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A class of Finsler measure spaces of constant weighted Ricci curvature

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Abstract: The weight Ricci curvature plays an important role in studying global Finsler geometry. In this paper, we study a class of Finsler measure spaces of constant weighted Ricci curvature. We explicitly construct new families of such complete Finsler measure spaces. In particular, we find an eigenfunction and its eigenvalue for such spaces, generalizing a result previously only known in the case of Gaussian shrinking soliton. Finally, we give necessary and sufficient conditions on the coordinate functions for these spaces to be Euclidean measure spaces.

Key words: Finsler measure space, constant weighted Ricci curvature, Finsler Gaussian soliton, eigenfunction, eigenvalue

1. Introduction

Finsler geometry is just the Riemannian geometry without the quadratic restriction. In Riemannian geometry, the volume of geodesic balls can be controlled by the lower bounds of the Ricci curvature. However the situation is much more complicated for Finsler metrics. In 1996, Z. Shen introduced the notion of the S-curvature (mean covariation in an alternation terminology in [13]) and proved that S-curvature and the Ricci curvature determine the local behavior of the Busemann–Hausdorff measure of small metric balls around a point [14]. In [10], Ohta introduced the weighted Ricci curvature Ric_N with $N \in [dimM, +\infty]$ which is the combination of the Ricci curvature and the S-curvature. The weighted Ricci curvature not only can control the volume of geodesic balls but also establish some interesting global results on a Finsler measure space [11, 12, 18], such as Bochner inequality and Poincaŕe-Lichnerowicz inequality. Note that $\mathrm{Ric}_N \geq k > 0$ for some $N \in [dimM, +\infty)$ implies the compactness of M by Bonnet–Myers Theorem [10]. In the Riemannian measure space case Ric_∞ reduces the Bakry-Émery Ricci curvature Ric_f where f is the potential function. Very recently, Yin-Mo obtain a logarithmic Sobolev inequality and a Lichnerowicz type theorem in complete Finsler measure spaces with $\mathrm{Ric}_\infty \geq k > 0$ [18]. Hence Finsler measure spaces of special weighted ∞ Ricci curvature properties deserve further study. In this paper, we construct and study complete noncompact Finsler measure spaces with constant weighted Ricci curvature Ric_∞. Let $F(x, y) = \varphi(y)$ be a Minkowski norm on $\mathbb{R}^n$ and σ_{BH} be its Busemann–Hausdorff measure function. We prove the following (see Section 3):

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Theorem 1.1 *For arbitrary real number* ρ*,* $(\mathbb{R}^n,\, F,\, e^{-\frac{\rho}{2}\varphi(x)^2}\sigma_{BH}dx)$ *is a complete Finsler measure spaces of constant weighted Ricci curvature* $\mathrm{Ric}_\infty = \rho$*.*

We have the following three interesting cases.
(a) When $F(x,\, y) = |y|,\; \rho = \frac{1}{2}$, then

$$(\mathbb{R}^n,\, F,\, dV) = (\mathbb{R}^n,\, g_{can},\, e^{\frac{|x|^2}{4}}dx)$$

is the Gaussian shrinking soliton [2, 3].
(b) When $F(x,\, y) = |y|, \rho = -\ln(2\pi)^{\frac{n}{2}}$, then

$$(\mathbb{R}^n,\, F,\, dV) = (\mathbb{R}^n, g_{can}, (2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx)$$

is the Gaussian probability space [5, 8].
(c) When $F(x,\, y) = |y|, \rho = -\frac{1}{2}$, then

$$(\mathbb{R}^n,\, F,\, dV) = (\mathbb{R}^n,\, g_{can},\, e^{-\frac{|x|^2}{4}}dx)$$

is the Gaussian expanding soliton [2].

It is worth mentioning that the Gaussian shrinking soliton, the Gaussian probability space and the Gaussian expanding soliton are important examples in the theory of Riemannian measure space. In [3], Cao and Zhou have proved that for any fixed point $p \in M$

$$\frac{1}{4}(r(x)-c)^2 \leq f(x) \leq \frac{1}{4}(r(x)+c)^2$$

on a nontrivial, noncompact, shrinking gradient Ricci soliton $(M, g, e^{-f}dV)$, where $r(x) = d(p,x)$ is the distance function from some fixed point $p \in M$. In view of the Gaussian shrinking soliton, the leading term $\frac{1}{4}r(x)^2$ for the lower and upper bounds on f in Cao-Zhou's result is optimal. The isoperimetric inequality and the Brann–Minkowski inequality in the Gaussian probability space have obtained by Eskenazis and Moschidis etc. [6, 9].

A Finsler Gaussian soliton $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ is a Minkowski space $(\mathbb{R}^n, F)$ together with a volume form $e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH}$ on $\mathbb{R}^n$, where $F(x,y) = \varphi(y)$, $d\mathrm{V}_{\mathrm{BH}}$ denotes the Busemann–Hausdorff volume form of F and ρ is a nonzero constant. Theorem 1.1 tells us that all Finsler Gaussian solitons have constant weighted Ricci curvature Ric_∞. Moreover we have the following

Theorem 1.2 *Let* $(\mathbb{R}^n,\, F,\, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ *be a Finsler Gaussian soliton. Then* $r^2 - \frac{n}{\rho}$ *is an eigenfunction corresponding eigenvalues* $\lambda = 2\rho$ *in* $(\mathbb{R}^n,\, F,\, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ *where* r *is the distance function from origin.*

Let us take a look at the special case: when $\varphi(y) = |y|,\; \rho = \frac{1}{2}$,

$$r^2 - \frac{n}{\rho} = |x|^2 - 2n = \sum_{k=1}^{n}\left[(x^k)^2 - 2\right]$$

is the eigenfunction of $\lambda_2(\mathbb{R}^n) = 1$ in $(\mathbb{R}^n,\ g_{can},\ e^{\frac{|x|^2}{4}}dx)$ (see (25) and Example 1 in [4]). By a direct computation we can obtain that $(x^k)^2 - 2$ are also eigenfunctions of $\lambda_2(\mathbb{R}^n) = 1$ for $k = 1, \cdots, n$. Thus it is a natural problem to study whether 2ρ is also the second eigenvalue of Finsler Laplacian on a Finsler Gaussian soliton $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ which φ is of non-Euclidean type (see Theorem 1.3 below).

We know that, in the Gaussian shrinking soliton $(\mathbb{R}^n,\ F,\ e^{\frac{|x|^2}{4}}dx)$, all coordinate functions are eigenfunctions of $\lambda_1(\mathbb{R}^n) = \frac{1}{2}$ [4]. A nature task for us is to determine all Finsler Gaussian solitons such that their coordinate functions are eigenfunctions of ρ. We show the following.

Theorem 1.3 *Let $(\mathbb{R}^n,\ F,\ e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ be a Finsler Gaussian soliton. Then, the following assertions are equivalent:*

(i) one of the coordinate functions is the eigenfunction of ρ;

(ii) all coordinate functions are the eigenfunctions of ρ;

(iii) φ is a Euclidean norm.

Theorem 1.3 tells us that on the Gaussian shrinking soliton $(\mathbb{R}^n,\ F,\ e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ each coordinate function is not the eigenfunction of ρ unless φ is Euclidean norm. This contrasts sharply with the situation in the Gaussian shrinking soliton.

2. Preliminaries

Let M be an n-dimensional manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i\partial/\partial x^i$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, y)$ is smooth in $TM \setminus \{0\}$;

(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;

(iii) Strong convexity: The fundamental tensor

$$g_y := g_{ij}(x, y)dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positively definite. A C^∞ manifold M with its Finsler metric F is said a Finsler manifold. A typical example of Finsler manifolds is defined on $\mathbb{R}^n$:

$$g_{ij} = g_{ij}(y).$$

The pair $(\mathbb{R}^n,\ F)$ is called a Minkowski space.

Let F be a Finsler metric on an n-dimensional manifold M. Given two linearly independent vectors $v, w \in T_xM\backslash\{0\}$, the flag curvature is defined by

$$K(v, w) := \frac{g_v(R^v(v, w)w, v)}{g_v(v, v)g_v(w, w) - g_v(v, w)^2},$$

where R^v is the Chern curvature [17]. Then the Ricci curvature for (M, F) is defined by

$$\mathrm{Ric}(x, v) = \sum_{\alpha=1}^{n-1} K(v, e_\alpha),$$

where $e_1, \cdots, e_{n-1}, \frac{v}{F(v)}$ form an orthonormal basis of T_xM with respect to g_v.

Let $(M, F, d\mu)$ be an n-dimensional Finsler measure manifold. Let G^i be the geodesic coefficients of F and $d\mu := \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. The S-curvature of $(F, d\mu)$ is given by

$$S := \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial \log \sigma}{\partial x^i}.$$

It is a globally defined scalar function on $TM\backslash\{0\}$. In the Riemannian measure space case $(M, F, d\mu)$, the S-curvature is just the differential of the potential function f, where $F = \sqrt{g_{ij}(x)y^iy^j}$ and $d\mu = e^{-f}\sqrt{\det(g_{ij}(x))}dx^1 \wedge \cdots \wedge dx^n$. Give a vector $y \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$, $\dot{\gamma}(0) = y$. Define

$$\dot{S}(x, y) := F^{-2} \frac{d}{dt}[S(\gamma(t), \dot{\gamma}(t))]_{t=0}.$$

Given a vector $y \in T_xM$, The weighted Ricci curvature Ric_∞ of $(M, F, d\mu)$ at y is defined by (see [10])

$$\mathrm{Ric}_\infty(x, y) := \mathrm{Ric}(x, y) + \dot{S}(x, y).$$

This is the Bakry-Émery Ricci curvature Ric_f in the Riemannian measure space case where f is the potential function. For definitions of other weighted Ricci curvatures see [10].

We mention that Ric_∞ is different from the weighted Ricci curvature $\mathbf{WPRic}_0$ introduced by Tabatabeifar et al. [16]. However, both Ric_∞ and $\mathbf{WPRic}_0$ are (a, b)-weighted Ricci curvature (defined by Z. Shen at 2020 Conference on Riemannian-Finsler geometry at Ningbo, China).

For a smooth function u, the gradient vector of u at x is defined by

$$\nabla u(x) := \mathcal{L}^{-1}(du), \tag{2.1}$$

where $\mathcal{L} : T_xM \to T_x^*M$ is the Legendre transformation. Let $v = v^i \frac{\partial}{\partial x^i}$ be a smooth vector field on M. The divergence of v with respect to an arbitrary volume form $d\mu := \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is defined by

$$\mathrm{div} v := \sum_{i=1}^{n} \left(\frac{\partial v^i}{\partial x^i} + v^i \frac{\partial \log \sigma(x)}{\partial x^i} \right).$$

Then the Finsler–Laplacian of u can be defined by [7]

$$\Delta u := \mathrm{div}(\nabla u),$$

where the equality is in the weak $W^{1,2}(M)$ sense (see [14], p.209). It is a nonlinear elliptic differential operator. Finsler–Laplacian is just the drift Laplacian if F is a Riemannian metric.

Let $\lambda \in \mathbb{R}$ and $f \in H_0^1(M, F, d\mu)$. Recall that λ and f are called an eigenvalue and an eigenfunction of $(M, F, d\mu)$ respectively if [14]

$$\Delta f = -\lambda f.$$

In particular, the smallest positive eigenvalue on Finsler Laplacian is called the first eigenvalue.

3. Proofs of Theorems 1.1 and 1.2

Let F be a Finsler metric on an n-dimensional manifold M. We say $\mathrm{Ric}_\infty = \rho$ for some $\rho \in \mathbb{R}$ if, for any $y \in T_xM$ [15]

$$\mathrm{Ric}_\infty(x, y) = \rho F(x, y)^2.$$

Let $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}\sigma_{BH}dx)$ be a Finsler measure space where $F(x, y) = \varphi(y)$ is a Minkowski norm on $\mathbb{R}^n$, σ_{BH} is the Busemann–Hausdorff measure function of F and ρ is a constant.

Note that $(\mathbb{R}^n, F)$ is a Minkowski space. Hence F is a projectively flat Finsler metric [1]. Let $\gamma(t) = vt$ be the geodesic passing through origin o. Then

$$\begin{aligned} r(x) &= d(o, x) \\ &= \int_0^s F(\gamma(t), \dot{\gamma}(t))dt \\ &= \int_0^s \varphi(v)dt \\ &= \varphi(v)s = \varphi(x), \end{aligned} \tag{3.1}$$

where $r(x)$ is the distance function from origin o and $x = \gamma(s)$. Since the geodesic coefficient $G^i = 0$ for $i \in \{1, \cdots, n\}$, F has vanishing Ricci curvature. Put

$$y = \dot{\gamma}(s) = \frac{d}{dt}(vt) \mid_{t=s} = v, \ u := t - s.$$

Then

$$\gamma(t) = (u + s)v = uv + x := \sigma(u).$$

It follows that

$$\sigma(0) = x, \ \dot{\sigma}(0) = v = y.$$

According to [14], the Hessian of the function $f = \frac{\rho}{2}\varphi^2$ is

$$\begin{aligned} Hessf(y) &= \frac{d^2}{du^2}(f \circ \sigma) \mid_{u=0} \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\dot{\sigma}^i(0)\dot{\sigma}^j(0) + \frac{\partial f}{\partial x^i}\ddot{\sigma}^i(0) \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\dot{\sigma}^i(0)\dot{\sigma}^j(0) - 2\frac{\partial f}{\partial x^i}G^i(v) \\ &= \rho\left(\frac{1}{2}\varphi^2(x)\right)_{x^i x^j} v^i v^j \\ &= \rho g_{ij}(v)v^i v^j = \rho\varphi^2(v) = \rho F(x, y)^2. \end{aligned}$$

It follows that

$$\mathrm{Ric}_\infty(x, y) = \mathrm{Ric}(x, y) + Hessf(y) = \rho F(x, y)^2.$$

Hence $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}\sigma_{BH}dx)$ has constant weighted Ricci curvature $\mathrm{Ric}_\infty = \rho$. Therefore we complete the proof of Theorem 1.1.

Now we are going to show Theorem 1.2. Let r be the distance functions from origin. By (14.7) and (3.7) in [14], we have

$$\Delta r = \frac{1}{\sigma(x)}\frac{\partial}{\partial x^i}\left[\sigma(x)g^{*ij}(dr)\frac{\partial r}{\partial x^j}\right], \tag{3.2}$$

where

$$\sigma(x) = \sigma_{BH}e^{-\frac{\rho}{2}\varphi^2(x)} = \sigma_{BH}e^{-\frac{\rho}{2}r^2(x)} \tag{3.3}$$

and

$$g^{*ij}(dr) = g^{*ij}(\mathcal{L}(\nabla r)) = g^{ij}(\nabla r). \tag{3.4}$$

Note that

$$\sigma_{BH} = \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}\,\{y\in\mathbb{R}^n|F(y)<1\}} = constant \tag{3.5}$$

for a Minkowski space $(\mathbb{R}^n, F)$. It follows that

$$\begin{aligned}\Delta r &= \frac{1}{e^{-\frac{\rho}{2}r^2}\sigma_{BH}}\frac{\partial}{\partial x^i}\left[\sigma_{BH}e^{-\frac{\rho}{2}r^2}g^{ij}(\nabla r)\frac{\partial r}{\partial x^j}\right]\\ &= g^{ij}(\nabla r)\frac{\partial r}{\partial x^j}\frac{\partial}{\partial x^i}\left(-\frac{\rho}{2}r^2\right) + \frac{\partial}{\partial x^i}\left[g^{ij}(\nabla r)\frac{\partial r}{\partial x^j}\right]\\ &= -\rho r\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}g^{*ij}(dr) + \frac{\partial r}{\partial x^j}\frac{\partial}{\partial x^i}\left[g^{ij}(\nabla r)\right] + g^{ij}(\nabla r)\frac{\partial^2\varphi}{\partial x^i\partial x^j}\end{aligned} \tag{3.6}$$

where we have used (3.2), (3.3), (3.4) and (3.5). We know ∇r is a unit vector field [14]. Thus we have

$$\begin{aligned}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}g^{*ij}(dr) &= \frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}g^*_{dr}(dx^i, dx^j)\\ &= g^*_{dr}(dr, dr)\\ &= [F^*(dr)]^2\\ &= [F(\nabla r)]^2 = 1.\end{aligned} \tag{3.7}$$

For each $x\in\mathbb{R}^n$, identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$. Combining this with (2.1) we have

$$\nabla r = (\nabla^i r)\frac{\partial}{\partial x^i} = (\nabla^1 r, \nabla^2 r, \cdots, \nabla^n r),$$

where

$$\nabla^i r = \frac{\partial r}{\partial x^j}g^{ji}(\nabla r), \tag{3.8}$$

where

$$(g^{ij}) = (g_{_{ij}})^{-1},\;\; g_{ij} := \frac{1}{2}\frac{\partial^2\varphi^2}{\partial x^i\partial x^j}.$$

It follows that

$$\frac{\partial g_{jk}(\nabla r)}{\partial x^i} = \frac{\partial}{\partial x^i}\left[g_{jk}(\nabla^1 r, \nabla^2 r, \cdots, \nabla^n r)\right] = 2C_{jkl}(\nabla r)\frac{\partial(\nabla^l r)}{\partial x^i}, \tag{3.9}$$

where $C_{jkl} := \frac{1}{2}\frac{\partial g_{kl}}{\partial x^j}$. From (3.9) we have

$$\begin{aligned}\frac{\partial g^{il}(\nabla r)}{\partial x^i} &= -g^{ij}(\nabla r)\frac{\partial g_{jk}(\nabla r)}{\partial x^i}g^{kl}(\nabla r)\\ &= -2g^{ij}(\nabla r)C_{jks}(\nabla r)\frac{\partial(\nabla^s r)}{\partial x^i}g^{kl}(\nabla r)\end{aligned} \tag{3.10}$$

where we have used the following formula

$$g^{ij}(\nabla r)g_{jk}(\nabla r) = \delta^i_k.$$

Note that

$$(\nabla^k r)C_{lks}(\nabla r) = 0.$$

Together with (3.10) and (3.8), we obtain

$$\begin{aligned}\frac{\partial r}{\partial x^j}\frac{\partial}{\partial x^i}\left[g^{ij}(\nabla r)\right] &= -2\frac{\partial r}{\partial x^j}g^{il}(\nabla r)C_{lks}(\nabla r)\frac{\partial(\nabla^s r)}{\partial x^i}g^{kj}(\nabla r)\\ &= -2g^{il}(\nabla r)\frac{\partial(\nabla^s r)}{\partial x^i}(\nabla^k r)C_{lks}(\nabla r) = 0.\end{aligned} \tag{3.11}$$

It follows from (3.1) that

$$dr = d\varphi = \frac{\partial \varphi}{\partial x^j}dx^j. \tag{3.12}$$

For each $x \in \mathbb{R}^n$, identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$. Together this with (3.12) and Proposition 14.8.1 in [1] we get

$$\mathcal{L}(x) = \mathcal{L}\left(x^i\frac{\partial}{\partial x^i}\right) = \varphi\frac{\partial \varphi}{\partial x^j}dx^j = rdr \tag{3.13}$$

for $x \in \mathbb{R}^n$. This gives

$$x = \mathcal{L}^{-1}(rdr) = r\nabla r.$$

We obtain $g^{ij}(\nabla r) = g^{ij}(x)$. Combining this with (3.1), (3.4) and (3.7) we get

$$\begin{aligned}g^{ij}(\nabla r)\frac{\partial^2 \varphi}{\partial x^i \partial x^j} &= g^{ij}(x)\left(\varphi\frac{\partial^2 \varphi}{\partial x^i \partial x^j} + \frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j} - \frac{\partial \varphi}{\partial x^i}\frac{\partial \varphi}{\partial x^j}\right)/\varphi\\ &= \frac{1}{\varphi}\left[g^{ij}g_{ij} - g^{ij}(\nabla r)\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}\right] = \frac{n-1}{r}.\end{aligned} \tag{3.14}$$

Plugging (3.7), (3.11) and (3.14) into (3.6) yields

$$\Delta r = -\rho r + \frac{n-1}{r}. \tag{3.15}$$

Without much difficulty, one can show that

$$\nabla r^2 = \mathcal{L}^{-1}(dr^2) = 2r\mathcal{L}^{-1}(dr) = 2r\nabla r.$$

This gives

$$\begin{aligned}\Delta\left(r^2 - \frac{n}{\rho}\right) &= \Delta r^2 \\ &= div(\nabla r^2) \\ &= 2r div(\nabla r) + 2dr(\nabla r) \\ &= 2r\Delta r + 2\left[F(\nabla r)\right]^2 \\ &= 2r\left(\frac{n-1}{r} - \rho r\right) + 2 = -2\rho\left(r^2 - \frac{n}{\rho}\right)\end{aligned}$$

where we have used (3.7) and (3.15). Thus $r^2 - \frac{n}{\rho}$ is an eigenfunction corresponding eigenvalue $\lambda = 2\rho$ in $(\mathbb{R}^n, F, e^{-\frac{\rho}{4}F(x)^2}\sigma_{BH}dx)$.

4. A new characterization of Minkowski norm of Euclidean type

Now we are going to establish the Lemmas required in the proof of Theorem 1.3.

Lemma 4.1 *Let $(\mathbb{R}^n, F)$ be a Minkowski space. Then the gradient of x^k is a constant vector, that is*

$$\nabla x^k = constant \tag{4.1}$$

for $k = 1, \cdots, n$. Therefore, $\left(g^{ij}(\nabla x^k)\right)$ is a constant matrix.

Proof Note that $(\mathbb{R}^n, F)$ is a Minkowski space. Hence we get g^{*ij} are independent of x, where $g^{*ij} = \left[\frac{1}{2}F^{*2}(x, p)\right]_{p_i p_j}$. For each $x \in \mathbb{R}^n$, identity $T_x\mathbb{R}^n$ and $T_x(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. Then

$$\nabla x^k = \mathcal{L}^{-1}(dx^k) = g^{*lk}(e_k)\frac{\partial}{\partial x^l} = (g^{*1k}(e_k), g^{*2k}(e_k)\cdots, g^{*nk}(e_k))$$

where e_k is the k-th standard base of $\mathbb{R}^n$. It follows that the section ∇x^k is a constant section in $T\mathbb{R}^n$. We obtain that $(g^{ij}(\nabla x^k))$ is a constant matrix, where

$$(g^{ij}) := (g_{ij})^{-1}, \;\; g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial x^i \partial x^j}.$$

□

Lemma 4.2 *Let $(\mathbb{R}^n, F, e^{-\frac{\rho}{2}\varphi(x)^2}dV_{BH})$ be a Finsler Gaussian soliton. Then for $k \in \{1, \cdots, n\}$*

$$\Delta x^k = -\rho g^{ik}(\nabla x^k)g_{il}(x)x^l. \tag{4.2}$$

Proof By using Lemma 4.1, we have

$$\frac{\partial g^{*ik}(dx^k)}{\partial x^i} = \frac{\partial g^{ik}(\nabla x^k)}{\partial x^i} = 0.$$

Together with (14.7) in [14], (3.3) and (3.5), we have

$$\begin{aligned} \Delta x^k &= \frac{1}{e^{-\frac{\rho}{2}\varphi^2}} \frac{\partial}{\partial x^i} \left[e^{-\frac{\rho}{2}\varphi^2} g^{*ij}(dx^k) \frac{\partial x^k}{\partial x^j} \right] \\ &= \frac{1}{e^{-\frac{\rho}{2}\varphi^2}} \frac{\partial}{\partial x^i} \left[e^{-\frac{\rho}{2}\varphi^2} g^{*ik}(dx^k) \right] \\ &= g^{*ik}(dx^k) \frac{\partial}{\partial x^i} \left(-\frac{\rho}{2}\varphi^2 \right) + \frac{\partial}{\partial x^i} \left[g^{*ik}(dx^k) \right] \\ &= -\rho x^j g^{ik}(\nabla x^k) \frac{\partial^2}{\partial x^i \partial x^j} \frac{[\varphi(x)]^2}{2} = -\rho g^{ik}(\nabla x^k) g_{ij}(x) x^j. \end{aligned} \tag{4.3}$$

□

Let $(\mathbb{R}^n, F)$ be an n-dimensional Minkowski space and

$$C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k}(y)$$

be its Cartan torsion [14]. Suppose that there is an i_0 such that $C_{i_0 jk} = 0$. Without lose of generality we can assume that $i_0 = 1$. It follows that

$$\frac{\partial g_{ij}}{\partial y^1} = 2C_{ij1} = 0,$$

where g_{ij} are the components of the fundamental tensor g_y in the direction y [1]. Thus we obtain

$$g_{ij} = g_{ij}(y^2, \cdots, y^n).$$

In particular, we have

$$g_{ij}(1, y^2, \cdots, y^n) = g_{ij}(y^2, \cdots, y^n).$$

Since g_{ij} is positively homogeneous of degree 0, we have, for $y \neq 0$,

$$\begin{aligned} [F(y)]^2 &= g_{ij}(y) y^i y^j \\ &= g_{ij}(ty^1, ty^2, \cdots, ty^n) y^i y^j = g_{ij}(1, ty^2, \cdots, ty^n) y^i y^j, \end{aligned}$$

where $t > 0$. Letting t go to zero, by smoothness on $\mathbb{R}^n \setminus \{0\}$ we obtain

$$F(y)^2 = g_{ij}(1, 0, \cdots, 0) y^i y^j.$$

It is then easy to check that setting $\tilde{g}_{ij}(y) = g_{ij}(1, 0, \cdots, 0)$ one gets a Euclidean norm on $\mathbb{R}^n$, whose associated norm is exactly F. We have obtained the following:

Lemma 4.3 *Let $(\mathbb{R}^n, F)$ be an n-dimensional Minkowski space and C_{ijk} be its Cartan torsion. Assume that there is an i_0 such that $C_{i_0jk} = 0$. Then F comes from an Euclidean norm on $\mathbb{R}^n$.*

Proof of Theorem 1.3.

$(i) \Rightarrow (iii)$. Suppose that one of the coordinate functions x^{i_0} is the eigenfunction corresponding the eigenvalue ρ. Without lose of generality we can assume that $i_0 = 1$. Hence

$$\Delta x^1 = -\rho x^1.$$

Combining this with (4.2) yields

$$x^1 = g^{i1}(\nabla x^1) g_{il}(x) x^l \tag{4.4}$$

where we have used that $\rho \neq 0$. Differentiating (4.4) with respect to x^1, we obtain

$$\begin{aligned} 1 &= g^{i1}(\nabla x^1)\frac{\partial}{\partial x^1}\left[g_{il}(x)x^l\right] \\ &= g^{i1}(\nabla x^1)\left[g_{i1}(x) + x^l\frac{\partial^3\varphi^2}{\partial x^1\partial x^i\partial x^l}\right] = g^{i1}(\nabla x^1)g_{i1}(x) \end{aligned} \tag{4.5}$$

where we have used the fact $\frac{\partial^2\varphi^2}{\partial x^1\partial x^i}$ is positively homogeneous of degree 0. Similarly differentiating (4.4) with respect to x^α, we have

$$g^{i1}(\nabla x^1)g_{i\alpha}(x) = 0, \qquad \alpha = 2, \cdots, n. \tag{4.6}$$

Let

$$A := \begin{pmatrix} g^{11}(\nabla x^1) & g^{21}(\nabla x^1) & \cdots & g^{n1}(\nabla x^1) \\ g^{12}(x) & g^{22}(x) & \cdots & g^{n2}(x) \\ \vdots & \vdots & & \vdots \\ g^{1n}(x) & g^{2n}(x) & \cdots & g^{nn}(x) \end{pmatrix}, \ G := (g_{ij}(x)). \tag{4.7}$$

Using (4.5), (4.6) and the following fact

$$g^{i\alpha}(x)g_{ij}(x) = \delta^\alpha_j,$$

we have

$$AG = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

It follows that

$$A = G^{-1} = \left(g^{ij}(x)\right).$$

Taking this together with the first equation of (4.7) yields

$$g^{11}(\nabla x^1) = g^{11}(x), \cdots, g^{n1}(\nabla x^1) = g^{n1}(x). \tag{4.8}$$

For each $x \in \mathbb{R}^n$, we identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$. Then

$$g^{*ij}(\mathcal{L}(x)) = g^{ij}(x) \tag{4.9}$$

and

$$(x_1, \cdots, x_n) = \mathcal{L}(x^1, \cdots, x^n) = \mathcal{L}(x),$$

where $x_k = x^j g_{jk}(x)$. We take $(x_1, \cdots, x_n)$ as a coordinate system in $(\mathbb{R}^n)^*$. Then

$$g^{*ij}(x_1, \cdots, x_n) = \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(\varphi^*)^2,$$

where φ^* is the dual Minkowski norm of φ [14]. From (4.8), (4.9), (4.10) and Lemma 4.1 we have

$$g^{*1j}(x_1, \cdots, x_n) \ = \ g^{1j}(x) = g^{1j}(\nabla x^1) = constant.$$

It follows that the Cartan torsion C^{*ijk} of φ^* satisfy

$$C^{*1jk} = 0.$$

By Lemma 4.3, φ^* is a Euclidean norm on $(\mathbb{R}^n)^*$. Thus φ is a Euclidean norm on $\mathbb{R}^n$.

$(iii) \Rightarrow (ii)$. Assume that $F(x, y) = \sqrt{a_{ij}y^i y^j}$, where (a_{ij}) is a constant matrix. By (4.2) we have

$$\Delta x^k = -\rho x^k, \quad k = 1, \cdots, n.$$

$(ii) \Rightarrow (i)$. It is obvious.

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