{3}\right)},$$
$$\sum_{n=0}^{\infty}\binom{1/4}{n}^{2}\frac{(-1)^{n}(8n-1)}{(8n-5)(8n+3)}=\frac{\sqrt[4]{2}\sqrt{\pi}}{25\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{5}{8}\right)},$$
$$\sum_{n=0}^{\infty}\binom{3/2}{n}^{2}\frac{(-1)^{n}(4n-3)}{(4n-5)(4n-1)}=\frac{6\sqrt{2\pi}}{25\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)},$$
$$\sum_{n=0}^{\infty}\binom{-3/2}{n}^{2}\frac{(-1)^{n}(4n+3)}{(4n+5)(4n+1)}=\frac{\sqrt{\pi}}{2\sqrt{2}\Gamma^{2}\left(\frac{3}{4}\right)}.$$

Note that the first identity can be also proved using the Fourier-Legendre expansion of $(x(1-x))^{1/4}$ *(cf. [11]).*

It is interesting to note that we can repeat the process every time we have a "closed form" for $\sum_{n=1}^{\infty}\binom{x}{n}^{k}\frac{(-1)^{n}}{(n+1)^{j}}$ or $\sum_{n=1}^{\infty}\binom{x}{n}^{k}\frac{1}{(n+1)^{j}}$ for each positive integer k and for every $j\in 1,\dots,k$. Indeed, taking $B_{n}=\binom{x}{n}^{k}$ we have

$$B_{n+1}-B_{n}=B_{n}\left(1-\left(\frac{x-n}{n+1}\right)^{k}\right)$$

and clearly

$$1-\left(\frac{x-n}{n+1}\right)^{k}=\frac{(2n-x+1)\sum_{j=0}^{k-1}(n+1)^{k-j-1}(x-n)^{j}}{(n+1)^{k}}. \tag{3.20}$$

Hence it is obvious to note, using the partial fraction decomposition of the right-hand side of (3.20) and taking $A_{n}:=\frac{1}{n+\frac{1-x}{2}}$ or $A_{n}:=\frac{(-1)^{n}}{n+\frac{1-x}{2}}$, that $\sum_{n=1}^{\infty}A_{n}\Delta B_{n}$ is a combination of terms of the type $\sum_{n=1}^{\infty}\binom{x}{n}^{k}\frac{(-1)^{n}}{(n+1)^{j}}$ or $\sum_{n=1}^{\infty}\binom{x}{n}^{k}\frac{1}{(n+1)^{j}}$.

3.4. Further results

With regard to our application of the ${}_2F_1$-identity in (3.17) via (3.16), this leads us to consider how our technique in Section 1.2 may be applied to ${}_2F_1(x)$-functions in full generality.

If we set

$$B_{n}=\frac{(a)_{n}(b)_{n}}{(c)_{n}n!}x^{n},$$

then, by the modified Abel lemma, we get

$$\sum_{n\geq 1}\frac{(a)_{n}(b)_{n}}{(c)_{n}n!}x^{n}(A_{n}-A_{n-1})=-A_{0}\frac{ab}{c}x+\sum_{n\geq 1}\frac{(a)_{n}(b)_{n}}{(c)_{n}n!}x^{n}A_{n}\left(1-\frac{(n+a)(n+b)x}{(n+c)(n+1)}\right).$$

Now, if we assume that $A_n = A_{n,x} = \frac{1}{(n+c)(n+1)-(n+a)(n+b)x}$, then

$$\sum_{n\geq 1} \frac{(a)_n (b)_n}{(c)_n n!} x^n (A_n - A_{n-1}) = -\frac{abx}{c^2 - abcx} + \sum_{n\geq 1} \frac{(a)_n (b)_n}{(c)_{n+1} (n+1)!} x^n$$

$$= -\frac{abx}{c^2 - abcx} + \frac{1}{x(a-1)(b-1)} \sum_{n\geq 2} \frac{(a-1)_n (b-1)_n}{(c)_n n!} x^n. \tag{3.21}$$

Although $\sum_{n\geq 1} \frac{(a)_n (b)_n}{(c)_n n!} x^n (A_n - A_{n-1})$ is not expressible as a single hypergeometric function, the generality of the above identity suggests that there is much in the way of possibility for obtaining new closed forms. For example, if we set $c = b - k$, $k \in \mathbb{N}$, on left-hand side we have

$$\sum_{n\geq 1} \frac{(a)_n (b)_n}{(b-k)_n n!} x^n (A_n - A_{n-1})$$

and Mathematica, Maple, MATLAB, etc., are not able to recognize this function in symbolic form. On the other hand,

$$\sum_{n\geq 0} \frac{(a-1)_n (b-1)_n}{(b-k)_n n!} x^n = {}_2F_1\left[\begin{matrix} a-1, b-1; \\ b-k; \end{matrix}\; x\right]$$

can be evaluated in a closed form using the following known identity:

$${}_2F_1\left[\begin{matrix} \alpha, \beta; \\ \beta-k; \end{matrix}\; x\right] = (1-x)^{-\alpha-k} \sum_{j=0}^{k} \frac{(-k)_j (\beta-\alpha-k)_j x^j}{(\beta-k)_j j!}.$$

4. Future research

Our evaluation technique, as described in Section 1.2 is very versatile in the evaluation of harmonic sums. For example, we may obtain the interesting formula

$$\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{H_n}{(4n-1)(4n+3)} = \frac{4\ln(2)}{\pi} - \frac{2}{\pi} - \frac{2\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)}$$

using the technique in Section 1.2. However, we have focused on hypergeometric identities in this article, and we leave it to a future project to use the method from Section 1.2 in the evaluation of Euler-type sums and the like.

By differentiating (3.19) and setting $x = -\frac{1}{2}$, we may obtain the following lemniscate-like constant:

$$\sum_{n=0}^{\infty} \left(-\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{(4n+1)\, O_n}{(4n-1)(4n+3)}, \tag{4.1}$$

letting $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$ denote the n^{th} odd harmonic number. Letting $A_n = (-1)^n \frac{O_n}{4n+3}$ and $B_n = \left(-\frac{1}{16}\right)^n \binom{2n}{n}^2$, and by making use of a harmonic sum introduced in [19], we can show that the problem of evaluating (4.1) is equivalent to the problem of evaluating the following lemniscate-like constant:

$$\sum_{n=0}^{\infty}\left(-\frac{1}{16}\right)^{n}\binom{2n}{n}^{2}\frac{1}{4n+3}.$$

How can this be evaluated, in view of the WZ proofs from [7]? Applying Carlson's theorem together with a differential operator to the hypergeometric identity given as Lemma 3 in [7], we can prove that the following lemniscate-like constant evaluation holds:

$$\frac{2\pi^2(\pi+4\ln(2)-4)}{\Gamma^4\left(\frac{1}{4}\right)}+\frac{4-6\ln(2)}{\pi}=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^{k}\binom{2k}{k}^{2}\frac{H_{2k}}{4k+3}.$$

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