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Sequences of polynomials satisfying the Pascal property

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Abstract: Since one of the most important properties of binomial coefficients is the Pascal's triangle identity (referred to as the Pascal property) and since the sequence of binomial polynomials forms a regular basis for integer-valued polynomials, it is natural to ask whether the Pascal property holds in some more general setting, and what types of integer-valued polynomials possess the Pascal property. After defining the general Pascal property, a sequence of polynomials which satisfies the Pascal property is characterized with the classical case as an example. In connection with integer-valued polynomials, characterizations are derived for a sequence of polynomials which satisfies the Pascal property and also forms a regular basis of integer-valued polynomials; this is done both in a discrete valuation domain and in a Dedekind domain. Several classical cases are worked out as examples.

Key words: Pascal property, integer-valued polynomial, discrete valuation domain with finite residue field, Dedekind domain

1. Introduction

In the classical case (i.e. in $\mathbb{Z}$, the ring of integers), the Pascal's triangle identity states that

$$\binom{k}{n-1} = \binom{k+1}{n} - \binom{k}{n}, \tag{1.1}$$

where $\binom{k}{n}$ is the usual binomial coefficient with integers $k \geq 0$ and $n \geq 1$. Viewing this as an identity among polynomials evaluated over $\mathbb{Z}$ leads us to the following definition.

Let K be a field, let

- $\mathcal{U} := \{u_k\}_{k\geq 0}$ be a sequence of distinct elements in K, and let
- $\mathcal{P} := \{P_n(x)\}_{n\geq 0} \subseteq K[x]$, with $P_0(x) = 1$, and $\deg P_n(x) = n \quad (n \geq 1)$.

We say that the sequence $\mathcal{P}$ satisfies the Pascal property (with respect to $\mathcal{U}$), or the pair $(\mathcal{P},\mathcal{U})$ is a Pascal pair, if, for each $n \geq 1$ and for all $k \geq 0$, there holds the identity

$$P_{n-1}(u_k) = P_n(u_{k+1}) - P_n(u_k). \tag{1.2}$$

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When $u_k = k$ and $P_n(x)$ is the binomial polynomial $\binom{x}{n}$, the relation (1.2) clearly becomes the classical Pascal's triangle identity (1.1). In another direction, it is well-known that the binomial polynomials defined by

$$\binom{x}{0} = 1, \qquad \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \quad (n \geq 1)$$

form a regular basis for the $\mathbb{Z}$-module $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials over $\mathbb{Z}$, [4, Proposition I.1.1]. This leads us to the problem of determining conditions on the pair $(\mathcal{P},\mathcal{U})$ which is both a Pascal pair and a regular basis for the integer-valued polynomials.

The objectives of our work here are:

- to determine condition(s) for a pair $(\mathcal{P},\mathcal{U})$ to be a Pascal pair;
- for a discrete valuation domain V, to determine condition(s) on a Pascal pair $(\mathcal{P},\mathcal{U})$ for which the sequence $\mathcal{P}$ forms a regular basis for the the module of integer-valued polynomials $\mathrm{Int}(V)$;
- to extend the localized results obtained in the second objective from a discrete valuation domain to a Dedekind domain.

2. The Pascal property

Let K, $\{u_k\}$ and $\mathcal{P} = \{P_n(x)\}$ be as set out above. In this section, we determine condition(s) for a pair $(\mathcal{P},\mathcal{U})$ to be a Pascal pair. For convenience, set

$$P_0(x) = 1, \qquad P_n(x) = \frac{1}{d_n}\sum_{i=0}^{n} a_{n,i}x^i \qquad (n \geq 1), \tag{2.1}$$

where $d_n \in K^* := K \setminus \{0\}$, and $a_{n,n} = 1$.

Theorem 2.1 *The pair $(\mathcal{P},\mathcal{U})$ is a Pascal pair if and only if the following conditions hold for k, $n \in \mathbb{N}$, the set of natural numbers:*

$$u_k = u_0 + kd_1 \tag{2.2}$$

$$d_n = n! \cdot d_1^n \tag{2.3}$$

$$a_{n,m} = \frac{1}{n+1}\sum_{i=0}^{n-m}\binom{m+1+i}{m} a_{n+1,m+1+i}d_1^i \qquad (0 \leq m \leq n). \tag{2.4}$$

Proof Assume $(\mathcal{P},\mathcal{U})$ is a Pascal pair. Then for $k \geq 1$, we have

$$1 = P_0(u_{k-1}) = P_1(u_k) - P_1(u_{k-1}) = \frac{u_k + a_{1,0}}{d_1} - \frac{u_{k-1} + a_{1,0}}{d_1} = \frac{u_k - u_{k-1}}{d_1},$$

and so

$$u_k = u_{k-1} + d_1. \tag{2.5}$$

Iterating the relation (2.5), we get

$$u_k = u_{k-1} + d_1 = (u_{k-2} + d_1) + d_1 = \cdots = u_0 + kd_1,$$

which proves (2.2). Next, for $n \geq 1$ and $k \geq 0$, using (2.1) and the Pascal property, we have

$$
\begin{aligned}
\frac{1}{d_n}\sum_{m=0}^{n} a_{n,m}u_k^m &= P_n(u_k) = P_{n+1}(u_{k+1}) - P_{n+1}(u_k) \\
&= \frac{1}{d_{n+1}}\sum_{j=0}^{n+1} a_{n+1,j}(u_k+d_1)^j - \frac{1}{d_{n+1}}\sum_{j=0}^{n+1} a_{n+1,j}(u_k)^j \\
&= \frac{1}{d_{n+1}}\sum_{j=1}^{n+1} a_{n+1,j}((u_k+d_1)^j - u_k^j) = \frac{1}{d_{n+1}}\sum_{j=1}^{n+1} a_{n+1,j}\left(\sum_{r=0}^{j-1}\binom{j}{r} d_1^{j-r}u_k^r\right) \\
&= \frac{1}{d_{n+1}}\sum_{j=0}^{n}\left(\sum_{r=0}^{j}\binom{j+1}{r} a_{n+1,j+1}\ d_1^{j+1-r}u_k^r\right) = \frac{1}{d_{n+1}}\sum_{m=0}^{n}\left(\sum_{i=m+1}^{n+1}\binom{i}{m} a_{n+1,i}\ d_1^i\right)u_k^m \\
&= \frac{1}{d_{n+1}}\sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\binom{m+1+i}{m} a_{n+1,m+1+i}d_1^{i+1}\right)u_k^m, \qquad (2.6)
\end{aligned}
$$

which yields

$$\sum_{m=0}^{n} a_{n,m}u_k^m = \frac{d_n}{d_{n+1}}\sum_{m=0}^{n}\left(\sum_{i=0}^{n-m}\binom{m+1+i}{m} a_{n+1,m+1+i}d_1^{i+1}\right)u_k^m \quad (k \geq 0). \tag{2.7}$$

Rewrite (2.7) as

$$A_0 + A_1u_k + \cdots + A_nu_k^n = 0 \quad (k \geq 0), \tag{2.8}$$

where

$$A_m = a_{n,m} - \frac{d_n}{d_{n+1}}\sum_{i=0}^{n-m}\binom{m+1+i}{m} a_{n+1,m+1+i}\ d_1^{i+1} \quad (0 \leq m \leq n). \tag{2.9}$$

Taking $k \in \{0,1,\ldots,n\}$, the relation (2.8) yields the following homogeneous system of equations in $A_0,\ldots,A_n$:

$$
\begin{gathered}
A_0 + A_1u_0 + \cdots + A_nu_0^n = 0 \\
\vdots \\
A_0 + A_1u_n + \cdots + A_nu_n^n = 0,
\end{gathered}
$$

whose coefficient matrix is Vandermonde in the u_k's. Since the elements u_k's are all distinct, the determinant of the coefficient matrix does not vanish, resulting in

$$0 = A_0 = A_1 = \cdots = A_n,$$

and so (2.9) becomes

$$a_{n,m} = \frac{d_n}{d_{n+1}}\sum_{i=0}^{n-m}\binom{m+1+i}{m} a_{n+1,m+1+i}\ d_1^{i+1} \qquad (0 \leq m \leq n). \tag{2.10}$$

Since $a_{n,n} = 1$, taking $m = n$ in (2.10) again, we obtain $1 = \frac{d_n}{d_{n+1}}(n+1)d_1$, which yields $d_{n+1} = (n+1)\cdot d_1d_n$. Iterating this last relation, we get

$$d_n = n\cdot d_1d_{n-1} = n(n-1)\cdot d_1^2d_{n-2} = \cdots = n!\cdot d_1^n \qquad (n \geq 1),$$

which proves (2.3). Substituting $d_{n+1} = (n+1)d_1 d_n$ into (2.10), we get

$$a_{n,m} = \frac{1}{n+1} \sum_{i=0}^{n-m} \binom{m+1+i}{m} a_{n+1,m+1+i}\ d_1^i \qquad (0 \le m \le n),$$

which proves (2.4).

Conversely, assume that (2.2), (2.3) and (2.4) hold. Using the identity derived in (2.6), (2.3) and (2.4), we have

$$\begin{aligned}
P_{n+1}(u_{k+1}) - P_{n+1}(u_k) &= \frac{1}{d_{n+1}} \sum_{m=0}^{n} \left(\sum_{i=0}^{n-m} \binom{m+1+i}{m} a_{n+1,m+1+i}\ d_1^{i+1} \right) u_k^m \\
&= \frac{1}{(n+1)! \cdot d_1^{n+1}} \sum_{m=0}^{n} \left(\sum_{i=0}^{n-m} \binom{m+1+i}{m} a_{n+1,m+1+i}\ d_1^{i+1} \right) u_k^m \\
&= \frac{1}{(n+1)! \cdot d_1^{n}} \sum_{m=0}^{n} \left(\sum_{i=0}^{n-m} \binom{m+1+i}{m} a_{n+1,m+1+i}\ d_1^{i} \right) u_k^m \\
&= \frac{1}{(n+1)! \cdot d_1^{n}} \sum_{m=0}^{n} (n+1) a_{n,m} u_k^m = \frac{1}{n! \cdot d_1^n} \sum_{m=0}^{n} a_{n,m} u_k^m = \frac{1}{d_n} \sum_{m=0}^{n} a_{n,m} u_k^m = P_n(u_k),
\end{aligned}$$

confirming that the pair $(\mathcal{P}, \mathcal{U})$ satisfies the Pascal property. □

Remark. From the relation (2.2) in Theorem 2.1, the validity of the Pascal property necessarily forces the field K to be of characteristic 0; for otherwise, the elements $\{u_k\}$ cannot all be distinct. Henceforth, it is implicit throughout that the field characteristic $\mathrm{char}(K) = 0$.

An immediate consequence of Theorem 2.1 is the fact that in the classical case only binomial polynomials can form a Pascal pair, as we show now.

Corollary 2.2 *Let $K = \mathbb{Q}$, the field of rational numbers. Assume that $(\mathcal{P}, \mathcal{U})$ is a Pascal pair. If $u_0 = 0$, $d_1 = 1$ and $a_{n,0} = 0$ for all $n \in \mathbb{N}_0$, the set of nonnegative integers, then*

$$P_n(x) = \binom{x}{n} \qquad (n \in \mathbb{N}_0). \tag{2.11}$$

Proof The expression (2.11) holds for $n = 0$ by definition. Since $d_1 = 1, u_0 = 0$, by Theorem 2.1, we get

$$d_n = n!,\ u_k = k \qquad (n, k \in \mathbb{N}_0),$$

which implies at once that $P_1(x) = \binom{x}{1}$. Assume now that $P_{n-1}(x) = \binom{x}{n-1}$ for $n \ge 2$. By the Pascal property, for $k \in \mathbb{N}_0,\ n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{k(k-1)\cdots(k-(n-2))}{(n-1)!} = \binom{k}{n-1} &= P_n(k+1) - P_n(k) \qquad (2.12)\\
&= \frac{1}{n!} \sum_{r=0}^{n} a_{n,r}((k+1)^r - k^r).
\end{aligned}$$

Using $a_{n,0} = 0$, we get

$$n \cdot k(k-1)\cdots(k-(n-2)) = \sum_{r=1}^{n} a_{n,r}((k+1)^r - k^r) \qquad (k \geq 0). \tag{2.13}$$

Substituting $k = 0,\ 1, \ldots, n-2$ into (2.13), we get a system of equations in $a_{n,1}, \ldots, a_{n,n-1}$, whose coefficient matrix is

$$\begin{bmatrix} 1^1 - 0^1 & 1^2 - 0^2 & \cdots & 1^{n-1} - 0^{n-1} \\ 2^1 - 1^1 & 2^2 - 1^2 & \cdots & 2^{n-1} - 1^{n-1} \\ \vdots & & & \\ (n-1)^1 - (n-2)^1 & (n-1)^2 - (n-2)^2 & \cdots & (n-1)^{n-1} - (n-2)^{n-1} \end{bmatrix}.$$

Using elementary operations, this matrix can be put into a Vandermonde form, in the elements $1, 2, \ldots, n-1$, whose determinant is nonzero. The system then has a unique solution in $a_{n,1}, \ldots, a_{n,n-1}$. On the other hand, it is well-known that the relation (2.12) holds for the binomial polynomials $P_n(k) = \binom{k}{n}$. Thus, the unique set of coefficients of $P_n(x)$ must coincide with that of $P_n(k)$ yielding $P_n(x) = \binom{x}{n}$ as desired. □

Since binomial polynomials are intimately related to Stirling numbers, as a by-product we apply Theorem 2.1 to derive some seemingly new identities about Stirling numbers. Indeed, there are some relationships between binomial polynomials and certain well-known numbers, such as, Stirling numbers and Catalan numbers (see [7], [8]). Recall that the falling factorial of degree $n \in \mathbb{N}$, [5, Definition 3.1],

$$(x)_n := n! \cdot \binom{x}{n} = x(x-1)\cdots(x-n+1), \qquad (x)_0 := 1$$

has a power series expansion of the form

$$(x)_n = \sum_{m=0}^{n} s(n,m)\ x^m,$$

where the coefficients $s(n,m)$ are known as Stirling numbers of the first kind. Clearly, the definition implies $s(n,m) = 0$ if $m > n$ and $s(n,n) = 1$ for all $n \geq 0$.

Corollary 2.3 *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. We have*

$$s(n,m) = \frac{1}{n+1} \sum_{i=0}^{n-m} \binom{m+1+i}{m} s(n+1, m+1+i) \tag{2.14}$$

$$= \frac{1}{n-m} \sum_{i=1}^{n-m} \left(\binom{m+1+i}{m} - n\binom{m+i}{m} \right) s(n, m+i). \tag{2.15}$$

Proof From Corollary 2.2, we know that the binomial polynomial $\binom{x}{n}$ arises from taking $d_1 = 1$, $u_0 = 0$, $a_{n,0} = 0$ in the Pascal pair $(\mathcal{P}, \mathcal{U})$, and this in turn shows that $s(n,m)$ is identical with the corresponding $a_{n,m}$. The identity (2.14) thus follows from the relation (2.4).

To prove (2.15), recall from [5, Theorem 8.7] that Stirling numbers of the first kind $s(n,m)$ satisfy a triangular recurrence relation of the form

$$s(n+1,m) = s(n,m-1) - n \cdot s(n,m). \tag{2.16}$$

Combining (2.14) and (2.16), we get

$$\begin{aligned}(n+1) \cdot s(n,m) &= \sum_{i=0}^{n-m} \binom{m+1+i}{m} s(n+1,m+1+i) \\ &= \sum_{i=0}^{n-m} \binom{m+1+i}{m} \left(s(n,m+i) - n \cdot s(n,m+1+i)\right) \\ &= (m+1)s(n,m) + \sum_{i=1}^{n-m} \left(\binom{m+1+i}{m} - n\binom{m+i}{m}\right) s(n,m+i),\end{aligned}$$

and the result follows. □

3. Integer-valued polynomials

3.1. Discrete valuation domains

Throughout this subsection, let V be a discrete valuation domain with a finite residue field, and let Q_V be its quotient field. Denote by $\mathfrak{m}$ the unique maximal principal ideal of V, which is generated by a uniformizer t, and let q be the cardinal of residue field $V/\mathfrak{m}$. Let ν be the corresponding valuation on V, so that for each element $x \in V^* := V \setminus \{0\}$, the number $\nu(x)$ is the largest integer k such that $x \in \mathfrak{m}^k$.

Not only does the binomial coefficients satisfy the Pascal's triangle identity, but the set of binomial polynomials $\{\binom{x}{n}\}_{n\geq 0}$ also forms a regular basis for the $\mathbb{Z}$-module $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials over $\mathbb{Q}$. It is natural to ask for conditions guaranteeing that a Pascal pair $(\mathcal{P},\mathcal{U})$ is also a regular basis for the V-module $\mathrm{Int}(V)$ of integer-valued polynomials over V.

We start by recalling relevant notion and results taken from [4].

Definition 3.1 *Let A be an integral domain with its quotient field Q_A and B a domain for which $A[x] \subset B \subset Q_A[x]$. For $n \in \mathbb{N}_0$, the nth characteristic ideal of B is the fractional ideal*

$$J_n(B) = \{0\} \cup \{\alpha \in Q_A : \text{there is an } f \in B \text{ of the form } f = \alpha x^n + \alpha_{n-1}x^{n-1} + \cdots\},$$

which is the set of all leading coefficients of polynomials in B of degree $\leq n$ including 0.

If $B =$ Int(V), we write J_n instead of $J_n($Int$(V))$.

We shall make use of the following information about regular bases and $J_n(B)$.

Lemma 3.2 *Let $A, Q_A, B, J_n(B)$ be as defined in Definition 3.1.*

1) *[4, Proposition II.1.4] A sequence $\{f_n(x)\}_{n\in\mathbb{N}_0}$ of elements of B is a regular basis of B if and only if, for each n, the polynomial f_n is of degree n whose leading coefficient generates $J_n(B)$ as an A-module.*

2) *[4, Corollary II.2.9] In the case* $B = Int(V)$, *define an arithmetic function* $w_q(n)$ *on the positive integers by*

$$w_q(n) = \sum_{k\geq 1} \left\lfloor \frac{n}{q^k} \right\rfloor. \tag{3.1}$$

Then

$$J_n = t^{-w_q(n)}V,$$

i.e. J_n *is a fractional ideal generated by* $t^{-w_q(n)}$.

Adopting the notation (2.1), if, for each $x_0 \in V$, we have

$$\nu(x_0^n + x_0^{n-1}a_{n,n-1} + \cdots + a_{n,0}) \geq \nu(d_n),$$

then $P_n(x) = \frac{1}{d_n}\sum_{i=0}^{n} a_{n,i}x^i \in Int(V)$.

An answer to the question posed above is contained in the next theorem.

Theorem 3.3 *Let* $(\mathcal{P},\mathcal{U})$ *be a Pascal pair. Assume that* $\mathcal{P} \subset Int(V)$. *Then the sequence* $\mathcal{P}$ *is a regular basis for the* V*-module* $Int(V)$ *if and only if the following two conditions are fulfilled :*

1) *the element* d_1 *(in the first degree polynomial of* $\mathcal{P}$*) is a unit in* V*;*

2) *the valuation values* $\nu(k!) = w_q(k)$ *hold for all* $k \in \mathbb{N}$.

Proof Assume that $\mathcal{P}$ is a regular basis for the V-module Int(V). By Lemma 3.2, the leading coefficients $1/d_n$ generates J_n. Since generators of a fractional ideal are unique up to multiplication by units, this shows that $\nu(d_n) = w_q(n)$. Thus, $\nu(d_1) = w_q(1) = 0$, i.e. d_1 is a unit in V, which proves part 1. Furthermore, for each n, being a Pascal pair we have

$$w_q(n) = \nu(d_n) = \nu(d_1^n \cdot n!) = \nu(d_1^n) + \nu(n!) = \nu(n!),$$

which proves part 2.

Conversely, assume that the conditions 1) and 2) hold. We have $w_q(n) = \nu(n! \cdot d_1^n) = \nu(d_n)$, which by Lemma 3.2 implies that $1/d_n$ is a generator of J_n. This shows, by Lemma 3.2, that the set $\mathcal{P}$ forms a regular basis for V-module Int(V). □

3.2. Dedekind domains

In this subsection, we proceed to extend the results in the last subsection from a discrete valuation domain to a Dedekind domain. Throughout this subsection, let D be a Dedekind domain of characteristic 0, and let Q_D be its quotient field. Let Max(D) be the set of all maximal ideals of D. For $M \in$ Max(D), the localization of D at M is the set

$$D_M = \left\{ \frac{a}{b} \in Q_D : a \in D \text{ and } b \in D \setminus M \right\}.$$

It is easily checked that D_M is a discrete valuation domain with a valuation arising from the maximal ideal M, and this in turn implies that it has a unique maximal ideal. The structure of this unique maximal ideal is described in the next proposition.

Proposition 3.4 *For $M \in Max(D)$, the unique maximal ideal of D_M is*

$$\mathcal{M} := \left\{ \frac{a}{b} \in Q_{D_M} \mid a \in M \text{ and } b \in D \setminus M \right\},$$

where Q_{D_M} is the quotient field of the domain D_M.

Proof That $\mathcal{M}$ is an ideal of D_M is easily verified. There remains to check its maximality. Let I be an ideal of D_M which contains the ideal $\mathcal{M}$. Assume that $I \neq \mathcal{M}$. Then there exists $a/b \in I \setminus \mathcal{M}$. Thus, both a and b must be elements of $D \setminus M$, which implies that both a/b and b/a belong to D_M forcing a/b to be a unit in D_M, and so $I = D_M$ showing that $\mathcal{M}$ is a maximal ideal of D_M. □

There is a simple relation between fractional ideals of D and those of D_M as seen in the next proposition.

Proposition 3.5 *Let $M \in Max(D)$ and let $\mathcal{M}$ be the unique maximal ideal of D_M. The following two fractional ideals are equal*

$$\frac{1}{M}D = \frac{1}{\mathcal{M}}D_M.$$

Proof Since $D \subset D_M$, by Proposition 3.4, the fractional ideal inclusion that $\frac{1}{M} \subset \frac{1}{\mathcal{M}}$ holds, which in turn leads to the inclusion $\frac{1}{M}D \subset \frac{1}{\mathcal{M}}D_M$.

On the other hand, let $x \in \frac{1}{\mathcal{M}}D_M$. Then $x = (b/a) \cdot (c/d)$ for some $a \in M$, $c \in D$ and $b,\ d \in D \setminus M$. Thus, $x = (1/ad) \cdot bc \in \frac{1}{M}D$, showing that $\frac{1}{M}D \supset \frac{1}{\mathcal{M}}D_M$. □

Recalling the notion of characteristic ideal from Definition 3.1 and applying the result in [4, Proposition II.1.8], we have

$$(J_n(\mathrm{Int}(D)))_M = J_n(\mathrm{Int}(D_M)).$$

Denoting $J_n(\mathrm{Int}(D))$ by $\mathcal{J}_n$, we need the following result from [4]. For each $M \in \mathrm{Max}(D)$, we denote the cardinality of the residue field D/M by $N(M)$.

Lemma 3.6 *[4, Proposition II.3.1] The characteristic ideal $\mathcal{J}_n$ of $Int(D)$ can be represented in the form:*

$$\mathcal{J}_n = \prod_{M \in \mathrm{Max}(D)} M^{-w_{N(M)}(n)},$$

where the function w is as defined in (3.1); if the cardinality of the residue field D/M is infinite, as a convention, take $w_{N(M)} \equiv 0$.

An extension of the sufficiency part of Theorem 3.3 to Dedekind domains now reads:

Theorem 3.7 *Adopting the above notation, let $\mathcal{P}$ and $\mathcal{U} := \{u_k\}_{k \geq 0}$ be defined as in Section 1. Assume that*

- *$(\mathcal{P},\mathcal{U})$ is a Pascal pair;*
- *the polynomials $P_n(x) \in \mathcal{P}$ are an integer-valued over D for all n.*

If d_1 is a unit in D and

$$\frac{1}{n!}D = \prod_{M\in \mathrm{Max}(D)} M^{-w_{N(M)}(n)}, \tag{3.2}$$

then the sequence $\mathcal{P}$ is a regular basis of Int(D).

Proof If d_1 is a unit and (3.2) holds, then Lemma 3.6 shows that

$$\mathcal{J}_n = \prod_{M\in \mathrm{Max}(D)} M^{-w_{N(M)}(n)} = \frac{1}{n!}D = \frac{1}{n!\cdot d_1^n}D = \frac{1}{d_n}D.$$

The desired result now follows by invoking upon part 1) of Lemma 3.2. □

Regarding an extension of the necessity part of Theorem 3.3, i.e. the converse of Theorem 3.7, an extra condition related to a local property is needed for its validity, as we show now.

Theorem 3.8 *Adopting the notation of Theorem 3.7 and assume that*

- *$(\mathcal{P},\mathcal{U})$ is a Pascal pair;*
- *the polynomials $P_n(x)$ are an integer-valued over D for all n;*
- *the sequence $\mathcal{P} = \{P_n(x)\}$ forms a regular basis for Int(D_M) for all $M \in$ Max(D).*

If the sequence $\mathcal{P}$ is a regular basis of Int(D), then d_1 is a unit in D and the relation (3.2) holds.

Proof Since D_M is a discrete valuation domain, from the three hypotheses and the assumption that $\mathcal{P}$ is a regular basis for Int(D_M) for all $M \in \mathrm{Max}(D)$, by Theorem 3.3 the element d_1 must be a unit in D_M for all $M \in \mathrm{Max}(D)$. From Proposition 3.4, we may write $d_1 = a/b$, where both a and b are elements of $D \setminus M$ for all $M \in \mathrm{Max}(D)$.

We assert that both a and b must be units in D. For assuming that a is not a unit in D, since D is a Dedekind domain, the principal ideal (a) generated by a is contained in some maximal ideal of D, contradicting the fact that $a \in D \setminus M$ for all maximal ideal M in D. Consequently, both a and b are units in D yielding d_1 to be a unit in D. If $\mathcal{P}$ is a regular basis of Int(D), by Lemma 3.2 and the Pascal property (Theorem 2.1), we get

$$\mathcal{J}_n = \frac{1}{d_n}D = \frac{1}{d_1^n\cdot n!}D = \frac{1}{n!}D,$$

and the relation (3.2) follows at once from Lemma 3.6. □

The last two theorems (Theorems 3.7 and 3.8) provide intrinsic information on a Pascal pair to be a regular basis for integer-valued polynomials and vice versa. The next example illustrates that the second hypotheses in Theorem 3.7 cannot be removed.

Example. If we take $Q_D = \mathbb{Q}$, $n_0 = 0$ and $d_1 = 1 = a_{k,0}$ for all $k \geq 1$ in Theorem 2.1, then by (2.4)

$$a_{k,k-1} = \frac{1}{k+1}\sum_{i=0}^{1}\binom{k+i}{k-1}a_{k+1,k+i} = \frac{1}{k+1}\left(k\cdot a_{k+1,k} + \binom{k+1}{k-1}\right),$$

and so,

$$a_{k+1,k} = \frac{k+1}{k} \cdot a_{k,k-1} - \frac{1}{k}\binom{k+1}{k-1}. \tag{3.3}$$

By setting $a_{1,0} = 1$ and substituting $k = 1,\ 2,\ 3, \dots$, we obtain the values of $a_{2,1},\ a_{3,2},\ a_{4,3}, \dots$, respectively. Similarly,

$$\begin{aligned} a_{k,k-2} &= \frac{1}{k+1}\sum_{i=0}^{2}\binom{k-1+i}{k-2}a_{k+1,k-1+i} \\ &= \frac{1}{k+1}\left(\binom{k-1}{k-2}a_{k+1,k-1} + \binom{k}{k-2}a_{k+1,k} + \binom{k+1}{k-2}\right), \end{aligned}$$

and so,

$$a_{k+1,k-1} = \frac{k+1}{k-1} \cdot a_{k,k-2} - \frac{1}{k-1}\binom{k+1}{k-2} - \frac{1}{k-1}\binom{k}{k-2}a_{k+1,k}. \tag{3.4}$$

Using the values of $a_{k+1,k}$ from (3.3), the values of $a_{3,1},\ a_{4,2},\ a_{5,3}, \dots$ can then be computed from (3.4). Continuing in the same manner, all the coefficients of $\{P_n(x)\}$ in the corresponding Pascal pair $(\mathcal{P}, \mathcal{U})$ can be determined. For instance, we get $a_{2,1} = 1$ and $a_{2,0} = 1$ showing that $P_2(x) = (x^2 + x + 1)/2$. Since $P_2(1) = 3/2$, the polynomial $P_2(x)$ is not integer-valued. This gives us an example of a Pascal pair that satisfies the condition of (3.2) but does not form a regular basis of $\mathrm{Int}(\mathbb{Z})$.

4. Examples

In this section, we work out more examples starting with polynomials of Lagrange-interpolation form.

4.1. Lagrange-type polynomials

Let $\mathcal{U} := \{u_k\}_{k\geq 0}$ be a sequence of distinct elements in a discrete valuation domain V with $u_0 = 0$, and define the sequence of Lagrange-type polynomials $\mathcal{C} = \{C_n(x)\}_{n\geq 0}$ over the quotient field Q_V by

$$C_0(x) = 1, \qquad C_n(x) = \prod_{i=0}^{n-1}\frac{x - u_i}{u_n - u_i} \quad (n \geq 1). \tag{4.1}$$

Theorem 4.1 *The pair $(\mathcal{C}, \mathcal{U})$ is a Pascal pair if and only if $u_k = ku_1$ for all $k \in \mathbb{N}_0$.*

Proof If the sequence $\{C_n(x)\}$ satisfies the Pascal property with respect to $\mathcal{U}$, then

$$C_n(u_k) + C_{n-1}(u_k) = C_n(u_{k+1}) \tag{4.2}$$

for all $n \geq 1$ and $k \geq 0$. Substituting $n = 1$ and using the explicit form in (4.1), we get

$$\frac{u_k}{u_1} + 1 = C_1(u_k) + C_0(u_k) = C_1(u_{k+1}) = \frac{u_{k+1}}{u_1} \quad (k \geq 0),$$

which gives $u_k + u_1 = u_{k+1} \quad (k \geq 0)$. A simple iteration yields

$$u_k = u_{k-1} + u_1 = u_{k-2} + 2u_1 = \cdots = u_0 + ku_1 = ku_1 \quad (k \geq 0).$$

Conversely, assume that $u_k = ku_1 \quad (k \geq 0)$. Since

$$C_1(u_k) + C_0(u_k) = \frac{u_k}{u_1} + 1 = \frac{(k+1)u_1}{u_1} = \frac{u_{k+1}}{u_1} = C_1(u_{k+1}),$$

i.e. the Pascal property (4.2) holds for $n = 1$. Now, for $n \geq 2$, from (4.1) we get

$$C_n(u_k) + C_{n-1}(u_k) = \prod_{i=0}^{n-1} \frac{ku_1 - iu_1}{nu_1 - iu_1} + \prod_{j=0}^{n-2} \frac{ku_1 - ju_1}{(n-1)u_1 - ju_1} = \prod_{i=0}^{n-1} \frac{k-i}{n-i} + \prod_{j=0}^{n-2} \frac{k-j}{n-1-j}$$

$$= \binom{k}{n} + \binom{k}{n-1} = \binom{k+1}{n} = \prod_{i=0}^{n-1} \frac{(k+1-i)u_1}{(n-i)u_1} = \prod_{i=0}^{n-1} \frac{u_{k+1} - u_i}{u_n - u_i} = C_n(u_{k+1}).$$

□

Remark. As already observed in the remark after Theorem 2.1, here also for Lagrange-type polynomials to satisfy the Pascal property, the underlying field is necessarily of characteristic 0.

We push forward from Theorem 4.1 to deduce more information about Pascal pairs in this setting. Let $u_1 := u \in V^*$ so that $u_k = ku$. Theorem 4.1 shows then that the polynomials

$$\binom{x}{0}_u := 1 \quad \text{and} \quad \binom{x}{n}_u := \prod_{i=0}^{n-1} \frac{x - iu}{(n-i)u} = \frac{1}{n!u^n} \prod_{i=0}^{n-1} (x - iu) \qquad (n \geq 1) \tag{4.3}$$

satisfy the Pascal property with respect to the sequence

$$\mathcal{U}_0 := \{u_0 = 0, u_k = ku \ (k = 1, 2, \ldots)\};$$

henceforth, we refer to the polynomials in (4.3) as $\mathcal{U}_0$-binomials. We now determine a necessary and sufficient condition for these $\mathcal{U}_0$-binomials to form a regular basis of the V-module $\mathrm{Int}(V)$. To do so, we make use of Bhargava's notion of t-ordering ([2] and [1]).

Definition 4.2 *A sequence $\{u_k\}_{k \geq 0}$ of elements of V is a t-ordering of V if*

$$\nu\left(\prod_{k=0}^{n-1} (u_n - u_k)\right) \leq \nu\left(\prod_{k=0}^{n-1} (x_0 - u_k)\right) \quad \textit{for all } n \geq 1,\ x_0 \in V.$$

The next lemma displays two regular bases for $\mathrm{Int}(V)$ constructed using t-ordering sequences.

Lemma 4.3 *Let $\{u_k\}_{k \geq 0}$ be a t-ordering of V.*

I) ([2, Theorem 19]) A regular basis of V-module $\mathrm{Int}(V)$ is given by

$$f_n(x) = \prod_{k=0}^{n-1} \frac{x - u_k}{u_n - u_k} \quad (n \geq 0).$$

Conversely, the set of polynomials $\{f_n(x)\}$ forms a regular basis for $\mathrm{Int}(V)$ only if $\{u_k\}$ is a t-ordering of V.

II) ([1, Propostitition7]) The sequence of polynomials associated to the t-ordering $\{u_k\}$

$$f_n(x) = t^{-w_q(n)} \prod_{k=0}^{n-1} (x - u_k)$$

is a regular basis of $Int(V)$.

We pause to make some relevant observations that will be used later.

(Obs 1) Any two t-orderings $\{u_k\}$ and $\{w_k\}$ of V result in the same minimum condition:

$$\nu\left(\prod_{k=0}^{n-1}(u_n - u_k)\right) = \nu\left(\prod_{k=0}^{n-1}(w_n - w_k)\right) \quad (n \geq 1).$$

(Obs 2) By Lemmas 4.3, and 3.2, both $t^{-w_q(n)}$ and $\prod_{k=0}^{n-1} \frac{1}{u_n - u_k}$ with $\{u_k\}$ being a t-ordering, are generators of J_n. Thus,

$$w_q(n) = \nu\left(\prod_{k=0}^{n-1}(u_n - u_k)\right) \quad (n \geq 1). \tag{4.4}$$

(Obs 3) If a sequence $\{u_k\}_{k\geq 0}$ satisfies (4.4), then the polynomials

$$f_n(x) = \prod_{k=0}^{n-1} \frac{x - u_k}{u_n - u_k}$$

form a regular basis of the V-module $\mathrm{Int}(V)$ (because its leading coefficient generates J_n), and so by Propositon 4.3, the sequence $\{u_k\}$ is a t-ordering.

We now give an explicit example of Theorem 3.3 using $\mathcal{U}_0$-binomials.

Theorem 4.4 *Let $\binom{x}{n}_u$ be $\mathcal{U}_0$-binomials as defined in (4.3). The sequence $\{\binom{x}{n}_u\}_{n\geq 0}$ is a regular basis of $Int(V)$ if and only if u is a unit in V and $\nu(n!) = w_q(n)$ for all $n \geq 1$.*

Proof If the $\mathcal{U}_0$-binomials $\binom{x}{n}_u$ form a regular basis of V-module $\mathrm{Int}(V)$, then by [6, Theorem 2.2], the elements $u_k = ku$ $(0 < k \leq q-1)$ are units in V; in particular, u is a unit in V, so that $\nu(u) = 0$. It remains to show that $\nu(n!) = w_q(n)$ for all $n \geq 1$. Since $\binom{x}{n}_u$ forms a regular basis, Lemma 3.2 tells us that its leading coefficient $1/(n!u^n)$ is a generator of the characteristic ideal J_n. By (Obs 2),

$$w_q(n) = \nu(u^n \cdot n!) = \nu(u^n) + \nu(n!) = \nu(n!).$$

On the other hand, if u is a unit in V and $\nu(n!) = w_q(n)$ for all $n \geq 0$, then we claim that the sequence $\{u_k = ku\}$ is a t-ordering in V. To prove this claim, from the (Obs 2) and (Obs 3), it suffices to show that

$$\nu\left(\prod_{i=0}^{n-1} (n-i)u\right) = w_q(n) \qquad \text{for all } n \geq 1,$$

and this last relation is immediate from

$$\nu\left(\prod_{i=0}^{n-1}(n-i)u\right)=\nu\left(\prod_{i=0}^{n-1}(n-i)\right)+\nu(u)=\nu(n!)+0=w_q(n).$$

By (Obs 3), we conclude that the $\mathcal{U}_0$-binomials

$$\binom{x}{n}_u=\frac{1}{u^n\cdot n!}\cdot\prod_{k=0}^{n-1}(x-u_k)$$

also forms a regular basis for the V-module $\mathrm{Int}(V)$. $\square$

Next, we derive an explicit example of Theorem 3.7 using $\mathcal{U}_0$-binomials.

Theorem 4.5 *Let D be a Dedekind domain of characteristic 0. Let $\binom{x}{n}_u$ be polynomials as defined in (4.3).*

I) If u is a unit in D and

$$\left(\frac{1}{n!}\right):=\frac{1}{n!}D=\prod_{M\in\mathrm{Max}(D)}M^{-w_{N(M)}(n)},$$

then the sequence $\{\binom{x}{n}_u\}_{n\geq 0}$ is a regular basis of $Int(D)$.

II) Assume that $\binom{x}{n}_u$ is an integer-valued polynomials over D for all n. If the sequence $\{\binom{x}{n}_u\}_{n\geq 0}$ is a subset of integer-valued polynomails over D_M for all $M\in Max(D)$ and forms a regular basis of $Int(D)$, then u is a unit in D and

$$\left(\frac{1}{n!}\right):=\frac{1}{n!}D=\prod_{M\in\mathrm{Max}(D)}M^{-w_{N(M)}(n)}.$$

Proof Appealing to Theorem 4.4, and replacing d_1 by u in the proofs of Theorems 3.7 and 3.8, we obtain the results of parts I) and II), respectively. $\square$

Specializing $D=\mathbb{Z}$, we immediately get

Corollary 4.6 *The two sequences of polynomials*

$$\binom{x}{n}_1=\binom{x}{n}=\frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!}$$

and

$$\binom{x}{n}_{-1}=\frac{x(x+1)(x+2)\cdots(x+(n-1))}{(-1)^n\cdot n!}$$

form two regular bases of $\mathbb{Z}$-module Int($\mathbb{Z}$) which satisfy the Pascal's triangle identity.

4.2. VWDWO sequences

Consider the case $K = \mathbb{Q}$, and let $V_{\mathbb{Q}}$ be a discrete valuation domain of $\mathbb{Q}$. Here, q, the cardinality of the residue field, is a prime p, and so $\nu = \nu_p$. Recall that a sequence $\{u_k\}_{k\geq 0} \subset V$ is said to be very well distributed and well ordered (VWDWO) if for all $\ell, m \in \mathbb{N}_0$, the sequence elements satisfy

$$\nu(u_\ell - u_m) = \nu_q(\ell - m), \tag{4.5}$$

where $\nu_q(\ell)$ is the largest power of q that divides ℓ. Keeping the notation of $C_n(x)$ in (4.1), from [4, Theorem II.2.7], the polynomials $C_n(x)$ form a regular basis for the V-module $\mathrm{Int}(V)$ if the corresponding sequence $\{u_n\}$ satisfies the VWDWO condition (4.5). Very well distributed and well ordered sequences play a vital role in the investigation of integer-valued polynomials satisfying the so-called Lucas property, see e.g., [3] and [6].

Our last result shows that $\mathcal{U}_0$-polynomials also possess this property. To do so, we make use of:

Proposition 4.7 *[3, Theorem 2.2] If $n = n_0 + n_1q + \cdots + n_kq^k$ is the q-adic expansion of a positive integer n, and if $x = x_0 + x_1t + \cdots + x_jt^j + \cdots$ is the t-adic expansion of an element x of V, then*

$$C_n(x) \equiv C_{n_0}(x_0)C_{n_1}(x_1)\cdots C_{n_s}(x_s) \pmod{\mathfrak{m}}.$$

Since

$$\nu(nu - mu) = \nu((n-m)u) = \nu(n-m) = \nu_p(n-m) \qquad (n, m \geq 0),$$

the sequence $\{0, u, 2u, \dots\}$ is indeed a VWDWO sequence. From Proposition 4.7, we at once obtain:

Theorem 4.8 *If*

$$n = n_0 + n_1p + \cdots + n_sp^s, \quad x = x_0 + x_1p + \cdots + x_jp^j + \cdots$$

are p-adic expansions of $n \in \mathbb{N}$ and $x \in V_{\mathbb{Q}}$, then

$$\binom{x}{n} \equiv \binom{x_0}{n_0}_u \binom{x_1}{n_1}_u \cdots \binom{x_s}{n_s}_u \pmod{pV_{\mathbb{Q}}}.$$

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